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Continuity of Multilinear Operator on Normed Linear Spaces

Kazuhisa Nakasho
Yamaguchi University
Yamaguchi, Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. In this article, the continuity definitions of multilinear operators on normed linear spaces are discussed in the Mizar formalism [8]. In the first chapter, several basic theorems are prepared to handle the norm of the multilinear operator, and then it is formalized that the completeness of the linear space of bounded multilinear operators that range is a Banach space.

In the last chapter, the continuity of the multilinear operator on finite normed spaces is addressed. Especially, it is formalized that the continuity at the origin can be extended to the continuity at every point in its whole domain here. We referred to [9], [22], [15], [16] in this formalization.

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1. Completeness of the Space of Multilinear Operators

Now we state the propositions:

(1) Let us consider a natural number n, and a real number r. Suppose $0 < r$. Then there exists a real number s such that

 (i) $0 < s < r$, and

 (ii) $\sqrt{s \cdot s \cdot n} < r$.

(2) Let us consider finite sequences R_1, R_2 of elements of $\mathbb{R}$, natural numbers n, i, and a real number r. Suppose $i \in \mathrm{dom}\, R_1$ and $R_1 = n \mapsto (1$ **qua** real number) and $R_2 = R_1 +\cdot (i, r)$. Then $\prod R_2 = r$.

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(3) Let us consider a finite sequence F of elements of $\mathbb{R}$. Suppose for every element k of $\mathbb{N}$ such that $k \in \operatorname{dom} F$ holds $0 \leqslant F(k)$. Then $0 \leqslant \prod F$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequence F of elements of $\mathbb{R}$ such that for every element k of $\mathbb{N}$ such that $k \in \operatorname{dom} F$ holds $0 \leqslant F(k)$ and $\operatorname{len} F = \$_1$ holds $0 \leqslant \prod F$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [3, (23), (15)], [2, (3), (40)]. $\mathcal{P}[0]$ by [6, (94)]. For every natural number n, $\mathcal{P}[n]$ from [1, Sch. 2]. □

From now on X, G denote real norm space sequences, Y denotes a real normed space, and f denotes a multilinear operator from X into Y.

Now we state the propositions:

(4) $\operatorname{dom} \overline{X} = \operatorname{dom} X$.

(5) Let us consider an element z of $\prod X$. If $z = 0_{\prod X}$, then for every element i of $\operatorname{dom} X$, $z(i) = 0_{X(i)}$. The theorem is a consequence of (4).

(6) $f(0_{\prod X}) = 0_Y$. The theorem is a consequence of (5).

(7) Let us consider a finite sequence F of elements of $\mathbb{R}$. If for every element i of $\operatorname{dom} F$, $F(i) > 0$, then $\prod F > 0$.

(8) Let us consider a real norm space sequence X, and a real normed space Y. Suppose Y is complete. Let us consider a sequence s_1 of R-NormSpace-of-BoundedMu
If s_1 is Cauchy sequence by norm, then s_1 is convergent.

PROOF: Define $\mathcal{P}[\text{set}, \text{set}] \equiv$ there exists a sequence x_1 of Y such that for every natural number n, $x_1(n) = (\text{PartFuncs}(vseq(n), X, Y))(\$_1)$ and x_1 is convergent and $\$_2 = \lim x_1$. For every element x of $\prod X$, there exists an element y of Y such that $\mathcal{P}[x, y]$ by [**?** , (52), (45), (36)]. Consider f being a function from the carrier of $\prod X$ into the carrier of Y such that for every element x of $\prod X$, $\mathcal{P}[x, f(x)]$ from [5, Sch. 3]. Reconsider $t_1 = f$ as a function from $\prod X$ into Y. For every point u of $\prod X$ and for every element i of $\operatorname{dom} X$ and for every point x of $X(i)$, there exists a sequence x_2 of Y such that for every natural number n, $x_2(n) = ((\text{PartFuncs}(vseq(n), X, Y)) \cdot (\text{reproj}(i, u)))(x)$ and x_2 is convergent and $(t_1 \cdot (\text{reproj}(i, u)))(x) = \lim x_2$ by [4, (13)]. t_1 is Lipschitzian by [17, (20)], [**?** , (45)], [10, (9)], [11, (8)]. For every real number e such that $e > 0$ there exists a natural number k such that for every natural number n such that $n \geqslant k$ for every point x of $\prod X$, $\|(\text{PartFuncs}(vseq(n), X, Y))(x) - t_1(x)\| \leqslant e \cdot (\text{NrProduct}\, x)$ by [18, (8)], [**?** , (52), (45)], [5, (12)]. Reconsider $t_2 = t_1$ as a point of R-NormSpace-of-BoundedMultOpers(X, Y). For every real number e such that $e > 0$ there exists a natural number k such that for every natural number n such that $n \geqslant k$ holds $\|vseq(n) - t_2\| \leqslant e$ by [**?** , (35), (52)], [12, (45)], [**?** , (43)]. For every real number e such that $e > 0$ there exists a natural number m such that for every natural number n

