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Bilinear Operators on Normed Linear Spaces

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Summary. The main topic of this article is the property of bilinear operator on normed linear spaces formalized with help of Mizar [1]. In the first two chapters, algebraic structures of bilinear operators on linear spaces are formalized. Especially, the space of bounded bilinear operators on normed linear spaces is discussed here. In the third chapter, it is remarked that the algebraic structure of bounded bilinear operators to a certain Banach space also constitutes a Banach space.

In the last chapter, the correspondence between the space of bilinear operators and the space of composition of linear opearators is formalized. We referred to [6], [15], [5], [11] and [12] in this formalization.

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1. Real Vector Space of Bilinear Operators

Let X, Y, Z be real linear spaces. The functor BilinOpers(X,Y,Z) yielding a subset of RealVectSpace((the carrier of $X \times Y$), Z) is defined by

(Def. 1) for every set x, $x \in it$ iff x is a bilinear operator from $X \times Y$ into Z.

Let us observe that BilinOpers(X,Y,Z) is non empty and functional and BilinOpers(X,Y,Z) is linearly closed.

The functor R-VectorSpace-of-BilinOpers(X,Y,Z) yielding a strict RLS structure is defined by the term

(Def. 2) $\langle$BilinOpers(X, Y, Z), Zero(BilinOpers(X, Y, Z), RealVectSpace((the carrier of $X \times Y$), Z)), Add(BilinOpers(X, Y, Z), RealVectSpace((the carrier of $X \times Y$), Z)), Mult(BilinOpers(X, Y, Z), RealVectSpace((the carrier of $X \times Y$), Z))$\rangle$.

Let us note that R-VectorSpace-of-BilinOpers(X, Y, Z) is non empty and R-VectorSpace-of-BilinOpers(X, Y, Z) is Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, and scalar unital and R-VectorSpace-of-BilinOpers(X, Y, Z) is constituted functions.

Now we state the proposition:

(1) Let us consider real linear spaces X, Y, Z. Then R-VectorSpace-of-BilinOpers($X, Y,$ is a subspace of RealVectSpace((the carrier of $X \times Y$), Z).

Let X, Y, Z be real linear spaces, f be an element of R-VectorSpace-of-BilinOpers($X, Y,$ v be a vector of X, and w be a vector of Y. Let us note that the functor $f(v, w)$ yields a vector of Z. Now we state the propositions:

(2) Let us consider real linear spaces X, Y, Z, and vectors f, g, h of R-VectorSpace-of-B Then $h = f + g$ if and only if for every vector x of X and for every vector y of Y, $h(x, y) = f(x, y) + g(x, y)$.

(3) Let us consider real linear spaces X, Y, Z, vectors f, h of R-VectorSpace-of-BilinOpe and a real number a. Then $h = a \cdot f$ if and only if for every vector x of X and for every vector y of Y, $h(x, y) = a \cdot f(x, y)$.

Let us consider real linear spaces X, Y, Z. Now we state the propositions:

(4) $0_{\text{R-VectorSpace-of-BilinOpers}(X,Y,Z)} =$ (the carrier of $X \times Y$) $\longmapsto 0_Z$.

(5) (The carrier of $X \times Y$) $\longmapsto 0_Z$ is a bilinear operator from $X \times Y$ into Z.

2. Real Normed Linear Space of Bounded Bilinear Operators

Let X, Y, Z be real normed spaces and I_1 be a bilinear operator from $X \times Y$ into Z. We say that I_1 is Lipschitzian if and only if

(Def. 3) there exists a real number K such that $0 \leqslant K$ and for every vector x of X and for every vector y of Y, $\|I_1(x, y)\| \leqslant K \cdot \|x\| \cdot \|y\|$.

Now we state the propositions:

(6) Let us consider real normed spaces X, Y, Z, and a bilinear operator f from $X \times Y$ into Z. Suppose for every vector x of X for every vector y of Y, $f(x, y) = 0_Z$. Then f is Lipschitzian.

(7) Let us consider real normed spaces X, Y, Z. Then (the carrier of $X \times Y$) $\longmapsto 0_Z$ is a bilinear operator from $X \times Y$ into Z.

