

# Fubini's Theorem

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**Summary.** Fubini theorem is an essential tool for the analysis of high-dimensional space [10], [3], [4], a theorem about the multiple integral and iterated integral. The author has been working on formalizing Fubini's theorem over the past few years [7], [9] in the Mizar system [1], [2]. As a result, Fubini's theorem (30) was proved in complete form by this article.

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## 1. PRELIMINARIES

From now on  $X$  denotes a set.

Now we state the proposition:

- (1) Let us consider a subset  $A$  of  $X$ , and an  $X$ -defined binary relation  $f$ .  
Then  $f \upharpoonright A^c = f \upharpoonright (\text{dom } f \setminus A)$ .

Let us consider a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . Now we state the propositions:

- (2)  $\text{GTE-dom}(f, +\infty) = \text{EQ-dom}(f, +\infty)$ .
- (3)  $\text{LEQ-dom}(f, -\infty) = \text{EQ-dom}(f, -\infty)$ .

Now we state the propositions:

- (4) Let us consider a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ , and an extended real  $e$ . Then  $\text{GTE-dom}(f, e)$  misses  $\text{LE-dom}(f, e)$ .
- (5) Let us consider a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . Then  $\text{dom } f = (\text{EQ-dom}(f, -\infty) \cup \text{GT-dom}(f, -\infty) \cap \text{LE-dom}(f, +\infty)) \cup \text{EQ-dom}(f, +\infty)$ .

In the sequel  $X$ ,  $X_1$ ,  $X_2$  denote non empty sets.

Now we state the propositions:

- (6) Let us consider a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ , and an element  $x$  of  $X$ . Then
- (i)  $(\max_+(f))(x) \leq |f|(x)$ , and
  - (ii)  $(\max_-(f))(x) \leq |f|(x)$ .
- (7) Let us consider a partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ , an element  $x$  of  $X_1$ , and an element  $y$  of  $X_2$ . Then
- (i)  $\text{ProjPMap1}(|f|, x) = |\text{ProjPMap1}(f, x)|$ , and
  - (ii)  $\text{ProjPMap2}(|f|, y) = |\text{ProjPMap2}(f, y)|$ .

## 2. MARKOV'S INEQUALITY AND MISC.

From now on  $S$  denotes a  $\sigma$ -field of subsets of  $X$ ,  $S_1$  denotes a  $\sigma$ -field of subsets of  $X_1$ ,  $S_2$  denotes a  $\sigma$ -field of subsets of  $X_2$ ,  $M$  denotes a  $\sigma$ -measure on  $S$ ,  $M_1$  denotes a  $\sigma$ -measure on  $S_1$ , and  $M_2$  denotes a  $\sigma$ -measure on  $S_2$ .

Let  $X$  be a non empty set,  $S$  be a  $\sigma$ -field of subsets of  $X$ , and  $E$  be an element of  $S$ . One can verify that there exists a partial function from  $X$  to  $\overline{\mathbb{R}}$  which is  $E$ -measurable.

Now we state the proposition:

- (8) Let us consider an element  $E$  of  $S$ , and an  $E$ -measurable partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . Suppose  $\text{dom } f = E$ . Then  $\text{EQ-dom}(f, +\infty)$ ,  $\text{EQ-dom}(f, -\infty) \in S$ .

Let us consider an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$  and an  $E$ -measurable partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ . Now we state the propositions:

- (9) Suppose  $M_1$  is  $\sigma$ -finite and  $M_2$  is  $\sigma$ -finite and  $\text{dom } f = E$ . Then
- (i)  $\int \text{Integral2}(M_2, |f|) dM_1 = \int |f| d \text{ProdMeas}(M_1, M_2)$ , and
  - (ii)  $\int \text{Integral1}(M_1, |f|) dM_2 = \int |f| d \text{ProdMeas}(M_1, M_2)$ .
- (10) Suppose  $M_1$  is  $\sigma$ -finite and  $M_2$  is  $\sigma$ -finite and  $E = \text{dom } f$ . Then  $f$  is integrable on  $\text{ProdMeas}(M_1, M_2)$  if and only if  $\int \text{Integral1}(M_1, |f|) dM_2 < +\infty$ .
- (11) Suppose  $M_1$  is  $\sigma$ -finite and  $M_2$  is  $\sigma$ -finite and  $E = \text{dom } f$ . Then  $f$  is integrable on  $\text{ProdMeas}(M_1, M_2)$  if and only if  $\int \text{Integral2}(M_2, |f|) dM_1 < +\infty$ .

