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Preface

In recent years, several projects have aimed at providing computer assistance for doing mathematics. The project discussed here is called Mizar and concerns **computer oriented formalization of mathematics**, begun in 1973. The author of the Mizar language is Prof.Andrzej Trybulec (Warsaw University), who is also the leader of the group which prepared the majority of implementations. The project original goal was to design and implement a software environment to assist the process of preparing mathematical papers: the human writes mathematical texts and the machine verifies their correctness.

Université Catholique de Louvain and Foundation Philippe le Hodey (both of Belgium) has been conducting research related to the applications of the Mizar system since 1984. This research has involved an international group of mathematicians who first met at the Mizar Summer Workshop in Louvain-la-Neuve (Belgium) in 1985. Of these, the Polish group is the most active. These researchers cooperate within the framework of the Mizar Users Group (MizUG).

The papers published in the consecutive issues of "Formalized Mathematics *(a computer assisted approach)*" constitute the Main Mizar Library (MML). The power of the Mizar system lies in the automatic processing of cross–references among articles. This is done by the continuous actualization of MML. Before the theorems and definitions are included into the MML, they must be proved valid and correct. MML forms the basis of a Knowledge Management System for Mathematics supplied with Mizar articles. MML together with PC MIZAR are the systems for collecting, formalizing and verifying mathematical knowledge. The latest, the most advanced version of Mizar is PC MIZAR which together with MML runs on IBM PC under DOS 3.xx (implemented in Poland under the direction of Prof.A.Trybulec).

In the current issue of "Formalized Mathematics..." all the papers appear in chronological order, since they form the very beginning of MML. They concern both very general and very specialized, narrow subjects. In the future we intend to classify the papers into groups according to the mathematical domains i.e. foundations of mathematics, geometry, etc.

The MizTEX system, used for the automatic editing of this publication is constantly being developed.

Finally, I would like to add that for MizUG members we also publish the technical report "Main Mizar Library", containing the list of summaries of the articles; names of the authors, titles of the articles and names of files; publicity – ranking of theorems and articles as well as the list of contents of the articles so far published in the "Formalized Mathematics...".

Roman Matuszewski

Introduction

The Mizar project started many years ago and, as it developed, the emphasis on its different applications varied. It is therefore worthwhile to take this opportunity to recall that one of the main applications originally considered was using Mizar articles as source texts for mathematical publications. Of course, none of the following papers, or, rather, their abstracts, fulfils that expectation. Nevertheless, they let us see how close or how far are we still from our aim.In order to explain what exactly is published here it is necessary to at least give an outline of the project itself.

The Mizar language is a strongly standarized mathematial language, or, if one prefers, an extensively extended formalized language, for writing mathematical papers. Its structure allows for using a database; the final goal of the project is to provide a knowledge management system for mathematics. Thus it is possible to write mathematical papers in Mizar. They are usually 1000 - 2000 line texts corresponding to a short six- to nine-page publication or to one chapter of a textbook.

An article consists of two parts. The first, usually very short, is the description of the environment. It contains a list of publications where the notions used in the paper were introduced or where the theorems we refer to were proved, and other similar information. The second part is the text proper, where we define new notions, prove the correctness of the proposed definitions and where we prove new theorems. From the construction of the article follows that to write a new one we have to have access to the Mizar library of papers we can refer to. Obviously, to write the first papers we have to start with some axiomatics. The papers presented below make use of Main Mizar Library (MML), which was first created at the beginning of 1989 owing to the financial help obtained from the Ministry of National Education of the Republic Poland (grant RPBP III.24). The axiomatic foundation of this library is the Tarski-Grothendieck set theory which is quite a strong theory quaranteeing the existence of universal classes. To enable the Mizar processor to perform natural number computations, several additional axioms were also introduced, namely the axiomatics of strong real number arithmetic. So far the Main Mizar Library comprises of about 80 papers but their number is growing fast.

However, to verify the correctness of a paper the PC Mizar system used to build the library refers not to the library direct, but to a database automatically created from the papers there included. The data introduced into the database from a paper pass through an intermediate stage where the abstract of the paper is created. The abstractor program removes from the paper all data which are not stored in the database, i.e. justification of theorems, lemmas and private object definitions.

The evolution of the library requires writing many papers containing well-known theorems with uninteresting proofs. It seemed to us, therefore, that publication of whole papers is not justified. To tell the truth, only some of the authors were inclined to devote their time to the systematic development of the database; others agreed to write down only that part of mathematical folklore which makes work on an ambitious paper possible. Some of the papers submitted to the library concern new, unpublished mathematical results; thus the level of the papers varies.

As we prepared this collection, we wondered whether it should not be restricted to chosen, more interesting papers. There were doubts concerning the publication of such monotonous articles as, for example [2]. Actually, this paper was written mainly because, while justifying some trivial facts, the checker (system module checking inference correctness) exceeded certain quantitative limitations and we wanted to show how this can be overcome. Those who write in Mizar may have found the proofs in this paper interesting, they were removed, however, when the abstract was created.

Still, there are good reasons for publishing all papers. First, in this way we obtain a true picture of what the library looks like. It does not seem fair to remove trivial papers, even though the reader is warned that it's been done. Second, this publication will serve to write new papers, and Mizar authors need the information what has been proved and where. This actually was our original aim, similar to [1].

The abstracts of Mizar papers do not look as well as the present publication would lead to think, if only because they are ASCII files. These abstracts were automatically converted into source files of the TEX language. Some fragments were automatically translated into English, or to be precise, into a language which reminds English a little and others were left in original Mizar form with slight modifications, for example the keywords are in bold type. The obtained texts, with the exception of abstracts containing axiomatics, were not post-edited. The programs used were implemented by the following group: Grzegorz Bancerek, Czesław Byliński, Wojciech Leończuk, Krzysztof Prażmowski, Michał Muzalewski and the author. They include a program in Turbo Pascal converting Mizar into TEX and a special TEX format (a set of TEX macros).

Andrzej Trybulec

References

[1] Piotr Rudnicki and Andrzej Trybulec. *A Collection of TEXed Mizar Abstracts.* Technical Report University of Alberta, 1989.

[2] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1, 1990.

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Tarski Grothendieck Set Theory

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Summary. This is the first part of the axiomatics of the Mizar system. It includes the axioms of the Tarski Grothendieck set theory. They are: the axiom stating that everything is a set, the extensionality axiom, the definitional axiom of the singleton, the definitional axiom of the pair, the definitional axiom of the union of a family of sets, the definitional axiom of the boolean (the power set) of a set, the regularity axiom, the definitional axiom of the ordered pair, the Tarski's axiom A introduced in [2] (see also [1]), and the Frænkel scheme. Also, the definition of equinumerosity is introduced.

For simplicity we adopt the following convention: x, y, z, u will denote objects of the type Any; N, M, X, Y, Z will denote objects of the type set. Next we state two axioms:

$$x \textbf{ is } \mathrm{set}\,, \tag{1}$$

$$(\textbf{for } x \textbf{ holds } x \in X \textbf{ iff } x \in Y) \textbf{ implies } X = Y. \tag{2}$$

We now introduce two functors. Let us consider y. The functor

$$\{y\},$$

with values of the type set, is defined by

$$x \in \textbf{it} \textbf{ iff } x = y.$$

Let us consider z. The functor

$$\{y, z\},$$

with values of the type set, is defined by

$$x \in \textbf{it} \textbf{ iff } x = y \textbf{ or } x = z.$$

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The following axioms hold:

$$X = \{y\} \textbf{ iff for } x \textbf{ holds } x \in X \textbf{ iff } x = y, \tag{3}$$

$$X = \{y, z\} \textbf{ iff for } x \textbf{ holds } x \in X \textbf{ iff } x = y \textbf{ or } x = z. \tag{4}$$

Let us consider X, Y. The predicate

$$X \subseteq Y \qquad \text{is defined by} \qquad x \in X \textbf{ implies } x \in Y.$$

Let us consider X. The functor

$$\bigcup X,$$

with values of the type set, is defined by

$$x \in \textbf{it iff ex } Y \textbf{ st } x \in Y \;\&\; Y \in X.$$

Then we get

$$X = \bigcup Y \textbf{ iff for } x \textbf{ holds } x \in X \textbf{ iff ex } Z \textbf{ st } x \in Z \;\&\; Z \in Y, \tag{5}$$

$$X = \operatorname{bool} Y \textbf{ iff for } Z \textbf{ holds } Z \in X \textbf{ iff } Z \subseteq Y, \tag{6}$$

The regularity axiom claims that

$$x \in X \textbf{ implies ex } Y \textbf{ st } Y \in X \;\&\; \textbf{not ex } x \textbf{ st } x \in X \;\&\; x \in Y. \tag{7}$$

The scheme *Fraenkel* deals with a constant $\mathcal{A}$ that has the type set and a binary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex } X \textbf{ st for } x \textbf{ holds } x \in X \textbf{ iff ex } y \textbf{ st } y \in \mathcal{A} \;\&\; \mathcal{P}[y, x]$$

provided the parameters satisfy the following condition:

- $\textbf{for } x, y, z \textbf{ st } \mathcal{P}[x, y] \;\&\; \mathcal{P}[x, z] \textbf{ holds } y = z.$

Let us consider x, y. The functor

$$\langle x, y \rangle,$$

is defined by

$$\textbf{it} = \{\{x, y\}, \{x\}\}.$$

According to the definition

$$\langle x, y \rangle = \{\{x, y\}, \{x\}\}. \tag{8}$$

Let us consider X, Y. The predicate

$$X \approx Y$$

is defined by

$$\textbf{ex}\, Z \textbf{ st } (\textbf{for}\, x \textbf{ st } x \in X \textbf{ ex}\, y \textbf{ st } y \in Y \;\&\; \langle x, y\rangle \in Z) \;\&$$
$$(\textbf{for}\, y \textbf{ st } y \in Y \textbf{ ex}\, x \textbf{ st } x \in X \;\&\; \langle x, y\rangle \in Z)$$
$$\&\; \textbf{for}\, x,y,z,u \textbf{ st } \langle x, y\rangle \in Z \;\&\; \langle z, u\rangle \in Z \textbf{ holds } x = z \textbf{ iff } y = u.$$

The Tarski's axiom A claims that

(9) $$\textbf{ex}\, M \textbf{ st } N \in M \;\&\; (\textbf{for}\, X,Y \textbf{ holds } X \in M \;\&\; Y \subseteq X \textbf{ implies } Y \in M) \;\&$$
$$(\textbf{for}\, X \textbf{ holds } X \in M \textbf{ implies } \mathrm{bool}\, X \in M)$$
$$\&\; \textbf{for}\, X \textbf{ holds } X \subseteq M \textbf{ implies } X \approx M \textbf{ or } X \in M.$$

References

[1] Alfred Tarski. On well-ordered subsets of any set. *Fundamenta Mathematicae*, 32:176–183, 1939.

[2] Alfred Tarski. Über Unerreaichbare Kardinalzahlen. *Fundamenta Mathematicae*, 30:176–183, 1938.

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Built-in Concepts

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Summary. This abstract contains the second part of the axiomatics of the Mizar system (the first part is in abstract [1]). The axioms listed here characterize the Mizar built-in concepts that are automatically attached to every Mizar article. We give definitional axioms of the following concepts: element, subset, Cartesian product, domain (non empty subset), subdomain (non empty subset of a domain), set domain (domain consisting of sets). Axioms of strong arithmetics of real numbers are also included.

The notation and terminology used here have been introduced in the axiomatics [1]. For simplicity we adopt the following convention: x, y, z denote objects of the type Any; X, $X1$, $X2$, $X3$, $X4$, Y denote objects of the type set. The following axioms hold:

$$(\mathbf{ex}\, x\ \mathbf{st}\ x \in X)\ \mathbf{implies}\ (x\ \mathbf{is}\ \text{Element}\ \mathbf{of}\ X\ \mathbf{iff}\ x \in X), \tag{1}$$

$$X\ \mathbf{is}\ \text{Subset}\ \mathbf{of}\ Y\ \mathbf{iff}\ X \subseteq Y, \tag{2}$$

$$z \in [\!:X, Y:\!]\ \mathbf{iff}\ \mathbf{ex}\, x,y\ \mathbf{st}\ x \in X\ \&\ y \in Y\ \&\ z = \langle x, y \rangle, \tag{3}$$

$$X\ \mathbf{is}\ \text{DOMAIN}\ \mathbf{iff}\ \mathbf{ex}\, x\ \mathbf{st}\ x \in X, \tag{4}$$

$$[\!:X1,X2,X3:\!] = [\!:[\!:X1,X2:\!],X3:\!], \tag{5}$$

$$[\!:X1,X2,X3,X4:\!] = [\!:[\!:X1,X2,X3:\!],X4:\!]. \tag{6}$$

In the sequel $D1$, $D2$, $D3$, $D4$ will denote objects of the type DOMAIN. Let us introduce the consecutive axioms:

$$\mathbf{for}\ X\ \mathbf{being}\ \text{Element}\ \mathbf{of}\ [\!:D1,D2:\!]\ \mathbf{holds}\ X\ \mathbf{is}\ \text{TUPLE}\ \mathbf{of}\ D1,D2, \tag{7}$$

$$\mathbf{for}\ X\ \mathbf{being}\ \text{Element}\ \mathbf{of}\ [\!:D1,D2,D3:\!]\ \mathbf{holds}\ X\ \mathbf{is}\ \text{TUPLE}\ \mathbf{of}\ D1,D2,D3, \tag{8}$$

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(9) $$\textbf{for } X \textbf{ being } \text{Element } \textbf{of } [:D1,D2,D3,D4:] \\ \textbf{holds } X \textbf{ is } \text{TUPLE } \textbf{of } D1,D2,D3,D4.$$