Let X, Y, Z be real normed spaces. Let us observe that there exists a bilinear operator from $X \times Y$ into Z which is Lipschitzian.

Now we state the proposition:

(8) Let us consider real normed spaces X, Y, Z, and an object z. Then $z \in \mathrm{BilinOpers}(X, Y, Z)$ if and only if z is a bilinear operator from $X \times Y$ into Z.

Let X, Y, Z be real normed spaces. The functor $\mathrm{BoundedBilinOpers}(X, Y, Z)$ yielding a subset of R-VectorSpace-of-BilinOpers(X, Y, Z) is defined by

(Def. 4) for every set x, $x \in it$ iff x is a Lipschitzian bilinear operator from $X \times Y$ into Z.

Note that BoundedBilinOpers(X, Y, Z) is non empty and BoundedBilinOpers(X, Y, Z) is linearly closed.

The functor R-VectorSpace-of-BoundedBilinOpers(X, Y, Z) yielding a strict RLS structure is defined by the term

(Def. 5) $\langle$BoundedBilinOpers(X, Y, Z), Zero(BoundedBilinOpers(X, Y, Z), R-VectorSpace-o

Now we state the proposition:

(9) Let us consider real normed spaces X, Y, Z. Then R-VectorSpace-of-BoundedBilinO is a subspace of R-VectorSpace-of-BilinOpers(X, Y, Z).

Let X, Y, Z be real normed spaces. Note that R-VectorSpace-of-BoundedBilinOpers$(X,$ is non empty and R-VectorSpace-of-BoundedBilinOpers(X, Y, Z) is Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, and scalar unital and R-VectorSpace-of-BoundedBilinOpers is constituted functions.

Let f be an element of R-VectorSpace-of-BoundedBilinOpers(X, Y, Z), v be a vector of X, and w be a vector of Y. One can verify that the functor $f(v, w)$ yields a vector of Z. Now we state the propositions:

(10) Let us consider real normed spaces X, Y, Z, and vectors f, g, h of R-VectorSpace-of-BoundedBilinOpers(X, Y, Z). Then $h = f + g$ if and only if for every vector x of X and for every vector y of Y, $h(x, y) = f(x, y) + g(x, y)$. The theorem is a consequence of (2).

(11) Let us consider real normed spaces X, Y, Z, vectors f, h of R-VectorSpace-of-Bound and a real number a. Then $h = a \cdot f$ if and only if for every vector x of X and for every vector y of Y, $h(x, y) = a \cdot f(x, y)$. The theorem is a consequence of (3).

(12) Let us consider real normed spaces X, Y, Z. Then $0_{\text{R-VectorSpace-of-BoundedBilinOpers}(X}$ (the carrier of $X \times Y) \longmapsto 0_Z$. The theorem is a consequence of (4).

Let X, Y, Z be real normed spaces and f be an object. Assume $f \in$ BoundedBilinOpers(X, Y, Z). The functor modetrans(f, X, Y, Z) yielding a Lipschitzian bilinear operator from $X \times Y$ into Z is defined by the term

(Def. 6) f.

Let u be a bilinear operator from $X \times Y$ into Z. The functor PreNorms(u) yielding a non empty subset of $\mathbb{R}$ is defined by the term

(Def. 7) $\{\|u(t,s)\|$, where t is a vector of X, s is a vector of Y : $\|t\| \leqslant 1$ and $\|s\| \leqslant 1\}$.

Let g be a Lipschitzian bilinear operator from $X \times Y$ into Z. Observe that PreNorms(g) is upper bounded.

Now we state the proposition:

(13) Let us consider real normed spaces X, Y, Z, and a bilinear operator g from $X \times Y$ into Z. Then g is Lipschitzian if and only if PreNorms(g) is upper bounded.

Let X, Y, Z be real normed spaces. The functor BoundedBilinOpersNorm(X, Y, Z) yielding a function from BoundedBilinOpers(X, Y, Z) into $\mathbb{R}$ is defined by

(Def. 8) for every object x such that $x \in$ BoundedBilinOpers(X, Y, Z) holds $it(x) = \sup$ PreNorms(modetrans(x, X, Y, Z)).