such that $n \geqslant m$ holds $\|vseq(n) - t_2\| < e$. □

(9) Let us consider a real norm space sequence X, and a real Banach space Y. Then R-NormSpace-of-BoundedMultOpers(X, Y) is a real Banach space. The theorem is a consequence of (8).

Let X be a real norm space sequence and Y be a real Banach space. One can check that R-NormSpace-of-BoundedMultOpers(X, Y) is complete.

2. Equivalence of Continuity Definition of Multilinear Operator

Now we state the propositions:

(10) Let us consider a natural number n, an element F of $\mathcal{R}^n$, and a real number s. Suppose for every natural number i such that $i \in \operatorname{dom} F$ holds $0 \leqslant F(i) \leqslant s$. Then $|F| \leqslant \sqrt{s \cdot s \cdot (\operatorname{len} F)}$.
PROOF: Set $G = \operatorname{len} F \mapsto s$. Reconsider $F_0 = F$ as an element of $\mathbb{R}^{\operatorname{len} F}$. For every natural number j such that $j \in \operatorname{Seg} \operatorname{len} F_0$ holds $({}^2F_0)(j) \leqslant ({}^2G)(j)$ by [3, (57)], [20, (15)]. □

(11) Let us consider a real norm space sequence X, a real normed space Y, a multilinear operator f from X into Y, and a real number K. Suppose $0 \leqslant K$ and for every point x of $\prod X$, $\|f(x)\| \leqslant K \cdot (\operatorname{NrProduct} x)$. Let us consider points v_0, v_1 of $\prod X$, finite sequences C_0, C_1, and an element i of $\operatorname{dom} X$. Suppose $C_0 = v_0$ and $C_1 = v_1$ and $\|v_1 - v_0\| \leqslant 1$ and for every element j of $\operatorname{dom} X$ such that $i \neq j$ holds $C_1(j) = C_0(j)$. Then $\|f_{/v_1} - f_{/v_0}\| \leqslant (\|v_0\| + 1)^{\operatorname{len} X} \cdot K \cdot \|(v_1 - v_0)(i)\|$.
PROOF: For every object x such that $x \in \operatorname{dom} v_1$ holds $v_1(x) = (\operatorname{reproj}(i, v_0))(v_1(i))(x$ by (4), [? , (15), (16)]. Reconsider $v_3 = (\operatorname{reproj}(i, v_0))(v_1(i) - v_0(i))$ as a point of $\prod X$. Reconsider $R_1 = \operatorname{len} X \mapsto (1$ **qua** real number) as a finite sequence of elements of $\mathbb{R}$. Reconsider $N_1 = \|(v_1 - v_0)(i)\|$ as an element of $\mathbb{R}$. Reconsider $R_2 = R_1 +\cdot (i, N_1)$ as a finite sequence of elements of $\mathbb{R}$. Reconsider $R_3 = \operatorname{len} X \mapsto (\|v_0\| + 1)$ as a finite sequence of elements of $\mathbb{R}$. Set $R_4 = R_2 \bullet R_3$. $\prod R_2 = \|(v_1 - v_0)(i)\|$. Consider N_2 being a finite sequence of elements of $\mathbb{R}$ such that $\operatorname{dom} N_2 = \operatorname{dom} X$ and for every element i of $\operatorname{dom} X$, $N_2(i) = \|v_3(i)\|$ and $\operatorname{NrProduct} v_3 = \prod N_2$. For every element k of $\mathbb{N}$ such that $k \in \operatorname{dom} N_2$ holds $N_2(k) \leqslant R_4(k)$ and $0 \leqslant N_2(k)$ by [6, (60)], [19, (7)], [? , (15), (26)]. □