In the sequel D has the type DOMAIN. The following axioms hold:

(10) $$D1 \textbf{ is } \text{SUBDOMAIN } \textbf{of } D2 \textbf{ iff } D1 \subseteq D2,$$

(11) $$D \textbf{ is } \text{SET\_DOMAIN}.$$

In the sequel x, y, z denote objects of the type Element **of** REAL. The following axioms are true:

(12) $$x + y = y + x,$$

(13) $$x + (y + z) = (x + y) + z,$$

(14) $$x + 0 = x,$$

(15) $$x \cdot y = y \cdot x,$$

(16) $$x \cdot (y \cdot z) = (x \cdot y) \cdot z,$$

(17) $$x \cdot 1 = x,$$

(18) $$x \cdot (y + z) = x \cdot y + x \cdot z,$$

(19) $$\textbf{ex}\, y \textbf{ st } x + y = 0,$$

(20) $$x \neq 0 \textbf{ implies ex}\, y \textbf{ st } x \cdot y = 1,$$

(21) $$x \leq y \;\&\; y \leq x \textbf{ implies } x = y,$$

(22) $$x \leq y \;\&\; y \leq z \textbf{ implies } x \leq z,$$

(23) $$x \leq y \textbf{ or } y \leq x,$$

(24) $$x \leq y \textbf{ implies } x + z \leq y + z,$$

(25) $$x \leq y \;\&\; 0 \leq z \textbf{ implies } x \cdot z \leq y \cdot z,$$

(26) $$\textbf{for } X,Y \textbf{ being } \text{Subset } \textbf{of } \text{REAL } \textbf{st} \\ (\textbf{ex}\, x \textbf{ st } x \in X) \;\&\; (\textbf{ex}\, x \textbf{ st } x \in Y) \;\&\; \textbf{for } x,y \textbf{ st } x \in X \;\&\; y \in Y \textbf{ holds } x \leq y \\ \textbf{ex}\, z \textbf{ st for } x,y \textbf{ st } x \in X \;\&\; y \in Y \textbf{ holds } x \leq z \;\&\; z \leq y,$$

(27) $$x \textbf{ is } \text{Real},$$

(28) $$x \in \text{NAT} \textbf{ implies } x + 1 \in \text{NAT},$$

(29) $$\textbf{for } A \textbf{ being set of } \text{Real} \\ \textbf{st } 0 \in A \;\&\; \textbf{for } x \textbf{ st } x \in A \textbf{ holds } x + 1 \in A \textbf{ holds } \text{NAT} \subseteq A,$$

(30) $$x \in \text{NAT} \textbf{ implies } x \textbf{ is } \text{Nat}.$$

References

[1] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1, 1990.

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Boolean Properties of Sets

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Summary. The text includes a number of theorems about Boolean operations on sets: union, intersection, difference, symmetric difference; and relations on sets: meets (having non-empty intersection), misses (being disjoint) and subset (inclusion).

The terminology and notation used here are introduced in the article [1]. For simplicity we adopt the following convention: x will have the type Any; X, Y, Z, V will have the type set. The scheme *Separation* concerns a constant $\mathcal{A}$ that has the type set and a unary predicate $\mathcal{P}$ and states that the following holds

$$\mathbf{ex}\, X \;\mathbf{st}\; \mathbf{for}\, x \;\mathbf{holds}\; x \in X \;\mathbf{iff}\; x \in \mathcal{A} \;\&\; \mathcal{P}[x]$$

for all values of the parameters.

We now define several new constructions. The constant $\emptyset$ has the type set, and is defined by

$$\mathbf{not}\,\mathbf{ex}\, x \;\mathbf{st}\; x \in \mathbf{it}\,.$$

Let us consider X, Y. The functor

$$X \cup Y,$$

with values of the type set, is defined by

$$x \in \mathbf{it} \;\mathbf{iff}\; x \in X \;\mathbf{or}\; x \in Y.$$

The functor

$$X \cap Y,$$

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with values of the type set, is defined by

$$x \in \textbf{it} \textbf{ iff } x \in X \ \& \ x \in Y.$$

The functor

$$X \setminus Y,$$

yields the type set and is defined by

$$x \in \textbf{it} \textbf{ iff } x \in X \ \& \ \textbf{not}\, x \in Y.$$

The predicate

$$X \text{ meets } Y \qquad \text{is defined by} \qquad \textbf{ex}\, x \,\textbf{st}\, x \in X \ \& \ x \in Y.$$

The predicate

$$X \text{ misses } Y \qquad \text{is defined by} \qquad \textbf{for}\, x \,\textbf{holds}\, x \in X \textbf{ implies not}\, x \in Y.$$

Let us consider X, Y. The functor

$$X \dot{-} Y,$$

with values of the type set, is defined by

$$\textbf{it} = (X \setminus Y) \cup (Y \setminus X).$$

We now state several propositions:

$$Z = \emptyset \textbf{ iff not ex}\, x \,\textbf{st}\, x \in Z, \tag{1}$$

$$Z = X \cup Y \textbf{ iff for}\, x \,\textbf{holds}\, x \in Z \textbf{ iff } x \in X \textbf{ or } x \in Y, \tag{2}$$

$$Z = X \cap Y \textbf{ iff for}\, x \,\textbf{holds}\, x \in Z \textbf{ iff } x \in X \ \& \ x \in Y, \tag{3}$$

$$Z = X \setminus Y \textbf{ iff for}\, x \,\textbf{holds}\, x \in Z \textbf{ iff } x \in X \ \& \ \textbf{not}\, x \in Y, \tag{4}$$

$$X \subseteq Y \textbf{ iff for}\, x \,\textbf{holds}\, x \in X \textbf{ implies } x \in Y, \tag{5}$$

$$X \text{ meets } Y \textbf{ iff ex}\, x \,\textbf{st}\, x \in X \ \& \ x \in Y, \tag{6}$$

$$X \text{ misses } Y \textbf{ iff for}\, x \,\textbf{holds}\, x \in X \textbf{ implies not}\, x \in Y. \tag{7}$$

Let us consider X, Y. Let us note that one can characterize the predicate

$$X = Y$$

by the following (equivalent) condition:

$$X \subseteq Y \ \& \ Y \subseteq X.$$

The following propositions are true:

$$x \in X \cup Y \textbf{ iff } x \in X \textbf{ or } x \in Y, \tag{8}$$

$$x \in X \cap Y \textbf{ iff } x \in X \ \& \ x \in Y, \tag{9}$$

$$x \in X \setminus Y \textbf{ iff } x \in X \ \& \ \textbf{not}\, x \in Y, \tag{10}$$

$$x \in X \ \& \ X \subseteq Y \textbf{ implies } x \in Y, \tag{11}$$

$$x \in X \ \& \ X \text{ misses } Y \textbf{ implies not}\, x \in Y, \tag{12}$$

$$x \in X \ \& \ x \in Y \textbf{ implies } X \text{ meets } Y, \tag{13}$$

$$x \in X \textbf{ implies } X \neq \emptyset, \tag{14}$$

$$X \text{ meets } Y \textbf{ implies ex}\, x \textbf{ st } x \in X \ \& \ x \in Y, \tag{15}$$

$$(\textbf{for}\, x \textbf{ st } x \in X \textbf{ holds } x \in Y) \textbf{ implies } X \subseteq Y, \tag{16}$$

$$(\textbf{for}\, x \textbf{ st } x \in X \textbf{ holds not}\, x \in Y) \textbf{ implies } X \text{ misses } Y, \tag{17}$$

$$(\textbf{for}\, x \textbf{ holds } x \in X \textbf{ iff } x \in Y \textbf{ or } x \in Z) \textbf{ implies } X = Y \cup Z, \tag{18}$$

$$(\textbf{for}\, x \textbf{ holds } x \in X \textbf{ iff } x \in Y \ \& \ x \in Z) \textbf{ implies } X = Y \cap Z, \tag{19}$$

$$(\textbf{for}\, x \textbf{ holds } x \in X \textbf{ iff } x \in Y \ \& \ \textbf{not}\, x \in Z) \textbf{ implies } X = Y \setminus Z, \tag{20}$$

$$\textbf{not}\,(\textbf{ex}\, x \textbf{ st } x \in X) \textbf{ implies } X = \emptyset, \tag{21}$$

$$(\textbf{for}\, x \textbf{ holds } x \in X \textbf{ iff } x \in Y) \textbf{ implies } X = Y, \tag{22}$$

$$x \in X \dot{-} Y \textbf{ iff not}\,(x \in X \textbf{ iff } x \in Y), \tag{23}$$

$$x \in X \ \& \ x \in Y \textbf{ implies } X \cap Y \neq \emptyset, \tag{24}$$

$$(\textbf{for}\, x \textbf{ holds not}\, x \in X \textbf{ iff } (x \in Y \textbf{ iff } x \in Z)) \textbf{ implies } X = Y \dot{-} Z, \tag{25}$$

$$X \subseteq X, \tag{26}$$

$$\emptyset \subseteq X, \tag{27}$$

$$X \subseteq Y \ \& \ Y \subseteq X \textbf{ implies } X = Y, \tag{28}$$

$$X \subseteq Y \ \& \ Y \subseteq Z \textbf{ implies } X \subseteq Z, \tag{29}$$

$$X \subseteq \emptyset \textbf{ implies } X = \emptyset, \tag{30}$$

$$X \subseteq X \cup Y \ \& \ Y \subseteq X \cup Y, \tag{31}$$

$$X \subseteq Z \ \& \ Y \subseteq Z \textbf{ implies } X \cup Y \subseteq Z, \tag{32}$$

$$X \subseteq Y \textbf{ implies } X \cup Z \subseteq Y \cup Z \ \& \ Z \cup X \subseteq Z \cup Y, \tag{33}$$

$$X \subseteq Y \ \& \ Z \subseteq V \ \textbf{implies} \ X \cup Z \subseteq Y \cup V, \tag{34}$$

$$X \subseteq Y \ \textbf{implies} \ X \cup Y = Y \ \& \ Y \cup X = Y, \tag{35}$$

$$X \cup Y = Y \ \textbf{or} \ Y \cup X = Y \ \textbf{implies} \ X \subseteq Y, \tag{36}$$

$$X \cap Y \subseteq X \ \& \ X \cap Y \subseteq Y, \tag{37}$$

$$X \cap Y \subseteq X \cup Z, \tag{38}$$

$$Z \subseteq X \ \& \ Z \subseteq Y \ \textbf{implies} \ Z \subseteq X \cap Y, \tag{39}$$

$$X \subseteq Y \ \textbf{implies} \ X \cap Z \subseteq Y \cap Z \ \& \ Z \cap X \subseteq Z \cap Y, \tag{40}$$

$$X \subseteq Y \ \& \ Z \subseteq V \ \textbf{implies} \ X \cap Z \subseteq Y \cap V, \tag{41}$$

$$X \subseteq Y \ \textbf{implies} \ X \cap Y = X \ \& \ Y \cap X = X, \tag{42}$$

$$X \cap Y = X \ \textbf{or} \ Y \cap X = X \ \textbf{implies} \ X \subseteq Y, \tag{43}$$

$$X \subseteq Z \ \textbf{implies} \ X \cup Y \cap Z = (X \cup Y) \cap Z, \tag{44}$$

$$X \setminus Y = \emptyset \ \textbf{iff} \ X \subseteq Y, \tag{45}$$

$$X \subseteq Y \ \textbf{implies} \ X \setminus Z \subseteq Y \setminus Z, \tag{46}$$

$$X \subseteq Y \ \textbf{implies} \ Z \setminus Y \subseteq Z \setminus X, \tag{47}$$

$$X \subseteq Y \ \& \ Z \subseteq V \ \textbf{implies} \ X \setminus V \subseteq Y \setminus Z, \tag{48}$$

$$X \setminus Y \subseteq X, \tag{49}$$

$$X \subseteq Y \setminus X \ \textbf{implies} \ X = \emptyset, \tag{50}$$

$$X \subseteq Y \ \& \ X \subseteq Z \ \& \ Y \cap Z = \emptyset \ \textbf{implies} \ X = \emptyset, \tag{51}$$

$$X \subseteq Y \cup Z \ \textbf{implies} \ X \setminus Y \subseteq Z \ \& \ X \setminus Z \subseteq Y, \tag{52}$$

$$(X \cap Y) \cup (X \cap Z) = X \ \textbf{implies} \ X \subseteq Y \cup Z, \tag{53}$$

$$X \subseteq Y \ \textbf{implies} \ Y = X \cup (Y \setminus X) \ \& \ Y = (Y \setminus X) \cup X, \tag{54}$$

$$X \subseteq Y \ \& \ Y \cap Z = \emptyset \ \textbf{implies} \ X \cap Z = \emptyset, \tag{55}$$

$$X = Y \cup Z \ \textbf{iff} \ Y \subseteq X \ \& \ Z \subseteq X \ \& \ \textbf{for} \ V \ \textbf{st} \ Y \subseteq V \ \& \ Z \subseteq V \ \textbf{holds} \ X \subseteq V, \tag{56}$$

$$X = Y \cap Z \ \textbf{iff} \ X \subseteq Y \ \& \ X \subseteq Z \ \& \ \textbf{for} \ V \ \textbf{st} \ V \subseteq Y \ \& \ V \subseteq Z \ \textbf{holds} \ V \subseteq X, \tag{57}$$

$$X \setminus Y \subseteq X \dot{-} Y, \tag{58}$$

$$X \cup Y = \emptyset \textbf{ iff } X = \emptyset \ \& \ Y = \emptyset, \tag{59}$$