Now we state the propositions:

- (12) Let us consider an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ , an element  $U$  of  $S_1$ , and an  $E$ -measurable partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ . Suppose  $M_2$  is  $\sigma$ -finite and  $E = \text{dom } f$ . Then  $\text{Integral2}(M_2, |f|)$  is  $U$ -measurable.
- (13) Let us consider an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ , an element  $V$  of  $S_2$ , and an  $E$ -measurable partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ . Suppose  $M_1$  is  $\sigma$ -finite and  $E = \text{dom } f$ . Then  $\text{Integral1}(M_1, |f|)$  is  $V$ -measurable.

Let us consider a partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ . Now we state the propositions:

- (14) Suppose  $M_2$  is  $\sigma$ -finite and  $f$  is integrable on  $\text{ProdMeas}(M_1, M_2)$ . Then
- (i)  $\int \max_+(\text{Integral2}(M_2, |f|)) \, dM_1 = \int \text{Integral2}(M_2, |f|) \, dM_1$ , and
  - (ii)  $\int \max_-(\text{Integral2}(M_2, |f|)) \, dM_1 = 0$ .

The theorem is a consequence of (12).

- (15) Suppose  $M_1$  is  $\sigma$ -finite and  $f$  is integrable on  $\text{ProdMeas}(M_1, M_2)$ . Then
- (i)  $\int \max_+(\text{Integral1}(M_1, |f|)) \, dM_2 = \int \text{Integral1}(M_1, |f|) \, dM_2$ , and
  - (ii)  $\int \max_-(\text{Integral1}(M_1, |f|)) \, dM_2 = 0$ .

The theorem is a consequence of (13).

Now we state the proposition:

- (16) MARKOV'S INEQUALITY:

Let us consider an element  $E$  of  $S$ , an  $E$ -measurable partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ , and an extended real  $e$ . Suppose  $\text{dom } f = E$  and  $f$  is non-negative and  $e \geq 0$ . Then  $e \cdot M(\text{GTE-dom}(f, e)) \leq \int f \, dM$ .

PROOF:  $\text{GTE-dom}(f, +\infty) = \text{EQ-dom}(f, +\infty)$ . Reconsider  $E_3 = \text{GTE-dom}(f, e)$  as an element of  $S$ . For every element  $x$  of  $X$  such that  $x \in \text{dom}(\chi_{e, E_3, X} \upharpoonright E_3)$  holds  $(\chi_{e, E_3, X} \upharpoonright E_3)(x) \leq (f \upharpoonright E_3)(x)$  by [5, (47)], [11, (57)].  $\square$

### 3. FUBINI'S THEOREM

Now we state the propositions:

- (17) Let us consider partial functions  $f, g$  from  $X$  to  $\overline{\mathbb{R}}$ . Suppose  $f$  is integrable on  $M$  and  $g$  is integrable on  $M$ . Then
- (i)  $\int f + g \, dM = \int f \upharpoonright (\text{dom } f \cap \text{dom } g) \, dM + \int g \upharpoonright (\text{dom } f \cap \text{dom } g) \, dM$ ,  
and
  - (ii)  $\int f - g \, dM = \int f \upharpoonright (\text{dom } f \cap \text{dom } g) \, dM - \int g \upharpoonright (\text{dom } f \cap \text{dom } g) \, dM$ .
- (18) Let us consider a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . Then  $f$  is integrable on  $M$  if and only if  $\max_+(f)$  is integrable on  $M$  and  $\max_-(f)$  is integrable on  $M$ .

(19) Let us consider elements  $A, B$  of  $S$ , and a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . Suppose  $B \subseteq A$  and  $f \upharpoonright A$  is  $B$ -measurable. Then  $f$  is  $B$ -measurable.

Let  $X$  be a non empty set,  $S$  be a  $\sigma$ -field of subsets of  $X$ ,  $M$  be a  $\sigma$ -measure on  $S$ , and  $f$  be a partial function from  $X$  to  $\overline{\mathbb{R}}$ . We say that  $f$  is a.e.integrable on  $M$  if and only if

(Def. 1) there exists an element  $A$  of  $S$  such that  $M(A) = 0$  and  $A \subseteq \text{dom } f$  and  $f \upharpoonright A^c$  is integrable on  $M$ .

Let us consider a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . Now we state the propositions:

(20) If  $f$  is a.e.integrable on  $M$ , then  $\text{dom } f \in S$ .