(12) Let us consider a real norm space sequence X, a real normed space Y, a multilinear operator f from X into Y, and a real number K. Suppose $0 \leqslant K$ and for every point x of $\prod X$, $\|f(x)\| \leqslant K \cdot (\operatorname{NrProduct} x)$. Let us consider a point v_0 of $\prod X$. Then there exists a real number M such that

(i) $0 \leqslant M$, and

(ii) for every point v_1 of $\prod X$ such that $\|v_1 - v_0\| \leqslant 1$ there exists a finite sequence F of elements of $\mathbb{R}$ such that $\operatorname{dom} F = \operatorname{dom} X$ and $\|f_{/v_1} - f_{/v_0}\| \leqslant M \cdot K \cdot (\sum F)$ and for every element i of $\operatorname{dom} X$, $F(i) = \|(v_1 - v_0)(i)\|$.

PROOF: Consider g being a function such that $v_0 = g$ and $\operatorname{dom} g = \operatorname{dom} \overline{X}$ and for every object i such that $i \in \operatorname{dom} \overline{X}$ holds $g(i) \in \overline{X}(i)$. Reconsider $C_0 = v_0$ as a finite sequence. Define $\mathcal{P}$[natural number] $\equiv$ for every points v_0, v_1 of $\prod X$ for every finite sequences C_0, C_1 such that $\|v_1 - v_0\| \leqslant 1$ and $v_0 = C_0$ and $v_1 = C_1$ and $\$_1 \leqslant \operatorname{len} X$ and $C_1 \restriction (\operatorname{len} X -' \$_1) = C_0 \restriction (\operatorname{len} X -' \$_1)$ there exists a finite sequence F of elements of $\mathbb{R}$ such that $\operatorname{dom} F = \operatorname{Seg} \$_1$ and $\|f_{/v_1} - f_{/v_0}\| \leqslant (\|v_0\| + 3)^{\operatorname{len} X} \cdot K \cdot (\sum F)$ and for every natural number n such that $n \in \operatorname{Seg} \$_1$ there exists an element i of $\operatorname{dom} X$ such that $i = \operatorname{len} X -' \$_1 + n$ and $F(n) = \|(v_1 - v_0)(i)\|$. $\mathcal{P}[0]$ by [2, (58)], [21, (15)], [6, (72)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [1, (11)], [2, (1)], [1, (13)], [4, (49)]. For every natural number n, $\mathcal{P}[n]$ from [1, Sch. 2]. Consider g being a function such that $v_1 = g$ and $\operatorname{dom} g = \operatorname{dom} \overline{X}$ and for every object i such that $i \in \operatorname{dom} \overline{X}$ holds $g(i) \in \overline{X}(i)$. Consider F being a finite sequence of elements of $\mathbb{R}$ such that $\operatorname{dom} F = \operatorname{Seg} \operatorname{len} X$ and $\|f_{/v_1} - f_{/v_0}\| \leqslant (\|v_0\| + 3)^{\operatorname{len} X} \cdot K \cdot (\sum F)$ and for every natural number n such that $n \in \operatorname{Seg} \operatorname{len} X$ there exists an element i of $\operatorname{dom} X$ such that $i = \operatorname{len} X -' \operatorname{len} X + n$ and $F(n) = \|(v_1 - v_0)(i)\|$. For every element i of $\operatorname{dom} X$, $F(i) = \|(v_1 - v_0)(i)\|$. □

(13) Let us consider a point x of $\prod X$, and a real number r. Suppose $0 < r$. Then there exists a finite sequence s of elements of $\mathbb{R}$ and there exists a non empty, non-empty finite sequence Y such that $\operatorname{dom} s = \operatorname{dom} X$ and $\operatorname{dom} Y = \operatorname{dom} X$ and $\prod Y \subseteq \operatorname{Ball}(x, r)$ and for every element i of $\operatorname{dom} X$, $0 < s(i) < r$ and $Y(i) = \operatorname{Ball}(x(i), s(i))$.