Let f be a Lipschitzian bilinear operator from $X \times Y$ into Z. Let us note that modetrans(f, X, Y, Z) reduces to f.

Now we state the proposition:

(14) Let us consider real normed spaces X, Y, Z, and a Lipschitzian bilinear operator f from $X \times Y$ into Z. Then (BoundedBilinOpersNorm(X, Y, Z))(f) = $\sup$ PreNorms(f).

Let X, Y, Z be real normed spaces. The functor R-NormSpace-of-BoundedBilinOpers(
yielding a non empty normed structure is defined by the term

(Def. 9) ⟨BoundedBilinOpers(X, Y, Z), Zero(BoundedBilinOpers(X, Y, Z), R-VectorSpace-o

Now we state the propositions:

(15) Let us consider real normed spaces X, Y, Z. Then (the carrier of $X \times Y$) $\longmapsto 0_Z = 0_{\text{R-NormSpace-of-BoundedBilinOpers}(X,Y,Z)}$. The theorem is a consequence of (12).

(16) Let us consider real normed spaces X, Y, Z, a point f of R-NormSpace-of-BoundedB
and a Lipschitzian bilinear operator g from $X \times Y$ into Z. Suppose $g = f$. Let us consider a vector t of X, and a vector s of Y. Then $\|g(t,s)\| \leqslant \|f\| \cdot \|t\| \cdot \|s\|$. The theorem is a consequence of (14).

Let us consider real normed spaces X, Y, Z and a point f of R-NormSpace-of-BoundedB
Now we state the propositions:

(17) $0 \leqslant \|f\|$. The theorem is a consequence of (14).

(18) If $f = 0_{\text{R-NormSpace-of-BoundedBilinOpers}(X,Y,Z)}$, then $0 = \|f\|$. The theorem is a consequence of (15) and (14).

Let X, Y, Z be real normed spaces. One can verify that every element of R-NormSpace-of-BoundedBilinOpers(X, Y, Z) is function-like and relation-like.

Let f be an element of R-NormSpace-of-BoundedBilinOpers(X, Y, Z), v be a vector of X, and w be a vector of Y. Observe that the functor $f(v, w)$ yields a vector of Z. Now we state the propositions:

(19) Let us consider real normed spaces X, Y, Z, and points f, g, h of R-NormSpace-of-BoundedBilinOpers(X, Y, Z). Then $h = f + g$ if and only if for every vector x of X and for every vector y of Y, $h(x, y) = f(x, y) + g(x, y)$. The theorem is a consequence of (10).

(20) Let us consider real normed spaces X, Y, Z, points f, h of R-NormSpace-of-Boundec and a real number a. Then $h = a \cdot f$ if and only if for every vector x of X and for every vector y of Y, $h(x, y) = a \cdot f(x, y)$. The theorem is a consequence of (11).

(21) Let us consider real normed spaces X, Y, Z, points f, g of R-NormSpace-of-Boundec and a real number a. Then

(i) $\|f\| = 0$ iff $f = 0_{\text{R-NormSpace-of-BoundedBilinOpers}(X,Y,Z)}$, and

(ii) $\|a \cdot f\| = |a| \cdot \|f\|$, and

(iii) $\|f + g\| \leqslant \|f\| + \|g\|$.

Proof: $\|f+g\| \leqslant \|f\|+\|g\|$ by [8, (45)], (17), (16), (19). $\|a\cdot f\| = |a|\cdot\|f\|$ by [8, (45)], (17), (16), [2, (46)]. □

Let X, Y, Z be real normed spaces. Observe that R-NormSpace-of-BoundedBilinOpers(is non empty and R-NormSpace-of-BoundedBilinOpers(X, Y, Z) is reflexive, discernible, and real normed space-like.

Now we state the proposition:

(22) Let us consider real normed spaces X, Y, Z. Then R-NormSpace-of-BoundedBilinO is a real normed space.

Let X, Y, Z be real normed spaces. Let us note that R-NormSpace-of-BoundedBilinOpe is vector distributive, scalar distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, and right complementable.

Now we state the proposition:

(23) Let us consider real normed spaces X, Y, Z, and points f, g, h of R-NormSpace-of-BoundedBilinOpers(X, Y, Z). Then $h = f - g$ if and only if for every vector x of X and for every vector y of Y, $h(x, y) = f(x, y) - g(x, y)$. The theorem is a consequence of (19).