$$X \cup \emptyset = X \ \& \ \emptyset \cup X = X, \tag{60}$$

$$X \cap \emptyset = \emptyset \ \& \ \emptyset \cap X = \emptyset, \tag{61}$$

$$X \cup X = X, \tag{62}$$

$$X \cup Y = Y \cup X, \tag{63}$$

$$(X \cup Y) \cup Z = X \cup (Y \cup Z), \tag{64}$$

$$X \cap X = X, \tag{65}$$

$$X \cap Y = Y \cap X, \tag{66}$$

$$(X \cap Y) \cap Z = X \cap (Y \cap Z), \tag{67}$$

$$\begin{gathered} X \cap (X \cup Y) = X \\ \& \ (X \cup Y) \cap X = X \ \& \ X \cap (Y \cup X) = X \ \& \ (Y \cup X) \cap X = X, \end{gathered} \tag{68}$$

$$\begin{gathered} X \cup (X \cap Y) = X \\ \& \ (X \cap Y) \cup X = X \ \& \ X \cup (Y \cap X) = X \ \& \ (Y \cap X) \cup X = X, \end{gathered} \tag{69}$$

$$X \cap (Y \cup Z) = X \cap Y \cup X \cap Z \ \& \ (Y \cup Z) \cap X = Y \cap X \cup Z \cap X, \tag{70}$$

$$X \cup Y \cap Z = (X \cup Y) \cap (X \cup Z) \ \& \ Y \cap Z \cup X = (Y \cup X) \cap (Z \cup X), \tag{71}$$

$$(X \cap Y) \cup (Y \cap Z) \cup (Z \cap X) = (X \cup Y) \cap (Y \cup Z) \cap (Z \cup X), \tag{72}$$

$$X \setminus X = \emptyset, \tag{73}$$

$$X \setminus \emptyset = X, \tag{74}$$

$$\emptyset \setminus X = \emptyset, \tag{75}$$

$$X \setminus (X \cup Y) = \emptyset \ \& \ X \setminus (Y \cup X) = \emptyset, \tag{76}$$

$$X \setminus X \cap Y = X \setminus Y \ \& \ X \setminus Y \cap X = X \setminus Y, \tag{77}$$

$$(X \setminus Y) \cap Y = \emptyset \ \& \ Y \cap (X \setminus Y) = \emptyset, \tag{78}$$

$$X \cup (Y \setminus X) = X \cup Y \ \& \ (Y \setminus X) \cup X = Y \cup X, \tag{79}$$

$$X \cap Y \cup (X \setminus Y) = X \ \& \ (X \setminus Y) \cup X \cap Y = X, \tag{80}$$

$$X \setminus (Y \setminus Z) = (X \setminus Y) \cup X \cap Z, \tag{81}$$

$$X \setminus (X \setminus Y) = X \cap Y, \tag{82}$$

$$(X \cup Y) \setminus Y = X \setminus Y, \tag{83}$$

$$X \cap Y = \emptyset \textbf{ iff } X \setminus Y = X, \tag{84}$$

$$X \setminus (Y \cup Z) = (X \setminus Y) \cap (X \setminus Z), \tag{85}$$

$$X \setminus (Y \cap Z) = (X \setminus Y) \cup (X \setminus Z), \tag{86}$$

$$(X \cup Y) \setminus (X \cap Y) = (X \setminus Y) \cup (Y \setminus X), \tag{87}$$

$$(X \setminus Y) \setminus Z = X \setminus (Y \cup Z), \tag{88}$$

$$(X \cup Y) \setminus Z = (X \setminus Z) \cup (Y \setminus Z), \tag{89}$$

$$X \setminus Y = Y \setminus X \textbf{ implies } X = Y, \tag{90}$$

$$X \dot{-} Y = (X \setminus Y) \cup (Y \setminus X), \tag{91}$$

$$X \dot{-} \emptyset = X \ \& \ \emptyset \dot{-} X = X, \tag{92}$$

$$X \dot{-} X = \emptyset, \tag{93}$$

$$X \dot{-} Y = Y \dot{-} X, \tag{94}$$

$$X \cup Y = (X \dot{-} Y) \cup X \cap Y, \tag{95}$$

$$X \dot{-} Y = (X \cup Y) \setminus X \cap Y, \tag{96}$$

$$(X \dot{-} Y) \setminus Z = (X \setminus (Y \cup Z)) \cup (Y \setminus (X \cup Z)), \tag{97}$$

$$X \setminus (Y \dot{-} Z) = X \setminus (Y \cup Z) \cup X \cap Y \cap Z, \tag{98}$$

$$(X \dot{-} Y) \dot{-} Z = X \dot{-} (Y \dot{-} Z), \tag{99}$$

$$X \text{ meets } Y \cup Z \textbf{ iff } X \text{ meets } Y \textbf{ or } X \text{ meets } Z, \tag{100}$$

$$X \text{ meets } Y \ \& \ Y \subseteq Z \textbf{ implies } X \text{ meets } Z, \tag{101}$$

$$X \text{ meets } Y \cap Z \textbf{ implies } X \text{ meets } Y \ \& \ X \text{ meets } Z, \tag{102}$$

$$X \text{ meets } Y \textbf{ implies } Y \text{ meets } X, \tag{103}$$

$$\textbf{not } (X \text{ meets } \emptyset \textbf{ or } \emptyset \text{ meets } X), \tag{104}$$

$$X \text{ misses } Y \textbf{ iff not } X \text{ meets } Y, \tag{105}$$

$$X \text{ misses } Y \cup Z \textbf{ iff } X \text{ misses } Y \ \& \ X \text{ misses } Z, \tag{106}$$

$$X \text{ misses } Z \ \& \ Y \subseteq Z \textbf{ implies } X \text{ misses } Y, \tag{107}$$

$$X \text{ misses } Y \textbf{ or } X \text{ misses } Z \textbf{ implies } X \text{ misses } Y \cap Z, \tag{108}$$

$$X \text{ misses } \emptyset \;\&\; \emptyset \text{ misses } X, \tag{109}$$

$$X \text{ meets } X \textbf{ iff } X \neq \emptyset, \tag{110}$$

$$X \cap Y \text{ misses } X \setminus Y, \tag{111}$$

$$X \cap Y \text{ misses } X \dot{-} Y, \tag{112}$$

$$X \text{ meets } Y \setminus Z \textbf{ implies } X \text{ meets } Y, \tag{113}$$

$$X \subseteq Y \;\&\; X \subseteq Z \;\&\; Y \text{ misses } Z \textbf{ implies } X = \emptyset, \tag{114}$$

$$X \setminus Y \subseteq Z \;\&\; Y \setminus X \subseteq Z \textbf{ implies } X \dot{-} Y \subseteq Z, \tag{115}$$

$$X \cap (Y \setminus Z) = (X \cap Y) \setminus Z, \tag{116}$$

$$X \cap (Y \setminus Z) = X \cap Y \setminus X \cap Z \;\&\; (Y \setminus Z) \cap X = Y \cap X \setminus Z \cap X, \tag{117}$$

$$X \text{ misses } Y \textbf{ iff } X \cap Y = \emptyset, \tag{118}$$

$$X \text{ meets } Y \textbf{ iff } X \cap Y \neq \emptyset, \tag{119}$$

$$X \subseteq (Y \cup Z) \;\&\; X \cap Z = \emptyset \textbf{ implies } X \subseteq Y, \tag{120}$$

$$Y \subseteq X \;\&\; X \cap Y = \emptyset \textbf{ implies } Y = \emptyset, \tag{121}$$

$$X \text{ misses } Y \textbf{ implies } Y \text{ misses } X. \tag{122}$$

References

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Enumerated Sets

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Summary. We prove basic facts about enumerated sets: definitional theorems and their immediate consequences, some theorems related to the decomposition of an enumerated set into union of two sets, facts about removing elements that occur more than once, and facts about permutations of enumerated sets (with the length ≤ 4). The article includes also schemes enabling instantiation of up to nine universal quantifiers.

The terminology and notation used in this paper have been introduced in the papers [1] and [2]. For simplicity we adopt the following convention: x, $x1$, $x2$, $x3$, $x4$, $x5$, $x6$, $x7$, $x8$ have the type Any; X has the type set. In the article we present several logical schemes. The scheme *UI1* concerns a constant $\mathcal{A}$ and a unary predicate $\mathcal{P}$ and states that the following holds

$$\mathcal{P}[\mathcal{A}]$$

provided the parameters satisfy the following condition:

- **for** $x1$ **holds** $\mathcal{P}[x1]$.

The scheme *UI2* deals with a constant $\mathcal{A}$, a constant $\mathcal{B}$ and a binary predicate $\mathcal{P}$ and states that the following holds

$$\mathcal{P}[\mathcal{A},\mathcal{B}]$$

provided the parameters satisfy the following condition:

- **for** $x1$,$x2$ **holds** $\mathcal{P}[x1,x2]$.

The scheme *UI3* concerns a constant $\mathcal{A}$, a constant $\mathcal{B}$, a constant $\mathcal{C}$ and a ternary predicate $\mathcal{P}$ and states that the following holds

$$\mathcal{P}[\mathcal{A},\mathcal{B},\mathcal{C}]$$

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provided the parameters satisfy the following condition:

- **for** $x1$,$x2$,$x3$ **holds** $\mathcal{P}[x1,x2,x3]$.

The scheme *UI4* concerns a constant $\mathcal{A}$, a constant $\mathcal{B}$, a constant $\mathcal{C}$, a constant $\mathcal{D}$ and a 4-ary predicate $\mathcal{P}$ and states that the following holds

$$\mathcal{P}[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}]$$

provided the parameters satisfy the following condition:

- **for** $x1$,$x2$,$x3$,$x4$ **holds** $\mathcal{P}[x1,x2,x3,x4]$.

The scheme *UI5* deals with a constant $\mathcal{A}$, a constant $\mathcal{B}$, a constant $\mathcal{C}$, a constant $\mathcal{D}$, a constant $\mathcal{E}$ and a 5-ary predicate $\mathcal{P}$ and states that the following holds

$$\mathcal{P}[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D},\mathcal{E}]$$

provided the parameters satisfy the following condition:

- **for** $x1$,$x2$,$x3$,$x4$,$x5$ **holds** $\mathcal{P}[x1,x2,x3,x4,x5]$.

The scheme *UI6* deals with a constant $\mathcal{A}$, a constant $\mathcal{B}$, a constant $\mathcal{C}$, a constant $\mathcal{D}$, a constant $\mathcal{E}$, a constant $\mathcal{F}$ and a 6-ary predicate $\mathcal{P}$ and states that the following holds

$$\mathcal{P}[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D},\mathcal{E},\mathcal{F}]$$

provided the parameters satisfy the following condition:

- **for** $x1$,$x2$,$x3$,$x4$,$x5$,$x6$ **holds** $\mathcal{P}[x1,x2,x3,x4,x5,x6]$.

The scheme *UI7* concerns a constant $\mathcal{A}$, a constant $\mathcal{B}$, a constant $\mathcal{C}$, a constant $\mathcal{D}$, a constant $\mathcal{E}$, a constant $\mathcal{F}$, a constant $\mathcal{G}$ and a 7-ary predicate $\mathcal{P}$ and states that the following holds

$$\mathcal{P}[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D},\mathcal{E},\mathcal{F},\mathcal{G}]$$

provided the parameters satisfy the following condition:

- **for** $x1$,$x2$,$x3$,$x4$,$x5$,$x6$,$x7$ **holds** $\mathcal{P}[x1,x2,x3,x4,x5,x6,x7]$.

The scheme *UI8* concerns a constant $\mathcal{A}$, a constant $\mathcal{B}$, a constant $\mathcal{C}$, a constant $\mathcal{D}$, a constant $\mathcal{E}$, a constant $\mathcal{F}$, a constant $\mathcal{G}$, a constant $\mathcal{H}$ and a 8-ary predicate $\mathcal{P}$ and states that the following holds

$$\mathcal{P}[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D},\mathcal{E},\mathcal{F},\mathcal{G},\mathcal{H}]$$

provided the parameters satisfy the following condition:

- **for** $x1$,$x2$,$x3$,$x4$,$x5$,$x6$,$x7$,$x8$ **holds** $\mathcal{P}[x1,x2,x3,x4,x5,x6,x7,x8]$.