PROOF: Consider s_0 being a real number such that $0 < s_0 < r$ and $\sqrt{s_0 \cdot s_0 \cdot (\operatorname{len} X)} < r$. Set $C_2 = \operatorname{len} X \mapsto s_0$. For every element i of $\operatorname{dom} X$, $0 < C_2(i) < r$ by [3, (57)]. Define $\mathcal{P}_\infty$[object, object] $\equiv$ there exists an element i of $\operatorname{dom} X$ such that $\$_1 = i$ and $\$_2 = \operatorname{Ball}(x(i), C_2(i))$. For every natural number n such that $n \in \operatorname{Seg} \operatorname{len} X$ there exists an object d such that $P_1[n, d]$. Consider Y being a finite sequence such that $\operatorname{dom} Y = \operatorname{Seg} \operatorname{len} X$ and for every natural number n such that $n \in \operatorname{Seg} \operatorname{len} X$ holds $P_1[n, Y(n)]$ from [2, Sch. 1]. $\emptyset \notin \operatorname{rng} Y$ by [13, (14)]. For every element i of $\operatorname{dom} X$, $Y(i) = \operatorname{Ball}(x(i), C_2(i))$. For every object z such that $z \in \prod Y$ holds $z \in \operatorname{Ball}(x, r)$ by (4), [**?** , (26)], [3, (57)], (10). □

(14) Let us consider a real norm space sequence X, a real normed space Y, and a multilinear operator f from X into Y. Then

(i) f is continuous on the carrier of $\prod X$ iff f is continuous in $0_{\prod X}$, and

(ii) f is continuous on the carrier of $\prod X$ iff f is Lipschitzian.

Proof: $f_{/0_{\prod X}} = 0_Y$. If f is continuous in $0_{\prod X}$, then f is Lipschitzian by [14, (7)], (13), (4), (5). If f is Lipschitzian, then f is continuous on the carrier of $\prod X$ by (12), [6, (84)], [7, (10)], [3, (92), (112), (57)]. □

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**): 55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.

[7] Noboru Endou, Yasunari Shidama, and Keiichi Miyajima. The product space of real normed spaces and its properties. *Formalized Mathematics*, 15(**3**):81–85, 2007. doi:10.2478/v10037-007-0010-y.

[8] Adam Grabowski, Artur Korniłowicz, and Adam Naumowicz. Four decades of Mizar. *Journal of Automated Reasoning*, 55(3):191–198, 2015. doi:10.1007/s10817-015-9345-1.

[9] Miyadera Isao. *Functional Analysis*. Riko-Gaku-Sya, 1972.

[10] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[11] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.

[12] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.

[13] Kazuhisa Nakasho, Yuichi Futa, and Yasunari Shidama. Implicit function theorem. Part I. *Formalized Mathematics*, 25(**4**):269–281, 2017. doi:10.1515/forma-2017-0026.

[14] Takaya Nishiyama, Keiji Ohkubo, and Yasunari Shidama. The continuous functions on normed linear spaces. *Formalized Mathematics*, 12(**3**):269–275, 2004.

[15] Laurent Schwartz. *Théorie des ensembles et topologie, tome 1. Analyse*. Hermann, 1997.

[16] Laurent Schwartz. *Calcul différentiel, tome 2. Analyse*. Hermann, 1997.

[17] Yasunari Shidama. Banach space of bounded linear operators. *Formalized Mathematics*, 12(**1**):39–48, 2004.

[18] Yasumasa Suzuki, Noboru Endou, and Yasunari Shidama. Banach space of absolute summable real sequences. *Formalized Mathematics*, 11(**4**):377–380, 2003.

[19] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1 (**2**):329–334, 1990.

[20] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

[21] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[22] Kosaku Yoshida. *Functional Analysis*. Springer, 1980.

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