3. Real Banach Space of Bounded Bilinear Operators

Now we state the propositions:

(24) Let us consider real normed spaces X, Y, Z. Suppose Z is complete. Let us consider a sequence s_1 of R-NormSpace-of-BoundedBilinOpers(X, Y, Z). If s_1 is Cauchy sequence by norm, then s_1 is convergent.

Proof: Define $\mathcal{P}[\text{set}, \text{set}] \equiv$ there exists a sequence x_3 of Z such that for every natural number n, $x_3(n) = vseq(n)(\$_1)$ and x_3 is convergent and $\$_2 = \lim x_3$. For every element x_4 of $X \times Y$, there exists an element z of Z such that $\mathcal{P}[x_4, z]$ by [9, (18)], (23), (16), [? , (12)]. Consider f being a function from the carrier of $X \times Y$ into the carrier of Z such that for every element z of $X \times Y$, $\mathcal{P}[z, f(z)]$ from [4, Sch. 3]. Reconsider $t_1 = f$ as a function from $X \times Y$ into Z. For every points x_1, x_2 of X and for every point y of Y, $t_1(x_1+x_2, y) = t_1(x_1, y)+t_1(x_2, y)$ by [? , (12)], [10, (25)]. For every point x of X and for every point y of Y and for every real number a, $t_1(a \cdot x, y) = a \cdot t_1(x, y)$ by [? , (12)], [10, (28)]. For every point x of X and for every points y_1, y_2 of Y, $t_1(x, y_1+y_2) = t_1(x, y_1)+t_1(x, y_2)$ by [? , (12)], [10, (25)]. For every point x of X and for every point y of Y and for every real number a, $t_1(x, a \cdot y) = a \cdot t_1(x, y)$ by [? , (12)], [10, (28)]. t_1 is Lipschitzian by [9, (18)], [13, (20)], (16), [7, (9)]. For every real number e such that $e > 0$ there exists a natural number k such that for every natural number n such that $n \geqslant k$ for every point x of X for every point y of Y, $\|vseq(n)(x, y) - t_1(x, y)\| \leqslant e \cdot \|x\| \cdot \|y\|$ by [14, (8)], [9, (18)], (23), (16). Reconsider $t_2 = t_1$ as a point of R-NormSpace-of-BoundedBilinOpers(X, Y, Z). For every real number e such that $e > 0$ there exists a natural number k such that for every natural number n such that $n \geqslant k$ holds $\|vseq(n) - t_2\| \leqslant e$ by (23), [8, (45)], (14). For every real number e such that $e > 0$ there exists a natural number m such that for every natural number n such that $n \geqslant m$ holds $\|vseq(n) - t_2\| < e$. □

(25) Let us consider real normed spaces X, Y, and a real Banach space Z. Then R-NormSpace-of-BoundedBilinOpers(X, Y, Z) is a real Banach space. The theorem is a consequence of (24).

Let X, Y be real normed spaces and Z be a real Banach space. Let us note that R-NormSpace-of-BoundedBilinOpers(X, Y, Z) is complete.

4. Isomorphic Mappings between the Space of Bilinear Operators and the Space of Composition of Linear Operators

From now on X, Y, Z denote real linear spaces.

Now we state the proposition:

(26) There exists a linear operator I from RVectorSpaceOfLinearOperators(X, RVectorS
into R-VectorSpace-of-BilinOpers(X, Y, Z) such that

(i) I is bijective, and

(ii) for every point u of RVectorSpaceOfLinearOperators(X, RVectorSpaceOfLinear
and for every point x of X and for every point y of Y, $I(u)(x,y) = u(x)(y)$.