The scheme *UI9* concerns a constant $\mathcal{A}$, a constant $\mathcal{B}$, a constant $\mathcal{C}$, a constant $\mathcal{D}$, a constant $\mathcal{E}$, a constant $\mathcal{F}$, a constant $\mathcal{G}$, a constant $\mathcal{H}$, a constant $\mathcal{I}$ and a 9-ary predicate $\mathcal{P}$ and states that the following holds

$$\mathcal{P}[\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D},\mathcal{E},\mathcal{F},\mathcal{G},\mathcal{H},\mathcal{I}]$$

provided the parameters satisfy the following condition:

- $$\textbf{for } x1,x2,x3,x4,x5,x6,x7,x8,x9 \textbf{ being } \mathrm{Any} \\ \textbf{holds } \mathcal{P}[x1,x2,x3,x4,x5,x6,x7,x8,x9].$$

We now state a number of propositions:

$$\textbf{for } x1,X \textbf{ holds } X = \{x1\} \textbf{ iff for } x \textbf{ holds } x \in X \textbf{ iff } x = x1, \tag{1}$$

$$\textbf{for } x1,x \textbf{ holds } x \in \{x1\} \textbf{ iff } x = x1, \tag{2}$$

$$x \in \{x1\} \textbf{ implies } x = x1, \tag{3}$$

$$x \in \{x\}, \tag{4}$$

$$\textbf{for } x1,X \textbf{ st for } x \textbf{ holds } x \in X \textbf{ iff } x = x1 \textbf{ holds } X = \{x1\}, \tag{5}$$

$$\textbf{for } x1,x2,X \textbf{ holds } X = \{x1,x2\} \textbf{ iff for } x \textbf{ holds } x \in X \textbf{ iff } x = x1 \textbf{ or } x = x2, \tag{6}$$

$$\textbf{for } x1,x2 \textbf{ for } x \textbf{ holds } x \in \{x1,x2\} \textbf{ iff } x = x1 \textbf{ or } x = x2, \tag{7}$$

$$x \in \{x1,x2\} \textbf{ implies } x = x1 \textbf{ or } x = x2, \tag{8}$$

$$x = x1 \textbf{ or } x = x2 \textbf{ implies } x \in \{x1,x2\}, \tag{9}$$

$$\textbf{for } x1,x2,X \textbf{ st for } x \textbf{ holds } x \in X \textbf{ iff } x = x1 \textbf{ or } x = x2 \textbf{ holds } X = \{x1,x2\}. \tag{10}$$

Let us consider $x1$, $x2$, $x3$. The functor

$$\{x1,x2,x3\},$$

yields the type set and is defined by

$$x \in \textbf{it iff } x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3.$$

One can prove the following propositions:

$$\textbf{for } x1,x2,x3,X \\ \textbf{holds } X = \{x1,x2,x3\} \textbf{ iff for } x \textbf{ holds } x \in X \textbf{ iff } x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3, \tag{11}$$

$$\textbf{for } x1,x2,x3 \textbf{ for } x \textbf{ holds } x \in \{x1,x2,x3\} \textbf{ iff } x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3, \tag{12}$$

$$x \in \{x1,x2,x3\} \textbf{ implies } x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3, \tag{13}$$

(14) $$x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ implies } x \in \{x1,x2,x3\},$$

(15) $$\textbf{for } x1,x2,x3,X$$
$$\textbf{st for } x \textbf{ holds } x \in X \textbf{ iff } x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ holds } X = \{x1,x2,x3\}.$$

Let us consider $x1$, $x2$, $x3$, $x4$. The functor

$$\{x1,x2,x3,x4\},$$

with values of the type set, is defined by

$$x \in \textbf{it iff } x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4.$$

We now state several propositions:

(16) $$\textbf{for } x1,x2,x3,x4,X \textbf{ holds } X = \{x1,x2,x3,x4\}$$
$$\textbf{iff for } x \textbf{ holds } x \in X \textbf{ iff } x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4,$$

(17) $$\textbf{for } x1,x2,x3,x4$$
$$\textbf{for } x \textbf{ holds } x \in \{x1,x2,x3,x4\} \textbf{ iff } x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4,$$

(18) $$x \in \{x1,x2,x3,x4\} \textbf{ implies } x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4,$$

(19) $$x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4 \textbf{ implies } x \in \{x1,x2,x3,x4\},$$

(20) $$\textbf{for } x1,x2,x3,x4,X \textbf{ st}$$
$$\textbf{for } x \textbf{ holds } x \in X \textbf{ iff } x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4$$
$$\textbf{holds } X = \{x1,x2,x3,x4\}.$$

Let us consider $x1$, $x2$, $x3$, $x4$, $x5$. The functor

$$\{x1,x2,x3,x4,x5\},$$

yields the type set and is defined by

$$x \in \textbf{it iff } x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4 \textbf{ or } x = x5.$$

Next we state several propositions:

(21) $$\textbf{for } x1,x2,x3,x4,x5 \textbf{ for } X \textbf{ being } \text{set} \textbf{ holds } X = \{x1,x2,x3,x4,x5\}$$
$$\textbf{iff for } x \textbf{ holds } x \in X \textbf{ iff } x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4 \textbf{ or } x = x5,$$

(22) $$x \in \{x1,x2,x3,x4,x5\} \textbf{ iff } x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4 \textbf{ or } x = x5,$$

(23) $$x \in \{x1,x2,x3,x4,x5\}$$
$$\textbf{implies } x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4 \textbf{ or } x = x5,$$

(24) $$x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4 \textbf{ or } x = x5$$
$$\textbf{implies } x \in \{x1,x2,x3,x4,x5\},$$

(25)
$$\textbf{for } X \textbf{ being} \text{ set } \textbf{st}$$
$$\textbf{for } x \textbf{ holds } x \in X \textbf{ iff } x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4 \textbf{ or } x = x5$$
$$\textbf{holds } X = \{x1,x2,x3,x4,x5\}.$$

Let us consider $x1$, $x2$, $x3$, $x4$, $x5$, $x6$. The functor

$$\{x1,x2,x3,x4,x5,x6\},$$

with values of the type set, is defined by

$$x \in \textbf{it iff } x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4 \textbf{ or } x = x5 \textbf{ or } x = x6.$$

We now state several propositions:

(26) $\textbf{for } x1,x2,x3,x4,x5,x6 \textbf{ for } X \textbf{ being} \text{ set } \textbf{holds } X = \{x1,x2,x3,x4,x5,x6\} \textbf{ iff}$
$$\textbf{for } x$$
$$\textbf{holds } x \in X \textbf{ iff } x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4 \textbf{ or } x = x5 \textbf{ or } x = x6,$$

(27)
$$x \in \{x1,x2,x3,x4,x5,x6\}$$
$$\textbf{iff } x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4 \textbf{ or } x = x5 \textbf{ or } x = x6,$$

(28)
$$x \in \{x1,x2,x3,x4,x5,x6\}$$
$$\textbf{implies } x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4 \textbf{ or } x = x5 \textbf{ or } x = x6,$$

(29)
$$x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4 \textbf{ or } x = x5 \textbf{ or } x = x6$$
$$\textbf{implies } x \in \{x1,x2,x3,x4,x5,x6\},$$

(30)
$$\textbf{for } X \textbf{ being} \text{ set } \textbf{st}$$
$$\textbf{for } x$$
$$\textbf{holds } x \in X \textbf{ iff } x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4 \textbf{ or } x = x5 \textbf{ or } x = x6$$
$$\textbf{holds } X = \{x1,x2,x3,x4,x5,x6\}.$$

Let us consider $x1$, $x2$, $x3$, $x4$, $x5$, $x6$, $x7$. The functor

$$\{x1,x2,x3,x4,x5,x6,x7\},$$

yields the type set and is defined by

$$x \in \textbf{it iff } x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4 \textbf{ or } x = x5 \textbf{ or } x = x6 \textbf{ or } x = x7.$$

The following propositions are true:

(31) $\textbf{for } x1,x2,x3,x4,x5,x6,x7 \textbf{ for } X \textbf{ being} \text{ set } \textbf{holds } X = \{x1,x2,x3,x4,x5,x6,x7\}$
$$\textbf{iff for } x \textbf{ holds } x \in X$$
$$\textbf{iff } x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4 \textbf{ or } x = x5 \textbf{ or } x = x6 \textbf{ or } x = x7,$$

(32)
$$x \in \{x1,x2,x3,x4,x5,x6,x7\}$$
$$\textbf{iff } x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4 \textbf{ or } x = x5 \textbf{ or } x = x6 \textbf{ or } x = x7,$$

(33)
$$x \in \{x1,x2,x3,x4,x5,x6,x7\} \textbf{ implies}$$
$$x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4 \textbf{ or } x = x5 \textbf{ or } x = x6 \textbf{ or } x = x7,$$

(34)
$$x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4 \textbf{ or } x = x5 \textbf{ or } x = x6 \textbf{ or } x = x7$$
$$\textbf{implies } x \in \{x1,x2,x3,x4,x5,x6,x7\},$$

(35)
$$\textbf{for } X \textbf{ being } \mathrm{set} \textbf{ st}$$
$$\textbf{for } x \textbf{ holds } x \in X$$
$$\textbf{iff } x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4 \textbf{ or } x = x5 \textbf{ or } x = x6 \textbf{ or } x = x7$$
$$\textbf{holds } X = \{x1,x2,x3,x4,x5,x6,x7\}.$$

Let us consider $x1$, $x2$, $x3$, $x4$, $x5$, $x6$, $x7$, $x8$. The functor

$$\{x1,x2,x3,x4,x5,x6,x7,x8\},$$

with values of the type set, is defined by

$$x \in \textbf{it}$$
$$\textbf{iff } x = x1 \textbf{ or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4 \textbf{ or } x = x5 \textbf{ or } x = x6 \textbf{ or } x = x7 \textbf{ or } x = x8.$$

Next we state a number of propositions:

(36)
$$\textbf{for } x1,x2,x3,x4,x5,x6,x7,x8 \textbf{ for } X \textbf{ being } \mathrm{set} \textbf{ holds}$$
$$X = \{x1,x2,x3,x4,x5,x6,x7,x8\} \textbf{ iff for } x \textbf{ holds } x \in X \textbf{ iff } x = x1$$
$$\textbf{or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4 \textbf{ or } x = x5 \textbf{ or } x = x6 \textbf{ or } x = x7 \textbf{ or } x = x8,$$

(37)
$$x \in \{x1,x2,x3,x4,x5,x6,x7,x8\} \textbf{ iff } x = x1$$
$$\textbf{or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4 \textbf{ or } x = x5 \textbf{ or } x = x6 \textbf{ or } x = x7 \textbf{ or } x = x8,$$

(38)
$$x \in \{x1,x2,x3,x4,x5,x6,x7,x8\} \textbf{ implies } x = x1$$
$$\textbf{or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4 \textbf{ or } x = x5 \textbf{ or } x = x6 \textbf{ or } x = x7 \textbf{ or } x = x8,$$

(39)
$$x = x1$$
$$\textbf{or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4 \textbf{ or } x = x5 \textbf{ or } x = x6 \textbf{ or } x = x7 \textbf{ or } x = x8$$
$$\textbf{implies } x \in \{x1,x2,x3,x4,x5,x6,x7,x8\},$$

(40)
$$\textbf{for } X \textbf{ being } \mathrm{set} \textbf{ st}$$
$$\textbf{for } x \textbf{ holds } x \in X \textbf{ iff } x = x1$$
$$\textbf{or } x = x2 \textbf{ or } x = x3 \textbf{ or } x = x4 \textbf{ or } x = x5 \textbf{ or } x = x6 \textbf{ or } x = x7 \textbf{ or } x = x8$$
$$\textbf{holds } X = \{x1,x2,x3,x4,x5,x6,x7,x8\},$$

$$\{x1,x2\} = \{x1\} \cup \{x2\}, \tag{41}$$

$$\{x1,x2,x3\} = \{x1\} \cup \{x2,x3\}, \tag{42}$$

$$\{x1,x2,x3\} = \{x1,x2\} \cup \{x3\}, \tag{43}$$

$$\{x1,x2,x3,x4\} = \{x1\} \cup \{x2,x3,x4\}, \tag{44}$$

$$\{x1,x2,x3,x4\} = \{x1,x2\} \cup \{x3,x4\}, \tag{45}$$

$$\{x1,x2,x3,x4\} = \{x1,x2,x3\} \cup \{x4\}, \tag{46}$$

$$\{x1,x2,x3,x4,x5\} = \{x1\} \cup \{x2,x3,x4,x5\}, \tag{47}$$

$$\{x1,x2,x3,x4,x5\} = \{x1,x2\} \cup \{x3,x4,x5\}, \tag{48}$$

$$\{x1,x2,x3,x4,x5\} = \{x1,x2,x3\} \cup \{x4,x5\}, \tag{49}$$

$$\{x1,x2,x3,x4,x5\} = \{x1,x2,x3,x4\} \cup \{x5\}, \tag{50}$$

$$\{x1,x2,x3,x4,x5,x6\} = \{x1\} \cup \{x2,x3,x4,x5,x6\}, \tag{51}$$

$$\{x1,x2,x3,x4,x5,x6\} = \{x1,x2\} \cup \{x3,x4,x5,x6\}, \tag{52}$$

$$\{x1,x2,x3,x4,x5,x6\} = \{x1,x2,x3\} \cup \{x4,x5,x6\}, \tag{53}$$

$$\{x1,x2,x3,x4,x5,x6\} = \{x1,x2,x3,x4\} \cup \{x5,x6\}, \tag{54}$$

$$\{x1,x2,x3,x4,x5,x6\} = \{x1,x2,x3,x4,x5\} \cup \{x6\}, \tag{55}$$

$$\{x1,x2,x3,x4,x5,x6,x7\} = \{x1\} \cup \{x2,x3,x4,x5,x6,x7\}, \tag{56}$$

$$\{x1,x2,x3,x4,x5,x6,x7\} = \{x1,x2\} \cup \{x3,x4,x5,x6,x7\}, \tag{57}$$

$$\{x1,x2,x3,x4,x5,x6,x7\} = \{x1,x2,x3\} \cup \{x4,x5,x6,x7\}, \tag{58}$$

$$\{x1,x2,x3,x4,x5,x6,x7\} = \{x1,x2,x3,x4\} \cup \{x5,x6,x7\}, \tag{59}$$

$$\{x1,x2,x3,x4,x5,x6,x7\} = \{x1,x2,x3,x4,x5\} \cup \{x6,x7\}, \tag{60}$$

$$\{x1,x2,x3,x4,x5,x6,x7\} = \{x1,x2,x3,x4,x5,x6\} \cup \{x7\}, \tag{61}$$

$$\{x1,x2,x3,x4,x5,x6,x7,x8\} = \{x1\} \cup \{x2,x3,x4,x5,x6,x7,x8\}, \tag{62}$$

$$\{x1,x2,x3,x4,x5,x6,x7,x8\} = \{x1,x2\} \cup \{x3,x4,x5,x6,x7,x8\}, \tag{63}$$

$$\{x1,x2,x3,x4,x5,x6,x7,x8\} = \{x1,x2,x3\} \cup \{x4,x5,x6,x7,x8\}, \tag{64}$$

$$\{x1,x2,x3,x4,x5,x6,x7,x8\} = \{x1,x2,x3,x4\} \cup \{x5,x6,x7,x8\}, \tag{65}$$

$$\{x1,x2,x3,x4,x5,x6,x7,x8\} = \{x1,x2,x3,x4,x5\} \cup \{x6,x7,x8\}, \tag{66}$$

$$\{x1,x2,x3,x4,x5,x6,x7,x8\} = \{x1,x2,x3,x4,x5,x6\} \cup \{x7,x8\}, \tag{67}$$

$$\{x1,x2,x3,x4,x5,x6,x7,x8\} = \{x1,x2,x3,x4,x5,x6,x7\} \cup \{x8\}, \tag{68}$$

$$\{x1,x1\} = \{x1\}, \tag{69}$$

$$\{x1,x1,x2\} = \{x1,x2\}, \tag{70}$$

$$\{x1,x1,x2,x3\} = \{x1,x2,x3\}, \tag{71}$$

$$\{x1,x1,x2,x3,x4\} = \{x1,x2,x3,x4\}, \tag{72}$$

$$\{x1,x1,x2,x3,x4,x5\} = \{x1,x2,x3,x4,x5\}, \tag{73}$$

$$\{x1,x1,x2,x3,x4,x5,x6\} = \{x1,x2,x3,x4,x5,x6\}, \tag{74}$$

$$\{x1,x1,x2,x3,x4,x5,x6,x7\} = \{x1,x2,x3,x4,x5,x6,x7\}, \tag{75}$$

$$\{x1,x1,x1\} = \{x1\}, \tag{76}$$

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$$\{x1,x1,x1,x2,x3\} = \{x1,x2,x3\}, \tag{78}$$