(21) If  $f$  is integrable on  $M$ , then  $f$  is a.e.integrable on  $M$ . The theorem is a consequence of (1).

Let  $X$  be a non empty set,  $S$  be a  $\sigma$ -field of subsets of  $X$ ,  $M$  be a  $\sigma$ -measure on  $S$ , and  $f$  be a partial function from  $X$  to  $\overline{\mathbb{R}}$ . We say that  $f$  is a.e.finite of?  $M$  if and only if

(Def. 2) there exists an element  $A$  of  $S$  such that  $M(A) = 0$  and  $A \subseteq \text{dom } f$  and  $f \upharpoonright A^c$  is a partial function from  $X$  to  $\mathbb{R}$ .

Now we state the propositions:

(22) Let us consider an element  $E$  of  $S$ , and an  $E$ -measurable partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . Suppose  $\text{dom } f = E$ . Then  $f$  is a.e.finite of?  $M$  if and only if  $M(\text{EQ-dom}(f, +\infty) \cup \text{EQ-dom}(f, -\infty)) = 0$ . The theorem is a consequence of (8).

(23) Let us consider a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . Suppose  $f$  is integrable on  $M$ . Then

(i)  $M(\text{EQ-dom}(f, +\infty)) = 0$ , and

(ii)  $M(\text{EQ-dom}(f, -\infty)) = 0$ , and

(iii)  $f$  is a.e.finite of?  $M$ , and

(iv) for every real number  $r$  such that  $r > 0$  holds  $M(\text{GTE-dom}(|f|, r)) < +\infty$ .

The theorem is a consequence of (16).

(24) Let us consider a partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ . Suppose  $M_1$  is  $\sigma$ -finite and  $M_2$  is  $\sigma$ -finite and  $f$  is integrable on  $\text{ProdMeas}(M_1, M_2)$ . Then

(i)  $\text{Integral1}(M_1, \max_+(f))$  is integrable on  $M_2$ , and

(ii)  $\text{Integral2}(M_2, \max_+(f))$  is integrable on  $M_1$ , and

(iii)  $\text{Integral1}(M_1, \max_-(f))$  is integrable on  $M_2$ , and

- (iv)  $\text{Integral2}(M_2, \max_-(f))$  is integrable on  $M_1$ , and
  - (v)  $\text{Integral1}(M_1, |f|)$  is integrable on  $M_2$ , and
  - (vi)  $\text{Integral2}(M_2, |f|)$  is integrable on  $M_1$ .
- (25) Let us consider an element  $E$  of  $S$ , and an  $E$ -measurable partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . If  $\text{dom } f \subseteq E$  and  $f$  is a.e.integrable on  $M$ , then  $f$  is integrable on  $M$ . The theorem is a consequence of (20) and (1).
- (26) Let us consider an element  $A$  of  $S$ , and a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . Suppose  $M(A) = 0$  and  $A \subseteq \text{dom } f$  and  $f \upharpoonright A^c$  is integrable on  $M$ . Then there exists a partial function  $g$  from  $X$  to  $\overline{\mathbb{R}}$  such that
- (i)  $\text{dom } g = \text{dom } f$ , and
  - (ii)  $f \upharpoonright A^c = g \upharpoonright A^c$ , and
  - (iii)  $g$  is integrable on  $M$ , and
  - (iv)  $\int f \upharpoonright A^c \, dM = \int g \, dM$ .

PROOF: Consider  $B$  being an element of  $S$  such that  $B = \text{dom}(f \upharpoonright A^c)$  and  $f \upharpoonright A^c$  is  $B$ -measurable.  $f \upharpoonright A^c = f \upharpoonright (\text{dom } f \setminus A)$ . Define  $\mathcal{C}[\text{object}] \equiv \$1 \in A$ . Define  $\mathcal{F}(\text{object}) = +\infty$ . Define  $\mathcal{G}(\text{object}) = f(\$1)$ . Consider  $g$  being a function such that  $\text{dom } g = \text{dom } f$  and for every object  $x$  such that  $x \in \text{dom } f$  holds if  $\mathcal{C}[x]$ , then  $g(x) = \mathcal{F}(x)$  and if not  $\mathcal{C}[x]$ , then  $g(x) = \mathcal{G}(x)$  from [6, Sch. 1]. For every real number  $r$ ,  $(A \cup B) \cap \text{LE-dom}(g, r) \in S$  by [5, (49), (47)].  $\int f \upharpoonright A^c \, dM = \int g \upharpoonright (\text{dom } g \setminus A) \, dM$ .  $\square$