Proof: Set X_1 = the carrier of X. Set Y_1 = the carrier of Y. Set Z_1 = the carrier of Z. Consider I_0 being a function from $({Z_1}^{Y_1})^{X_1}$ into ${Z_1}^{X_1 \times Y_1}$ such that I_0 is bijective and for every function f from X_1 into ${Z_1}^{Y_1}$ and for every objects d, e such that $d \in X_1$ and $e \in Y_1$ holds $I_0(f)(d,e) = f(d)(e)$. Set L_1 = the carrier of RVectorSpaceOfLinearOperators(X, RVectorSpaceOfLinearC
Set B = the carrier of R-VectorSpace-of-BilinOpers(X, Y, Z). Reconsider $I = I_0 \restriction L_1$ as a function from L_1 into ${Z_1}^{X_1 \times Y_1}$. For every element x of L_1, for every point p of X and for every point q of Y, there exists a linear operator G from Y into Z such that $G = x(p)$ and $I(x)(p,q) = G(q)$ and $I(x) \in B$ by [3, (49)], [4, (66), (5)], [13, (16), (17)]. For every elements x_1, x_2 of L_1, $I(x_1 + x_2) = I(x_1) + I(x_2)$ by [13, (16)], (2). For every element x of L_1 and for every real number a, $I(a \cdot x) = a \cdot I(x)$ by [13, (17)], (3). For every point u of RVectorSpaceOfLinearOperators(X, RVectorSpaceOfLinearOperators(Y, Z)) and for every point x of X and for every point y of Y, $I(u)(x,y) = u(x)(y)$. For every object y such that $y \in B$ there exists an object x such that $x \in L_1$ and $y = I(x)$ by [4, (11), (66), (5)], [**?** , (11)]. □

In the sequel X, Y, Z denote real normed spaces.

Now we state the proposition:

(27) There exists a linear operator I from the real norm space of bounded linear operators from X into the real norm space of bounded linear operators from Y into Z into R-NormSpace-of-BoundedBilinOpers(X, Y, Z) such that

(i) I is bijective, and

(ii) for every point u of the real norm space of bounded linear operators from X into the real norm space of bounded linear operators from Y into Z, $\|u\| = \|I(u)\|$ and for every point x of X and for every point y of Y, $I(u)(x,y) = u(x)(y)$.

Proof: Set X_1 = the carrier of X. Set Y_1 = the carrier of Y. Set Z_1 = the carrier of Z. Consider I_0 being a function from $({Z_1}^{Y_1})^{X_1}$ into ${Z_1}^{X_1 \times Y_1}$ such that I_0 is bijective and for every function f from X_1 into ${Z_1}^{Y_1}$ and for every objects d, e such that $d \in X_1$ and $e \in Y_1$ holds $I_0(f)(d,e) = f(d)(e)$. Set L_1 = the carrier of the real norm space of bounded linear operators from X into the real norm space of bounded linear operators from Y into Z. Set $B =$ the carrier of R-NormSpace-of-BoundedBilinOpers(X, Y, Z). Set $L_2 =$ the carrier of the real norm space of bounded linear operators from Y into Z. ${L_2}^{X_1} \subseteq ({Z_1}^{Y_1})^{X_1}$. Reconsider $I = I_0 {\restriction} L_1$ as a function from L_1 into ${Z_1}^{X_1 \times Y_1}$. For every element x of L_1, for every point p of X and for every point q of Y, there exists a Lipschitzian linear operator G from Y into Z such that $G = x(p)$ and $I(x)(p, q) = G(q)$ and $I(x)$ is a Lipschitzian bilinear operator from $X \times Y$ into Z and $I(x) \in B$ and there exists a point I_2 of R-NormSpace-of-BoundedBilinOpers(X, Y, Z) such that $I_2 = I(x)$ and $\|x\| = \|I_2\|$ by [3, (49)], [4, (66), (5)], [13, (35), (36)]. For every elements x_1, x_2 of L_1, $I(x_1 + x_2) = I(x_1) + I(x_2)$ by [13, (35)], (19). For every element x of L_1 and for every real number a, $I(a \cdot x) = a \cdot I(x)$ by [13, (36)], (20). For every point u of the real norm space of bounded linear operators from X into the real norm space of bounded linear operators from Y into Z, $\|u\| = \|I(u)\|$ and for every point x of X and for every point y of Y, $I(u)(x, y) = u(x)(y)$. For every object y such that $y \in B$ there exists an object x such that $x \in L_1$ and $y = I(x)$ by [4, (11), (66), (5)], [**?** , (12)]. □

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