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$$\{x1,x1,x1,x2,x3,x4,x5\} = \{x1,x2,x3,x4,x5\}, \tag{80}$$

$$\{x1,x1,x1,x2,x3,x4,x5,x6\} = \{x1,x2,x3,x4,x5,x6\}, \tag{81}$$

$$\{x1,x1,x1,x1\} = \{x1\}, \tag{82}$$

$$\{x1,x1,x1,x1,x2\} = \{x1,x2\}, \tag{83}$$

$$\{x1,x1,x1,x1,x2,x3\} = \{x1,x2,x3\}, \tag{84}$$

$$\{x1,x1,x1,x1,x2,x3,x4\} = \{x1,x2,x3,x4\}, \tag{85}$$

$$\{x1,x1,x1,x1,x2,x3,x4,x5\} = \{x1,x2,x3,x4,x5\}, \tag{86}$$

$$\{x1,x1,x1,x1,x1\} = \{x1\}, \tag{87}$$

$$\{x1,x1,x1,x1,x1,x2\} = \{x1,x2\}, \tag{88}$$

$$\{x1,x1,x1,x1,x1,x2,x3\} = \{x1,x2,x3\}, \tag{89}$$

$$\{x1,x1,x1,x1,x1,x2,x3,x4\} = \{x1,x2,x3,x4\}, \tag{90}$$

$$\{x1{,}x1{,}x1{,}x1{,}x1{,}x1\} = \{x1\}, \tag{91}$$

$$\{x1{,}x1{,}x1{,}x1{,}x1{,}x1{,}x2\} = \{x1{,}x2\}, \tag{92}$$

$$\{x1{,}x1{,}x1{,}x1{,}x1{,}x1{,}x2{,}x3\} = \{x1{,}x2{,}x3\}, \tag{93}$$

$$\{x1{,}x1{,}x1{,}x1{,}x1{,}x1{,}x1\} = \{x1\}, \tag{94}$$

$$\{x1{,}x1{,}x1{,}x1{,}x1{,}x1{,}x1{,}x2\} = \{x1{,}x2\}, \tag{95}$$

$$\{x1{,}x1{,}x1{,}x1{,}x1{,}x1{,}x1{,}x1\} = \{x1\}, \tag{96}$$

$$\{x1{,}x2\} = \{x2{,}x1\}, \tag{97}$$

$$\{x1{,}x2{,}x3\} = \{x1{,}x3{,}x2\}, \tag{98}$$

$$\{x1{,}x2{,}x3\} = \{x2{,}x1{,}x3\}, \tag{99}$$

$$\{x1{,}x2{,}x3\} = \{x2{,}x3{,}x1\}, \tag{100}$$

$$\{x1{,}x2{,}x3\} = \{x3{,}x1{,}x2\}, \tag{101}$$

$$\{x1{,}x2{,}x3\} = \{x3{,}x2{,}x1\}, \tag{102}$$

$$\{x1{,}x2{,}x3{,}x4\} = \{x1{,}x2{,}x4{,}x3\}, \tag{103}$$

$$\{x1{,}x2{,}x3{,}x4\} = \{x1{,}x3{,}x2{,}x4\}, \tag{104}$$

$$\{x1{,}x2{,}x3{,}x4\} = \{x1{,}x3{,}x4{,}x2\}, \tag{105}$$

$$\{x1{,}x2{,}x3{,}x4\} = \{x1{,}x4{,}x2{,}x3\}, \tag{106}$$

$$\{x1{,}x2{,}x3{,}x4\} = \{x1{,}x4{,}x3{,}x2\}, \tag{107}$$

$$\{x1{,}x2{,}x3{,}x4\} = \{x2{,}x1{,}x3{,}x4\}, \tag{108}$$

$$\{x1{,}x2{,}x3{,}x4\} = \{x2{,}x1{,}x4{,}x3\}, \tag{109}$$

$$\{x1{,}x2{,}x3{,}x4\} = \{x2{,}x3{,}x1{,}x4\}, \tag{110}$$

$$\{x1{,}x2{,}x3{,}x4\} = \{x2{,}x3{,}x4{,}x1\}, \tag{111}$$

$$\{x1{,}x2{,}x3{,}x4\} = \{x2{,}x4{,}x1{,}x3\}, \tag{112}$$

$$\{x1{,}x2{,}x3{,}x4\} = \{x2{,}x4{,}x3{,}x1\}, \tag{113}$$

$$\{x1{,}x2{,}x3{,}x4\} = \{x3{,}x1{,}x2{,}x4\}, \tag{114}$$

$$\{x1{,}x2{,}x3{,}x4\} = \{x3{,}x1{,}x4{,}x2\}, \tag{115}$$

$$\{x1,x2,x3,x4\} = \{x3,x2,x1,x4\}, \tag{116}$$

$$\{x1,x2,x3,x4\} = \{x3,x2,x4,x1\}, \tag{117}$$

$$\{x1,x2,x3,x4\} = \{x3,x4,x1,x2\}, \tag{118}$$

$$\{x1,x2,x3,x4\} = \{x3,x4,x2,x1\}, \tag{119}$$

$$\{x1,x2,x3,x4\} = \{x4,x1,x2,x3\}, \tag{120}$$

$$\{x1,x2,x3,x4\} = \{x4,x1,x3,x2\}, \tag{121}$$

$$\{x1,x2,x3,x4\} = \{x4,x2,x1,x3\}, \tag{122}$$

$$\{x1,x2,x3,x4\} = \{x4,x2,x3,x1\}, \tag{123}$$

$$\{x1,x2,x3,x4\} = \{x4,x3,x1,x2\}, \tag{124}$$

$$\{x1,x2,x3,x4\} = \{x4,x3,x2,x1\}. \tag{125}$$

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Basic Properties of Real Numbers

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Summary. Basic facts of arithmetics of real numbers are presented: definitions and properties of the complement element, the inverse element, subtraction and division; some basic properties of the set REAL (e.g. density), and the scheme of separation for sets of reals.

For simplicity we adopt the following convention: x, y, z, t will denote objects of the type Real; r will denote an object of the type Any. Let us consider x, y. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$x + y \quad \text{is} \quad \text{Real},$$

$$x \cdot y \quad \text{is} \quad \text{Real}.$$

One can prove the following propositions:

(1) $$r \textbf{ is } \text{Real } \textbf{iff } r \in \text{REAL},$$

(2) $$x + y = y + x,$$

(3) $$x + (y + z) = (x + y) + z,$$

(4) $$x + 0 = x \ \& \ 0 + x = x,$$

(5) $$x \cdot y = y \cdot x,$$

(6) $$x \cdot (y \cdot z) = (x \cdot y) \cdot z,$$

(7) $$x \cdot 1 = x \ \& \ 1 \cdot x = x,$$

(8) $$(x + y) \cdot z = x \cdot z + y \cdot z \ \& \ z \cdot (x + y) = z \cdot x + z \cdot y,$$

(9) $z \neq 0 \ \& \ x \neq y$ **implies** $x \cdot z \neq y \cdot z \ \& \ z \cdot x \neq y \cdot z \ \& \ z \cdot x \neq z \cdot y \ \& \ x \cdot z \neq z \cdot y,$

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$$z + x = z + y \textbf{ or } x + z = y + z \textbf{ or } z + x = y + z \textbf{ or } x + z = z + y \textbf{ implies } x = y, \tag{10}$$

$$x \neq y \textbf{ iff } x + z \neq y + z, \tag{11}$$

$$z \neq 0 \ \& \ (x \cdot z = y \cdot z \textbf{ or } z \cdot x = z \cdot y \textbf{ or } x \cdot z = z \cdot y \textbf{ or } z \cdot x = y \cdot z) \textbf{ implies } x = y. \tag{12}$$

We now define two new functors. Let us consider x. The functor

$$-x,$$

with values of the type Real, is defined by

$$x + \textbf{it} = 0.$$

Assume that the following holds

$$x \neq 0.$$

The functor

$$x^{-1},$$

yields the type Real and is defined by

$$x \cdot \textbf{it} = 1.$$

We now define two new functors. Let us consider x, y. The functor

$$x - y,$$

yields the type Real and is defined by

$$\textbf{it} = x + (-y).$$

Assume that the following holds

$$y \neq 0.$$

The functor

$$x/y,$$

yields the type Real and is defined by

$$\textbf{it} = x \cdot y^{-1}.$$

The following propositions are true:

$$x + -x = 0 \ \& \ -x + x = 0, \tag{13}$$

$$x - y = x + -y, \tag{14}$$

(15) $$x \neq 0 \textbf{ implies } x \cdot x^{-1} = 1 \ \& \ x^{-1} \cdot x = 1,$$

(16) $$y \neq 0 \textbf{ implies } x/y = x \cdot y^{-1} \ \& \ x/y = y^{-1} \cdot x,$$

(17) $$x + y - z = x + (y - z),$$

(18) $$-(-x) = x,$$

(19) $$0 - x = -x,$$

(20) $$x \cdot 0 = 0 \ \& \ 0 \cdot x = 0,$$

(21) $$(-x) \cdot y = -(x \cdot y) \ \& \ x \cdot (-y) = -(x \cdot y) \ \& \ (-x) \cdot y = x \cdot (-y),$$

(22) $$x \neq 0 \textbf{ iff } -x \neq 0,$$

(23) $$x \cdot y = 0 \textbf{ iff } x = 0 \textbf{ or } y = 0,$$

(24) $$x \neq 0 \ \& \ y \neq 0 \textbf{ implies } x^{-1} \cdot y^{-1} = (x \cdot y)^{-1},$$

(25) $$x - 0 = x,$$

(26) $$-0 = 0,$$

(27) $$x - (y + z) = x - y - z,$$

(28) $$x - (y - z) = x - y + z,$$

(29) $$x \cdot (y - z) = x \cdot y - x \cdot z \ \& \ (y - z) \cdot x = y \cdot x - z \cdot x,$$

(30) $$x + z = y \textbf{ implies } x = y - z \ \& \ z = y - x,$$

(31) $$x \neq 0 \textbf{ implies } x^{-1} \neq 0,$$

(32) $$x \neq 0 \textbf{ implies } x^{-1\,-1} = x,$$

(33) $$x \neq 0 \textbf{ implies } 1/x = x^{-1} \ \& \ 1/x^{-1} = x,$$

(34) $$x \neq 0 \textbf{ implies } x \cdot (1/x) = 1 \ \& \ (1/x) \cdot x = 1,$$

(35) $$y \neq 0 \ \& \ t \neq 0 \textbf{ implies } (x/y) \cdot (z/t) = (x \cdot z)/(y \cdot t),$$

(36) $$x - x = 0,$$

(37) $$x \neq 0 \textbf{ implies } x/x = 1,$$

(38) $$y \neq 0 \ \& \ z \neq 0 \textbf{ implies } x/y = (x \cdot z)/(y \cdot z),$$

(39) $$y \neq 0 \textbf{ implies } -x/y = (-x)/y \ \& \ x/(-y) = -x/y,$$

(40) $$z \neq 0 \textbf{ implies } x/z + y/z = (x+y)/z \;\&\; x/z - y/z = (x-y)/z,$$

(41) $$y \neq 0 \;\&\; t \neq 0$$
$$\textbf{implies } x/y + z/t = (x \cdot t + z \cdot y)/(y \cdot t) \;\&\; x/y - z/t = (x \cdot t - z \cdot y)/(y \cdot t),$$

(42) $$y \neq 0 \;\&\; z \neq 0 \textbf{ implies } x/(y/z) = (x \cdot z)/y,$$

(43) $$y \neq 0 \textbf{ implies } x/y \cdot y = x,$$

(44) $$\textbf{for } x,y \textbf{ ex } z \textbf{ st } x = y + z \;\&\; x = z + y,$$

(45) $$\textbf{for } x,y \textbf{ st } y \neq 0 \textbf{ ex } z \textbf{ st } x = y \cdot z \;\&\; x = z \cdot y,$$

(46) $$x \leq y \;\&\; y \leq x \textbf{ implies } x = y,$$

(47) $$x \leq y \;\&\; y \leq z \textbf{ implies } x \leq z,$$

(48) $$x \leq y \textbf{ or } y \leq x,$$

(49) $$x \leq y \textbf{ implies } x + z \leq y + z \;\&\; x - z \leq y - z,$$

(50) $$x \leq y \textbf{ iff } -y \leq -x,$$

(51) $$x \leq y \;\&\; 0 \leq z \textbf{ implies } x \cdot z \leq y \cdot z \;\&\; z \cdot x \leq z \cdot y \;\&\; z \cdot x \leq y \cdot z \;\&\; x \cdot z \leq z \cdot y,$$