- (27) Let us consider a partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ . Suppose  $M_1$  is  $\sigma$ -finite and  $M_2$  is  $\sigma$ -finite and  $f$  is integrable on  $\text{ProdMeas}(M_1, M_2)$ . Then
- (i)  $\int f \, d\text{ProdMeas}(M_1, M_2) = \int \text{Integral1}(M_1, \max_+(f)) \, dM_2 - \int \text{Integral1}(M_1, \max_-(f)) \, dM_2$  and
  - (ii)  $\int f \, d\text{ProdMeas}(M_1, M_2) = \int \text{Integral2}(M_2, \max_+(f)) \, dM_1 - \int \text{Integral2}(M_2, \max_-(f)) \, dM_1$
- (28) Let us consider an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ , and an element  $y$  of  $X_2$ . Then
- (i) if  $M_1(\text{MeasurableYsection}(E, y)) \neq 0$ , then  $(\text{Integral1}(M_1, \overline{\chi}_{E, X_1 \times X_2}))(y) = +\infty$ , and
  - (ii) if  $M_1(\text{MeasurableYsection}(E, y)) = 0$ , then  $(\text{Integral1}(M_1, \overline{\chi}_{E, X_1 \times X_2}))(y) = 0$ .
- (29) Let us consider an element  $E$  of  $\sigma(\text{MeasRect}(S_1, S_2))$ , and an element  $x$  of  $X_1$ . Then
- (i) if  $M_2(\text{MeasurableXsection}(E, x)) \neq 0$ , then  $(\text{Integral2}(M_2, \overline{\chi}_{E, X_1 \times X_2}))(x) = +\infty$ , and

- (ii) if  $M_2(\text{MeasurableXsection}(E, x)) = 0$ , then  $(\text{Integral2}(M_2, \bar{\chi}_{E, X_1 \times X_2}))(x) = 0$ .

Now we state the propositions:

(30) FUBINI'S THEOREM:

Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_1$  on  $S_1$ , a  $\sigma$ -measure  $M_2$  on  $S_2$ , a partial function  $f$  from  $X_1 \times X_2$  to  $\bar{\mathbb{R}}$ , and an element  $S_3$  of  $S_1$ . Suppose  $M_1$  is  $\sigma$ -finite and  $M_2$  is  $\sigma$ -finite and  $f$  is integrable on  $\text{ProdMeas}(M_1, M_2)$  and  $X_1 = S_3$ . Then there exists an element  $U$  of  $S_1$  such that

- (i)  $M_1(U) = 0$ , and
- (ii) for every element  $x$  of  $X_1$  such that  $x \in U^c$  holds  $\text{ProjPMap1}(f, x)$  is integrable on  $M_2$ , and
- (iii)  $\text{Integral2}(M_2, |f|)\upharpoonright U^c$  is a partial function from  $X_1$  to  $\mathbb{R}$ , and
- (iv)  $\text{Integral2}(M_2, f)$  is  $(S_3 \setminus U)$ -measurable, and
- (v)  $\text{Integral2}(M_2, f)\upharpoonright U^c$  is integrable on  $M_1$ , and
- (vi)  $\text{Integral2}(M_2, f)\upharpoonright U^c \in$  the  $L^1$  functions of  $M_1$ , and
- (vii) there exists a function  $g$  from  $X_1$  into  $\bar{\mathbb{R}}$  such that  $g$  is integrable on  $M_1$  and  $g\upharpoonright U^c = \text{Integral2}(M_2, f)\upharpoonright U^c$  and  $\int f \, d\text{ProdMeas}(M_1, M_2) = \int g \, dM_1$ .