(52) $$x \leq y \;\&\; z \leq 0 \textbf{ implies } y \cdot z \leq x \cdot z \;\&\; z \cdot y \leq z \cdot x \;\&\; y \cdot z \leq z \cdot x \;\&\; z \cdot y \leq x \cdot z,$$

(53) $$x \leq y \textbf{ iff } x + z \leq y + z,$$

(54) $$x \leq y \textbf{ iff } x - z \leq y - z,$$

(55) $$x \leq y \;\&\; z \leq t$$
$$\textbf{implies } x + z \leq y + t \;\&\; x + z \leq t + y \;\&\; z + x \leq t + y \;\&\; z + x \leq y + t,$$

(56) $$x \leq x.$$

Let us consider x, y. The predicate

$$x < y \qquad \text{is defined by} \qquad x \leq y \;\&\; x \neq y.$$

One can prove the following propositions:

(57) $$x < y \textbf{ iff } x \leq y \;\&\; x \neq y,$$

(58) $$x \leq y \;\&\; y < z \textbf{ or } x < y \;\&\; y \leq z \textbf{ or } x < y \;\&\; y < z \textbf{ implies } x < z,$$

(59) $$x < y \textbf{ implies } x + z < y + z$$
$$\&\; x - z < y - z \;\&\; z + x < z + y \;\&\; x + z < z + y \;\&\; z + x < y + z,$$

(60) $$x + z < y + z$$
$$\textbf{or } z + x < z + y \textbf{ or } x + z < z + y \textbf{ or } z + x < y + z \textbf{ or } x - z < y - z$$
$$\textbf{implies } x < y,$$

(61) $$x \neq y \textbf{ implies } x < y \textbf{ or } y < x,$$

(62) $$\textbf{not } x < y \textbf{ iff } y \leq x,$$

(63) $$x < y \textbf{ or } y < x \textbf{ or } x = y,$$

(64) $$x < y \textbf{ implies not } y < x,$$

(65) $$0 < 1,$$

(66) $$x < 0 \textbf{ iff } 0 < -x,$$

(67) $$x < y \;\&\; z \leq t \textbf{ or } x \leq y \;\&\; z < t \textbf{ or } x < y \;\&\; z < t$$
$$\textbf{implies } x + z < y + t \;\&\; z + x < y + t \;\&\; z + x < t + y \;\&\; x + z < t + y,$$

(68) $$x < y \textbf{ iff } -y < -x,$$

(69) $$\textbf{for } x,y \textbf{ st } 0 < x \textbf{ holds } y < y + x,$$

(70) $0 < z \;\&\; x < y$ **implies** $x \cdot z < y \cdot z \;\&\; z \cdot x < z \cdot y \;\&\; x \cdot z < z \cdot y \;\&\; z \cdot x < y \cdot z,$

(71) $z < 0 \;\&\; x < y$ **implies** $y \cdot z < x \cdot z \;\&\; z \cdot y < z \cdot x \;\&\; y \cdot z < z \cdot x \;\&\; z \cdot y < x \cdot z,$

(72) $$0 < z \textbf{ implies } 0 < z^{-1},$$

(73) $$0 < z \textbf{ implies } (x < y \textbf{ iff } x/z < y/z),$$

(74) $$z < 0 \textbf{ implies } (x < y \textbf{ iff } y/z < x/z),$$

(75) $$x < y \textbf{ implies ex } z \textbf{ st } x < z \;\&\; z < y,$$

(76) $$\textbf{for } x \textbf{ ex } y \textbf{ st } x < y,$$

(77) $$\textbf{for } x \textbf{ ex } y \textbf{ st } y < x,$$

(78) $$\textbf{for } X,Y \textbf{ being } \mathrm{Subset} \textbf{ of } \mathrm{REAL} \textbf{ st}$$
$$(\textbf{ex } x \textbf{ st } x \in X) \;\&\; (\textbf{ex } x \textbf{ st } x \in Y) \;\&\; \textbf{for } x,y \textbf{ st } x \in X \;\&\; y \in Y \textbf{ holds } x \leq y$$
$$\textbf{ex } z \textbf{ st for } x,y \textbf{ st } x \in X \;\&\; y \in Y \textbf{ holds } x \leq z \;\&\; z \leq y.$$

The scheme *SepReal* concerns a unary predicate $\mathcal{P}$ states that the following holds

$$\textbf{ex } X \textbf{ being set of } \mathrm{Real} \textbf{ st for } x \textbf{ holds } x \in X \textbf{ iff } \mathcal{P}[x]$$

for all values of the parameter.

The following propositions are true:

$$y = -x \textbf{ iff } x + y = 0, \tag{79}$$

$$\textbf{for } x,y \textbf{ st } x \neq 0 \textbf{ holds } y = x^{-1} \textbf{ iff } x \cdot y = 1, \tag{80}$$

$$\textbf{for } x,y \textbf{ st } x \neq 0 \;\&\; y \neq 0 \textbf{ holds } (x/y)^{-1} = y/x, \tag{81}$$

$$\textbf{for } x,y,z,t \textbf{ st } y \neq 0 \;\&\; z \neq 0 \;\&\; t \neq 0 \textbf{ holds } (x/y)/(z/t) = (x \cdot t)/(y \cdot z), \tag{82}$$

$$-(x - y) = y - x, \tag{83}$$

$$x + y \leq z \textbf{ iff } x \leq z - y, \tag{84}$$

$$x + y \leq z \textbf{ iff } y \leq z - x, \tag{85}$$

$$x \leq y + z \textbf{ iff } x - y \leq z, \tag{86}$$

$$x \leq y + z \textbf{ iff } x - z \leq y, \tag{87}$$

$$x + y < z \textbf{ iff } x < z - y, \tag{88}$$

$$x + y < z \textbf{ iff } y < z - x, \tag{89}$$

$$x < z + y \textbf{ iff } x - z < y, \tag{90}$$

$$x < y + z \textbf{ iff } x - z < y, \tag{91}$$

$$\begin{aligned}&(x \leq y \;\&\; z \leq t \textbf{ implies } x - t \leq y - z)\\ &\;\&\; (x < y \;\&\; z \leq t \textbf{ or } x \leq y \;\&\; z < t \textbf{ or } x < y \;\&\; z < t \textbf{ implies } x - t < y - z),\end{aligned} \tag{92}$$

$$0 \leq x \cdot x. \tag{93}$$

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The Fundamental Properties of Natural Numbers

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Summary. Some fundamental properties of addition, multiplication, order relations, exact division, the remainder, divisibility, the least common multiple, the greatest common divisor are presented. A proof of Euclid algorithm is also given.

The article [1] provides the terminology and notation for this paper. For simplicity we adopt the following convention: x will denote an object of the type Real; k, l, m, n will denote objects of the type Nat; X will denote an object of the type **set of** Real. One can prove the following propositions:

(1) $$x \textbf{ is } \mathrm{Nat} \textbf{ implies } x+1 \textbf{ is } \mathrm{Nat},$$

(2) $$\textbf{for } X \textbf{ st } 0 \in X \;\&\; \textbf{for } x \textbf{ st } x \in X \textbf{ holds } x+1 \in X \textbf{ for } k \textbf{ holds } k \in X,$$

(3) $$k+n=n+k,$$

(4) $$k+m+n=k+(m+n),$$

(5) $$k+0=k \;\&\; 0+k=k,$$

(6) $$k \cdot n = n \cdot k,$$

(7) $$k \cdot (m \cdot n) = (k \cdot m) \cdot n,$$

(8) $$k \cdot 1 = k \;\&\; 1 \cdot k = k,$$

(9) $$k \cdot (n+m) = k \cdot n + k \cdot m \;\&\; (n+m) \cdot k = n \cdot k + m \cdot k,$$

(10) $$k+m=n+m \textbf{ or } k+m=m+n \textbf{ or } m+k=m+n \textbf{ implies } k=n,$$

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$$k \cdot 0 = 0 \;\&\; 0 \cdot k = 0. \tag{11}$$

Let us consider n, k. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$n + k \qquad \text{is} \qquad \mathrm{Nat}\,.$$

The scheme *Ind* deals with a unary predicate $\mathcal{P}$ states that the following holds

$$\textbf{for}\, k \textbf{ holds } \mathcal{P}[k]$$

provided the parameter satisfies the following conditions:

- $$\mathcal{P}[0],$$
- $$\textbf{for}\, k \textbf{ st } \mathcal{P}[k] \textbf{ holds } \mathcal{P}[k+1].$$

Let us consider n, k. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$n \cdot k \qquad \text{is} \qquad \mathrm{Nat}\,.$$

One can prove the following propositions:

$$k \leq n \;\&\; n \leq k \textbf{ implies } k = n, \tag{12}$$

$$k \leq n \;\&\; n \leq m \textbf{ implies } k \leq m, \tag{13}$$

$$k \leq n \textbf{ or } n \leq k, \tag{14}$$

$$k \leq k, \tag{15}$$

$$\begin{gathered} k \leq n \textbf{ implies} \\ k + m \leq n + m \;\&\; k + m \leq m + n \;\&\; m + k \leq m + n \;\&\; m + k \leq n + m, \end{gathered} \tag{16}$$

$$\begin{gathered} k + m \leq n + m \textbf{ or } k + m \leq m + n \textbf{ or } m + k \leq m + n \textbf{ or } m + k \leq n + m \\ \textbf{implies } k \leq n, \end{gathered} \tag{17}$$

$$\textbf{for}\, k \textbf{ holds } 0 \leq k, \tag{18}$$

$$0 \neq k \textbf{ implies } 0 < k, \tag{19}$$

$$k \leq n \textbf{ implies } k \cdot m \leq n \cdot m \;\&\; k \cdot m \leq m \cdot n \;\&\; m \cdot k \leq n \cdot m \;\&\; m \cdot k \leq m \cdot n, \tag{20}$$

$$0 \neq k + 1, \tag{21}$$

$$k = 0 \textbf{ or ex}\, n \textbf{ st } k = n + 1, \tag{22}$$

$$k + n = 0 \textbf{ implies } k = 0 \;\&\; n = 0, \tag{23}$$

$$k \neq 0 \;\&\; (n \cdot k = m \cdot k \textbf{ or } n \cdot k = k \cdot m \textbf{ or } k \cdot n = k \cdot m) \textbf{ implies } n = m, \tag{24}$$

(25) $$k \cdot n = 0 \textbf{ implies } k = 0 \textbf{ or } n = 0.$$

The scheme *Def_by_Ind* concerns a constant $\mathcal{A}$ that has the type Nat, a binary functor $\mathcal{F}$ yielding values of the type Nat and a binary predicate $\mathcal{P}$ and states that the following holds

$$(\textbf{for } k \textbf{ ex } n \textbf{ st } \mathcal{P}[k,n]) \ \& \ \textbf{for } k,n,m \textbf{ st } \mathcal{P}[k,n] \ \& \ \mathcal{P}[k,m] \textbf{ holds } n = m$$

provided the parameters satisfy the following condition:

- $$\textbf{for } k,n \textbf{ holds}$$ $$\mathcal{P}[k,n] \textbf{ iff } k = 0 \ \& \ n = \mathcal{A} \textbf{ or ex } m,l \textbf{ st } k = m+1 \ \& \ \mathcal{P}[m,l] \ \& \ n = \mathcal{F}(k,l).$$

Next we state several propositions:

(26) $$\textbf{for } k,n \textbf{ st } k \leq n+1 \textbf{ holds } k \leq n \textbf{ or } k = n+1,$$

(27) $$\textbf{for } n,k \textbf{ st } n \leq k \ \& \ k \leq n+1 \textbf{ holds } n = k \textbf{ or } k = n+1,$$

(28) $$\textbf{for } k,n \textbf{ st } k \leq n \textbf{ ex } m \textbf{ st } n = k+m,$$

(29) $$k \leq k+m,$$

(30) $$k < n \textbf{ iff } k \leq n \ \& \ k \neq n,$$

(31) $$\textbf{not } k < 0.$$

Now we present three schemes. The scheme *Comp_Ind* deals with a unary predicate $\mathcal{P}$ states that the following holds

$$\textbf{for } k \textbf{ holds } \mathcal{P}[k]$$

provided the parameter satisfies the following condition:

- $$\textbf{for } k \textbf{ st for } n \textbf{ st } n < k \textbf{ holds } \mathcal{P}[n] \textbf{ holds } \mathcal{P}[k].$$

The scheme *Min* concerns a unary predicate $\mathcal{P}$ states that the following holds

$$\textbf{ex } k \textbf{ st } \mathcal{P}[k] \ \& \ \textbf{for } n \textbf{ st } \mathcal{P}[n] \textbf{ holds } k \leq n$$

provided the parameter satisfies the following condition:

- $$\textbf{ex } k \textbf{ st } \mathcal{P}[k].$$

The scheme *Max* concerns a unary predicate $\mathcal{P}$ and a constant $\mathcal{A}$ that has the type Nat, and states that the following holds

$$\textbf{ex } k \textbf{ st } \mathcal{P}[k] \ \& \ \textbf{for } n \textbf{ st } \mathcal{P}[n] \textbf{ holds } n \leq k$$

provided the parameters satisfy the following conditions:

- $$\textbf{for } k \textbf{ st } \mathcal{P}[k] \textbf{ holds } k \leq \mathcal{A},$$

- $$\textbf{ex}\, k \;\textbf{st}\; \mathcal{P}[k].$$

We now state a number of propositions:

(32) $$\textbf{not}\,(k < n \;\&\; n < k),$$

(33) $$k < n \;\&\; n < m \;\textbf{implies}\; k < m,$$

(34) $$k < n \;\textbf{or}\; k = n \;\textbf{or}\; n < k,$$

(35) $$\textbf{not}\, k < k,$$

(36) $$k < n \;\textbf{implies}$$
$$k + m < n + m \;\&\; k + m < m + n \;\&\; m + k < m + n \;\&\; m + k < n + m,$$

(37) $$k \leq n \;\textbf{implies}\; k \leq n + m,$$

(38) $$k < n + 1 \;\textbf{iff}\; k \leq n,$$

(39) $$k \leq n \;\&\; n < m \;\textbf{or}\; k < n \;\&\; n \leq m \;\textbf{or}\; k < n \;\&\; n < m \;\textbf{implies}\; k < m,$$

(40) $$k \cdot n = 1 \;\textbf{implies}\; k = 1 \;\&\; n = 1,$$

(41) $$k + 1 \leq n \;\textbf{iff}\; k < n.$$

The scheme *Regr* concerns a unary predicate $\mathcal{P}$ states that the following holds

$$\mathcal{P}[0]$$

provided the parameter satisfies the following conditions:

- $$\textbf{ex}\, k \;\textbf{st}\; \mathcal{P}[k],$$
- $$\textbf{for}\, k \;\textbf{st}\; k \neq 0 \;\&\; \mathcal{P}[k] \;\textbf{ex}\, n \;\textbf{st}\; n < k \;\&\; \mathcal{P}[n].$$

In the sequel $k1$, t, $t1$ will denote objects of the type Nat. The following propositions are true:

(42) $$\textbf{for}\, m \;\textbf{st}\; 0 < m \;\textbf{for}\, n \;\textbf{ex}\, k{,}t \;\textbf{st}\; n = (m \cdot k) + t \;\&\; t < m,$$

(43) $$\textbf{for}\, n{,}m{,}k{,}k1{,}t{,}t1$$
$$\textbf{st}\; n = m \cdot k + t \;\&\; t < m \;\&\; n = m \cdot k1 + t1 \;\&\; t1 < m \;\textbf{holds}\; k = k1 \;\&\; t = t1.$$

We now define two new functors. Let k, l have the type Nat. The functor

$$k \div l,$$

yields the type Nat and is defined by

$$(\textbf{ex}\, t \;\textbf{st}\; k = l \cdot \textbf{it} + t \;\&\; t < l) \;\textbf{or}\; \textbf{it} = 0 \;\&\; l = 0.$$

The functor

$$k \bmod l,$$

yields the type Nat and is defined by

$$(\mathbf{ex}\, t\ \mathbf{st}\ k = l \cdot t + \mathbf{it}\ \&\ \mathbf{it} < l)\ \mathbf{or}\ \mathbf{it} = 0\ \&\ l = 0.$$

Next we state four propositions:

(44) $$\mathbf{for}\, k,l,n\ \mathbf{being}\ \mathrm{Nat}\ \mathbf{holds}\ n = k \div l\ \mathbf{iff}\ (\mathbf{ex}\, t\ \mathbf{st}\ k = l \cdot n + t\ \&\ t < l)\ \mathbf{or}\ n = 0\ \&\ l = 0,$$

(45) $$\mathbf{for}\, k,l,n\ \mathbf{being}\ \mathrm{Nat}\ \mathbf{holds}\ n = k \bmod l\ \mathbf{iff}\ (\mathbf{ex}\, t\ \mathbf{st}\ k = l \cdot t + n\ \&\ n < l)\ \mathbf{or}\ n = 0\ \&\ l = 0,$$

(46) $$\mathbf{for}\, m,n\ \mathbf{st}\ 0 < m\ \mathbf{holds}\ n \bmod m < m,$$

(47) $$\mathbf{for}\, n,m\ \mathbf{st}\ 0 < m\ \mathbf{holds}\ n = m \cdot (n \div m) + (n \bmod m).$$

Let k, l have the type Nat. The predicate

$$k \mid l \qquad \text{is defined by} \qquad \mathbf{ex}\, t\ \mathbf{st}\ l = k \cdot t.$$

Next we state a number of propositions:

(48) $$\mathbf{for}\, k,l\ \mathbf{being}\ \mathrm{Nat}\ \mathbf{holds}\ k \mid l\ \mathbf{iff}\ \mathbf{ex}\, t\ \mathbf{st}\ l = k \cdot t,$$

(49) $$\mathbf{for}\, n,m\ \mathbf{holds}\ m \mid n\ \mathbf{iff}\ n = m \cdot (n \div m),$$

(50) $$\mathbf{for}\, n\ \mathbf{holds}\ n \mid n,$$

(51) $$\mathbf{for}\, n,m,l\ \mathbf{st}\ n \mid m\ \&\ m \mid l\ \mathbf{holds}\ n \mid l,$$

(52) $$\mathbf{for}\, n,m\ \mathbf{st}\ n \mid m\ \&\ m \mid n\ \mathbf{holds}\ n = m,$$

(53) $$k \mid 0\ \&\ 1 \mid k,$$

(54) $$\mathbf{for}\, n,m\ \mathbf{st}\ 0 < m\ \&\ n \mid m\ \mathbf{holds}\ n \leq m,$$

(55) $$\mathbf{for}\, n,m,l\ \mathbf{st}\ n \mid m\ \&\ n \mid l\ \mathbf{holds}\ n \mid m + l,$$

(56) $$n \mid k\ \mathbf{implies}\ n \mid k \cdot m,$$

(57) $$\mathbf{for}\, n,m,l\ \mathbf{st}\ n \mid m\ \&\ n \mid m + l\ \mathbf{holds}\ n \mid l,$$

(58) $$n \mid m\ \&\ n \mid k\ \mathbf{implies}\ n \mid m \bmod k.$$

Let us consider k, n. The functor

$$k\ \mathrm{lcm}\ n,$$

with values of the type Nat, is defined by

$$k \mid \textbf{it} \;\&\; n \mid \textbf{it} \;\&\; \textbf{for}\, m \;\textbf{st}\; k \mid m \;\&\; n \mid m \;\textbf{holds it} \mid m.$$

Next we state a proposition

(59) $$\textbf{for}\, M \;\textbf{being}\; \mathrm{Nat}$$
$$\textbf{holds}\; M = k \operatorname{lcm} n \;\textbf{iff}\; k \mid M \;\&\; n \mid M \;\&\; \textbf{for}\, m \;\textbf{st}\; k \mid m \;\&\; n \mid m \;\textbf{holds}\; M \mid m.$$

Let us consider k, n. The functor

$$k \gcd n,$$

yields the type Nat and is defined by

$$\textbf{it} \mid k \;\&\; \textbf{it} \mid n \;\&\; \textbf{for}\, m \;\textbf{st}\; m \mid k \;\&\; m \mid n \;\textbf{holds}\; m \mid \textbf{it}\,.$$

We now state a proposition

(60) $$\textbf{for}\, M \;\textbf{being}\; \mathrm{Nat}$$
$$\textbf{holds}\; M = k \gcd n \;\textbf{iff}\; M \mid k \;\&\; M \mid n \;\&\; \textbf{for}\, m \;\textbf{st}\; m \mid k \;\&\; m \mid n \;\textbf{holds}\; m \mid M.$$

The scheme *Euklides* deals with a unary functor $\mathcal{F}$ yielding values of the type Nat, a constant $\mathcal{A}$ that has the type Nat and a constant $\mathcal{B}$ that has the type Nat, and states that the following holds

$$\textbf{ex}\, n \;\textbf{st}\; \mathcal{F}(n) = \mathcal{A} \gcd \mathcal{B} \;\&\; \mathcal{F}(n+1) = 0$$

provided the parameters satisfy the following conditions:

- $$0 < \mathcal{B} \;\&\; \mathcal{B} < \mathcal{A},$$
- $$\mathcal{F}(0) = \mathcal{A} \;\&\; \mathcal{F}(1) = \mathcal{B},$$
- $$\textbf{for}\, n \;\textbf{holds}\; \mathcal{F}(n+2) = \mathcal{F}(n) \bmod \mathcal{F}(n+1).$$

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Some Basic Properties of Sets

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Summary. In this article some basic theorems about singletons, pairs, power sets, unions of families of sets, and the cartesian product of two sets are proved.

The articles [1] and [2] provide the terminology and notation for this paper. One can prove the following propositions:

$$\text{bool}\,\emptyset = \{\emptyset\}, \tag{1}$$

$$\bigcup \emptyset = \emptyset. \tag{2}$$

For simplicity we adopt the following convention: x, $x1$, $x2$, y, $y1$, $y2$, z will denote objects of the type Any; A, B, X, $X1$, $X2$, Y, $Y1$, $Y2$, Z will denote objects of the type set. One can prove the following propositions:

$$\{x\} \neq \emptyset, \tag{3}$$

$$\{x, y\} \neq \emptyset, \tag{4}$$

$$\{x\} = \{x, x\}, \tag{5}$$

$$\{x\} = \{y\} \textbf{ implies } x = y, \tag{6}$$

$$\{x1{,}x2\} = \{x2{,}x1\}, \tag{7}$$

$$\{x\} = \{y1{,}y2\} \textbf{ implies } x = y1 \ \&\ x = y2, \tag{8}$$

$$\{x\} = \{y1{,}y2\} \textbf{ implies } y1 = y2, \tag{9}$$

$$\{x1{,}x2\} = \{y1{,}y2\} \textbf{ implies } (x1 = y1 \textbf{ or } x1 = y2) \ \&\ (x2 = y1 \textbf{ or } x2 = y2), \tag{10}$$

$$\{x1{,}x2\} = \{x1\} \cup \{x2\}, \tag{11}$$

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$$\{x\} \subseteq \{x,y\} \;\&\; \{y\} \subseteq \{x,y\}, \tag{12}$$

$$\{x\} \cup \{y\} = \{x\} \textbf{ or } \{x\} \cup \{y\} = \{y\} \textbf{ implies } x = y, \tag{13}$$

$$\{x\} \cup \{x,y\} = \{x,y\} \;\&\; \{x,y\} \cup \{x\} = \{x,y\}, \tag{14}$$

$$\{y\} \cup \{x,y\} = \{x,y\} \;\&\; \{x,y\} \cup \{y\} = \{x,y\}, \tag{15}$$

$$\{x\} \cap \{y\} = \emptyset \textbf{ or } \{y\} \cap \{x\} = \emptyset \textbf{ implies } x \neq y, \tag{16}$$

$$x \neq y \textbf{ implies } \{x\} \cap \{y\} = \emptyset \;\&\; \{y\} \cap \{x\} = \emptyset, \tag{17}$$

$$\{x\} \cap \{y\} = \{x\} \textbf{ or } \{x\} \cap \{y\} = \{y\} \textbf{ implies } x = y, \tag{18}$$

$$\begin{gathered}\{x\} \cap \{x,y\} = \{x\} \\ \;\&\; \{y\} \cap \{x,y\} = \{y\} \;\&\; \{x,y\} \cap \{x\} = \{x\} \;\&\; \{x,y\} \cap \{y\} = \{y\},\end{gathered} \tag{19}$$

$$\{x\} \setminus \{y\} = \{x\} \textbf{ iff } x \neq y, \tag{20}$$

$$\{x\} \setminus \{y\} = \emptyset \textbf{ implies } x = y, \tag{21}$$

$$\{x\} \setminus \{x,y\} = \emptyset \;\&\; \{y\} \setminus \{x,y\} = \emptyset, \tag{22}$$

$$x \neq y \textbf{ implies } \{x,y\} \setminus \{y\} = \{x\} \;\&\; \{x,y\} \setminus \{x\} = \{y\}, \tag{23}$$

$$\{x\} \subseteq \{y\} \textbf{ implies } \{x\} = \{y\}, \tag{24}$$

$$\{z\} \subseteq \{x,y\} \textbf{ implies } z = x \textbf{ or } z = y, \tag{25}$$

$$\{x,y\} \subseteq \{z\} \textbf{ implies } x = z \;\&\; y = z, \tag{26}$$

$$\{x,y\} \subseteq \{z\} \textbf{ implies } \{x,y\} = \{z\}, \tag{27}$$

$$\{x1,x2\} \subseteq \{y1,y2\} \textbf{ implies } (x1 = y1 \textbf{ or } x1 = y2) \;\&\; (x2 = y1 \textbf{ or } x2 = y2), \tag{28}$$

$$x \neq y \textbf{ implies } \{x\} \dot{-} \{y\} = \{x,y\}, \tag{29}$$

$$\mathrm{bool}\{x\} = \{\emptyset,\{x\}\}, \tag{30}$$

$$\bigcup\{x\} = x, \tag{31}$$

$$\bigcup\{\{x\},\{y\}\} = \{x,y\}, \tag{32}$$

$$\langle x1,x2\rangle = \langle y1,y2\rangle \textbf{ implies } x1 = y1 \;\&\; x2 = y2, \tag{33}$$

$$\langle x,y\rangle \in [:\{x1\},\{y1\}:] \textbf{ iff } x = x1 \;\&\; y = y1, \tag{34}$$

$$[:\{x\},\{y\}:] = \{\langle x,y\rangle\}, \tag{35}$$

(36) $$[\!:\{x\},\{y,z\}:\!] = \{\langle x,y\rangle,\langle x,z\rangle\} \;\&\; [\!:\{x,y\},\{z\}:\!] = \{\langle x,z\rangle,\langle y,z\rangle\},$$