PROOF: Consider  $A$  being an element of  $\sigma(\text{MeasRect}(S_1, S_2))$  such that  $A = \text{dom } f$  and  $f$  is  $A$ -measurable.  $\text{Integral2}(M_2, |f|)$  is integrable on  $M_1$  and  $\text{Integral2}(M_2, \max_+(f))$  is integrable on  $M_1$  and  $\text{Integral2}(M_2, \max_-(f))$  is integrable on  $M_1$ .  $\text{Integral2}(M_2, |f|)$  is a.e. finite of?  $M_1$ . Consider  $U$  being an element of  $S_1$  such that  $M_1(U) = 0$  and  $\text{Integral2}(M_2, |f|)\upharpoonright U^c$  is a partial function from  $X_1$  to  $\mathbb{R}$ . For every element  $x$  of  $X_1$  such that  $x \in U^c$  holds  $\text{ProjPMap1}(f, x)$  is integrable on  $M_2$  by [11, (57)], [9, (47)], (7), [8, (31)]. Consider  $g_1$  being a partial function from  $X_1$  to  $\bar{\mathbb{R}}$  such that  $\text{dom } g_1 = \text{dom}(\text{Integral2}(M_2, \max_+(f)))$  and  $g_1\upharpoonright U^c = \text{Integral2}(M_2, \max_+(f))\upharpoonright U^c$  and  $g_1$  is integrable on  $M_1$  and  $\int g_1 \, dM_1 = \int \text{Integral2}(M_2, \max_+(f))\upharpoonright U^c \, dM_1$ . Consider  $g_2$  being a partial function from  $X_1$  to  $\bar{\mathbb{R}}$  such that  $\text{dom } g_2 = \text{dom}(\text{Integral2}(M_2, \max_-(f)))$  and  $g_2\upharpoonright U^c = \text{Integral2}(M_2, \max_-(f))\upharpoonright U^c$  and  $g_2$  is integrable on  $M_1$  and  $\int g_2 \, dM_1 = \int \text{Integral2}(M_2, \max_-(f))\upharpoonright U^c \, dM_1$ . Consider  $g$  being a partial function from  $X_1$  to  $\bar{\mathbb{R}}$  such that  $\text{dom } g = \text{dom}(\text{Integral2}(M_2, f))$  and  $g\upharpoonright U^c = \text{Integral2}(M_2, f)\upharpoonright U^c$  and  $g$  is integrable on  $M_1$  and  $\int g \, dM_1 = \int \text{Integral2}(M_2, f)\upharpoonright U^c \, dM_1 = \int f \, d\text{ProdMeas}(M_1, M_2) = \int g \upharpoonright U^c \, dM_1$ .  $\square$

- (31) Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_1$  on  $S_1$ , a  $\sigma$ -measure  $M_2$  on  $S_2$ ,

a partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ , and an element  $S_4$  of  $S_2$ . Suppose  $M_1$  is  $\sigma$ -finite and  $M_2$  is  $\sigma$ -finite and  $f$  is integrable on  $\text{ProdMeas}(M_1, M_2)$  and  $X_2 = S_4$ . Then there exists an element  $V$  of  $S_2$  such that

- (i)  $M_2(V) = 0$ , and
- (ii) for every element  $y$  of  $X_2$  such that  $y \in V^c$  holds  $\text{ProjPMap2}(f, y)$  is integrable on  $M_1$ , and
- (iii)  $\text{Integral1}(M_1, |f|)\upharpoonright V^c$  is a partial function from  $X_2$  to  $\mathbb{R}$ , and
- (iv)  $\text{Integral1}(M_1, f)$  is  $(S_4 \setminus V)$ -measurable, and
- (v)  $\text{Integral1}(M_1, f)\upharpoonright V^c$  is integrable on  $M_2$ , and
- (vi)  $\text{Integral1}(M_1, f)\upharpoonright V^c \in$  the  $L^1$  functions of  $M_2$ , and
- (vii) there exists a function  $g$  from  $X_2$  into  $\overline{\mathbb{R}}$  such that  $g$  is integrable on  $M_2$  and  $g\upharpoonright V^c = \text{Integral1}(M_1, f)\upharpoonright V^c$  and  $\int f \, d\text{ProdMeas}(M_1, M_2) = \int g \, dM_2$ .

PROOF: Consider  $A$  being an element of  $\sigma(\text{MeasRect}(S_1, S_2))$  such that  $A = \text{dom } f$  and  $f$  is  $A$ -measurable.  $\text{Integral1}(M_1, |f|)$  is integrable on  $M_2$  and  $\text{Integral1}(M_1, \max_+(f))$  is integrable on  $M_2$  and  $\text{Integral1}(M_1, \max_-(f))$  is integrable on  $M_2$ .  $\text{Integral1}(M_1, |f|)$  is a.e.finite of?  $M_2$ . Consider  $V$  being an element of  $S_2$  such that  $M_2(V) = 0$  and  $\text{Integral1}(M_1, |f|)\upharpoonright V^c$  is a partial function from  $X_2$  to  $\mathbb{R}$ . For every element  $y$  of  $X_2$  such that  $y \in V^c$  holds  $\text{ProjPMap2}(f, y)$  is integrable on  $M_1$  by [11, (57)], [9, (47)], (7), [8, (31)]. Consider  $g_1$  being a partial function from  $X_2$  to  $\overline{\mathbb{R}}$  such that  $\text{dom } g_1 = \text{dom}(\text{Integral1}(M_1, \max_+(f)))$  and  $g_1\upharpoonright V^c = \text{Integral1}(M_1, \max_+(f))\upharpoonright V^c$  and  $g_1$  is integrable on  $M_2$  and  $\int g_1 \, dM_2 = \int \text{Integral1}(M_1, \max_+(f))\upharpoonright V^c \, dM_2$ . Consider  $g_2$  being a partial function from  $X_2$  to  $\overline{\mathbb{R}}$  such that  $\text{dom } g_2 = \text{dom}(\text{Integral1}(M_1, \max_-(f)))$  and  $g_2\upharpoonright V^c = \text{Integral1}(M_1, \max_-(f))\upharpoonright V^c$  and  $g_2$  is integrable on  $M_2$  and  $\int g_2 \, dM_2 = \int \text{Integral1}(M_1, \max_-(f))\upharpoonright V^c \, dM_2$ . Consider  $g$  being a partial function from  $X_2$  to  $\overline{\mathbb{R}}$  such that  $\text{dom } g = \text{dom}(\text{Integral1}(M_1, f))$  and  $g\upharpoonright V^c = \text{Integral1}(M_1, f)\upharpoonright V^c$  and  $g$  is integrable on  $M_2$  and  $\int g \, dM_2 = \int \text{Integral1}(M_1, f)\upharpoonright V^c \, dM_2 = \int f \, d\text{ProdMeas}(M_1, M_2) = \int g\upharpoonright V^c \, dM_2$ .  $\square$

Let us consider non empty sets  $X_1, X_2$ , a  $\sigma$ -field  $S_1$  of subsets of  $X_1$ , a  $\sigma$ -field  $S_2$  of subsets of  $X_2$ , a  $\sigma$ -measure  $M_1$  on  $S_1$ , a  $\sigma$ -measure  $M_2$  on  $S_2$ , and a partial function  $f$  from  $X_1 \times X_2$  to  $\overline{\mathbb{R}}$ . Now we state the propositions:

- (32) Suppose  $M_1$  is  $\sigma$ -finite and  $M_2$  is  $\sigma$ -finite and  $f$  is integrable on  $\text{ProdMeas}(M_1, M_2)$  and for every element  $x$  of  $X_1$ ,  $(\text{Integral2}(M_2, |f|))(x) < +\infty$ . Then
  - (i) for every element  $x$  of  $X_1$ ,  $\text{ProjPMap1}(f, x)$  is integrable on  $M_2$ , and
  - (ii) for every element  $U$  of  $S_1$ ,  $\text{Integral2}(M_2, f)$  is  $U$ -measurable, and

- (iii)  $\text{Integral2}(M_2, f)$  is integrable on  $M_1$ , and
- (iv)  $\int f \, d\text{ProdMeas}(M_1, M_2) = \int \text{Integral2}(M_2, f) \, dM_1$ , and
- (v)  $\text{Integral2}(M_2, f) \in$  the  $L^1$  functions of  $M_1$ .

The theorem is a consequence of (7), (24), (6), and (17).

- (33) Suppose  $M_1$  is  $\sigma$ -finite and  $M_2$  is  $\sigma$ -finite and  $f$  is integrable on  $\text{ProdMeas}(M_1, M_2)$  and for every element  $y$  of  $X_2$ ,  $(\text{Integral1}(M_1, |f|))(y) < +\infty$ . Then
- (i) for every element  $y$  of  $X_2$ ,  $\text{ProjPMap2}(f, y)$  is integrable on  $M_1$ , and
  - (ii) for every element  $V$  of  $S_2$ ,  $\text{Integral1}(M_1, f)$  is  $V$ -measurable, and
  - (iii)  $\text{Integral1}(M_1, f)$  is integrable on  $M_2$ , and
  - (iv)  $\int f \, d\text{ProdMeas}(M_1, M_2) = \int \text{Integral1}(M_1, f) \, dM_2$ , and
  - (v)  $\text{Integral1}(M_1, f) \in$  the  $L^1$  functions of  $M_2$ .

The theorem is a consequence of (7), (24), (6), and (17).

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