(37) $$\{x\} \subseteq X \textbf{ iff } x \in X,$$

(38) $$\{x1,x2\} \subseteq Z \textbf{ iff } x1 \in Z \;\&\; x2 \in Z,$$

(39) $$Y \subseteq \{x\} \textbf{ iff } Y = \emptyset \textbf{ or } Y = \{x\},$$

(40) $$Y \subseteq X \;\&\; \textbf{not}\, x \in Y \textbf{ implies } Y \subseteq X \setminus \{x\},$$

(41) $$X \neq \{x\} \;\&\; x \in X \textbf{ implies ex}\, y \textbf{ st } y \in X \;\&\; y \neq x,$$

(42) $$Z \subseteq \{x1,x2\} \textbf{ iff } Z = \emptyset \textbf{ or } Z = \{x1\} \textbf{ or } Z = \{x2\} \textbf{ or } Z = \{x1,x2\},$$

(43) $$\{z\} = X \cup Y$$
$$\textbf{implies } X = \{z\} \;\&\; Y = \{z\} \textbf{ or } X = \emptyset \;\&\; Y = \{z\} \textbf{ or } X = \{z\} \;\&\; Y = \emptyset,$$

(44) $$\{z\} = X \cup Y \;\&\; X \neq Y \textbf{ implies } X = \emptyset \textbf{ or } Y = \emptyset,$$

(45) $$\{x\} \cup X = X \textbf{ or } X \cup \{x\} = X \textbf{ implies } x \in X,$$

(46) $$x \in X \textbf{ implies } \{x\} \cup X = X \;\&\; X \cup \{x\} = X,$$

(47) $$\{x,y\} \cup Z = Z \textbf{ or } Z \cup \{x,y\} = Z \textbf{ implies } x \in Z \;\&\; y \in Z,$$

(48) $$x \in Z \;\&\; y \in Z \textbf{ implies } \{x,y\} \cup Z = Z \;\&\; Z \cup \{x,y\} = Z,$$

(49) $$\{x\} \cup X \neq \emptyset \;\&\; X \cup \{x\} \neq \emptyset,$$

(50) $$\{x,y\} \cup X \neq \emptyset \;\&\; X \cup \{x,y\} \neq \emptyset,$$

(51) $$X \cap \{x\} = \{x\} \textbf{ or } \{x\} \cap X = \{x\} \textbf{ implies } x \in X,$$

(52) $$x \in X \textbf{ implies } X \cap \{x\} = \{x\} \;\&\; \{x\} \cap X = \{x\},$$

(53) $$x \in Z \;\&\; y \in Z \textbf{ implies } \{x,y\} \cap Z = \{x,y\} \;\&\; \{x,y\} = Z \cap \{x,y\},$$

(54) $$\{x\} \cap X = \emptyset \textbf{ or } X \cap \{x\} = \emptyset \textbf{ implies not}\, x \in X,$$

(55) $$\{x,y\} \cap Z = \emptyset \textbf{ or } Z \cap \{x,y\} = \emptyset \textbf{ implies not}\, x \in Z \;\&\; \textbf{not}\, y \in Z,$$

(56) $$\textbf{not}\, x \in X \textbf{ implies } \{x\} \cap X = \emptyset \;\&\; X \cap \{x\} = \emptyset,$$

(57) $$\textbf{not}\, x \in Z \;\&\; \textbf{not}\, y \in Z \textbf{ implies } \{x,y\} \cap Z = \emptyset \;\&\; Z \cap \{x,y\} = \emptyset,$$

(58) $$\{x\} \cap X = \emptyset \textbf{ or } \{x\} \cap X = \{x\} \;\&\; X \cap \{x\} = \{x\},$$

(59) $$\{x,y\} \cap X = \{x\} \textbf{ or } X \cap \{x,y\} = \{x\} \textbf{ implies not}\, y \in X \textbf{ or } x = y,$$

$$x \in X \mathbin{\&} (\textbf{not}\, y \in X \textbf{ or } x = y) \textbf{ implies } \{x,y\} \cap X = \{x\} \mathbin{\&} X \cap \{x,y\} = \{x\}, \tag{60}$$

$$\{x,y\} \cap X = \{y\} \textbf{ or } X \cap \{x,y\} = \{y\} \textbf{ implies not}\, x \in X \textbf{ or } x = y, \tag{61}$$

$$y \in X \mathbin{\&} (\textbf{not}\, x \in X \textbf{ or } x = y) \textbf{ implies } \{x,y\} \cap X = \{y\} \mathbin{\&} X \cap \{x,y\} = \{y\}, \tag{62}$$

$$\{x,y\} \cap X = \{x,y\} \textbf{ or } X \cap \{x,y\} = \{x,y\} \textbf{ implies } x \in X \mathbin{\&} y \in X, \tag{63}$$

$$z \in X \setminus \{x\} \textbf{ iff } z \in X \mathbin{\&} z \neq x, \tag{64}$$

$$X \setminus \{x\} = X \textbf{ iff not}\, x \in X, \tag{65}$$

$$X \setminus \{x\} = \emptyset \textbf{ implies } X = \emptyset \textbf{ or } X = \{x\}, \tag{66}$$

$$\{x\} \setminus X = \{x\} \textbf{ iff not}\, x \in X, \tag{67}$$

$$\{x\} \setminus X = \emptyset \textbf{ iff } x \in X, \tag{68}$$

$$\{x\} \setminus X = \emptyset \textbf{ or } \{x\} \setminus X = \{x\}, \tag{69}$$

$$\{x,y\} \setminus X = \{x\} \textbf{ iff not}\, x \in X \mathbin{\&} (y \in X \textbf{ or } x = y), \tag{70}$$

$$\{x,y\} \setminus X = \{y\} \textbf{ iff } (x \in X \textbf{ or } x = y) \mathbin{\&} \textbf{not}\, y \in X, \tag{71}$$

$$\{x,y\} \setminus X = \{x,y\} \textbf{ iff not}\, x \in X \mathbin{\&} \textbf{not}\, y \in X, \tag{72}$$

$$\{x,y\} \setminus X = \emptyset \textbf{ iff } x \in X \mathbin{\&} y \in X, \tag{73}$$

$$\begin{gathered}\{x,y\} \setminus X = \emptyset \\ \textbf{or } \{x,y\} \setminus X = \{x\} \textbf{ or } \{x,y\} \setminus X = \{y\} \textbf{ or } \{x,y\} \setminus X = \{x,y\},\end{gathered} \tag{74}$$

$$X \setminus \{x,y\} = \emptyset \textbf{ iff } X = \emptyset \textbf{ or } X = \{x\} \textbf{ or } X = \{y\} \textbf{ or } X = \{x,y\}, \tag{75}$$

$$\emptyset \in \operatorname{bool} A, \tag{76}$$

$$A \in \operatorname{bool} A, \tag{77}$$

$$\operatorname{bool} A \neq \emptyset, \tag{78}$$

$$A \subseteq B \textbf{ implies } \operatorname{bool} A \subseteq \operatorname{bool} B, \tag{79}$$

$$\{A\} \subseteq \operatorname{bool} A, \tag{80}$$

$$\operatorname{bool} A \cup \operatorname{bool} B \subseteq \operatorname{bool}(A \cup B), \tag{81}$$

$$\operatorname{bool} A \cup \operatorname{bool} B = \operatorname{bool}(A \cup B) \textbf{ implies } A \subseteq B \textbf{ or } B \subseteq A, \tag{82}$$

$$\operatorname{bool}(A \cap B) = \operatorname{bool} A \cap \operatorname{bool} B, \tag{83}$$

$$\text{bool}(A \setminus B) \subseteq \{\emptyset\} \cup (\text{bool}\, A \setminus \text{bool}\, B), \tag{84}$$

$$X \in \text{bool}(A \setminus B) \textbf{ iff } X \subseteq A \;\&\; X \text{ misses } B, \tag{85}$$

$$\text{bool}(A \setminus B) \cup \text{bool}(B \setminus A) \subseteq \text{bool}(A \mathbin{\dot{-}} B), \tag{86}$$

$$X \in \text{bool}(A \mathbin{\dot{-}} B) \textbf{ iff } X \subseteq A \cup B \;\&\; X \text{ misses } A \cap B, \tag{87}$$

$$X \in \text{bool}\, A \;\&\; Y \in \text{bool}\, A \textbf{ implies } X \cup Y \in \text{bool}\, A, \tag{88}$$

$$X \in \text{bool}\, A \textbf{ or } Y \in \text{bool}\, A \textbf{ implies } X \cap Y \in \text{bool}\, A, \tag{89}$$

$$X \in \text{bool}\, A \textbf{ implies } X \setminus Y \in \text{bool}\, A, \tag{90}$$

$$X \in \text{bool}\, A \;\&\; Y \in \text{bool}\, A \textbf{ implies } X \mathbin{\dot{-}} Y \in \text{bool}\, A, \tag{91}$$

$$X \in A \textbf{ implies } X \subseteq \bigcup A, \tag{92}$$

$$\bigcup\{X, Y\} = X \cup Y, \tag{93}$$

$$(\textbf{for } X \textbf{ st } X \in A \textbf{ holds } X \subseteq Z) \textbf{ implies } \bigcup A \subseteq Z, \tag{94}$$

$$A \subseteq B \textbf{ implies } \bigcup A \subseteq \bigcup B, \tag{95}$$

$$\bigcup(A \cup B) = \bigcup A \cup \bigcup B, \tag{96}$$

$$\bigcup(A \cap B) \subseteq \bigcup A \cap \bigcup B, \tag{97}$$

$$(\textbf{for } X \textbf{ st } X \in A \textbf{ holds } X \cap B = \emptyset) \textbf{ implies } \bigcup(A) \cap B = \emptyset, \tag{98}$$

$$\bigcup \text{bool}\, A = A, \tag{99}$$

$$A \subseteq \text{bool} \bigcup A, \tag{100}$$

$$\begin{gathered}(\textbf{for } X{,}Y \textbf{ st } X \neq Y \;\&\; X \in A \cup B \;\&\; Y \in A \cup B \textbf{ holds } X \cap Y = \emptyset)\\ \textbf{implies } \bigcup(A \cap B) = \bigcup A \cap \bigcup B,\end{gathered} \tag{101}$$

$$z \in [:X, Y:] \textbf{ implies ex } x{,}y \textbf{ st } \langle x, y\rangle = z, \tag{102}$$

$$A \subseteq [:X, Y:] \;\&\; z \in A \textbf{ implies ex } x{,}y \textbf{ st } x \in X \;\&\; y \in Y \;\&\; z = \langle x, y\rangle, \tag{103}$$

$$\begin{gathered}z \in [:X1{,}Y1:] \cap [:X2{,}Y2:]\\ \textbf{implies ex } x{,}y \textbf{ st } z = \langle x, y\rangle \;\&\; x \in X1 \cap X2 \;\&\; y \in Y1 \cap Y2,\end{gathered} \tag{104}$$

$$[:X, Y:] \subseteq \text{bool}\,\text{bool}(X \cup Y), \tag{105}$$

$$\langle x, y\rangle \in [:X, Y:] \textbf{ iff } x \in X \;\&\; y \in Y, \tag{106}$$

(107) $$\langle x, y\rangle \in [:X, Y:] \textbf{ implies } \langle y, x\rangle \in [:Y, X:],$$

(108) $$(\textbf{for } x,y \textbf{ holds } \langle x, y\rangle \in [:X1,Y1:] \textbf{ iff } \langle x, y\rangle \in [:X2,Y2:]) \\ \textbf{implies } [:X1,Y1:] = [:X2,Y2:],$$

(109) $$A \subseteq [:X, Y:] \;\&\; (\textbf{for } x,y \textbf{ st } \langle x, y\rangle \in A \textbf{ holds } \langle x, y\rangle \in B) \textbf{ implies } A \subseteq B,$$

(110) $$A \subseteq [:X1,Y1:] \;\&\; B \subseteq [:X2,Y2:] \;\&\; (\textbf{for } x,y \textbf{ holds } \langle x, y\rangle \in A \textbf{ iff } \langle x, y\rangle \in B) \\ \textbf{implies } A = B,$$

(111) $$(\textbf{for } z \textbf{ st } z \in A \textbf{ ex } x,y \textbf{ st } z = \langle x, y\rangle) \;\&\; (\textbf{for } x,y \textbf{ st } \langle x, y\rangle \in A \textbf{ holds } \langle x, y\rangle \in B) \\ \textbf{implies } A \subseteq B,$$

(112) $$(\textbf{for } z \textbf{ st } z \in A \textbf{ ex } x,y \textbf{ st } z = \langle x, y\rangle) \;\& \\ (\textbf{for } z \textbf{ st } z \in B \textbf{ ex } x,y \textbf{ st } z = \langle x, y\rangle) \;\&\; (\textbf{for } x,y \textbf{ holds } \langle x, y\rangle \in A \textbf{ iff } \langle x, y\rangle \in B) \\ \textbf{implies } A = B,$$

(113) $$[:X, Y:] = \emptyset \textbf{ iff } X = \emptyset \textbf{ or } Y = \emptyset,$$

(114) $$X \neq \emptyset \;\&\; Y \neq \emptyset \;\&\; [:X, Y:] = [:Y, X:] \textbf{ implies } X = Y,$$

(115) $$[:X, X:] = [:Y, Y:] \textbf{ implies } X = Y,$$

(116) $$X \subseteq [:X, X:] \textbf{ implies } X = \emptyset,$$

(117) $$Z \neq \emptyset \;\&\; ([:X, Z:] \subseteq [:Y, Z:] \textbf{ or } [:Z, X:] \subseteq [:Z, Y:]) \textbf{ implies } X \subseteq Y,$$

(118) $$X \subseteq Y \textbf{ implies } [:X, Z:] \subseteq [:Y, Z:] \;\&\; [:Z, X:] \subseteq [:Z, Y:],$$

(119) $$X1 \subseteq Y1 \;\&\; X2 \subseteq Y2 \textbf{ implies } [:X1,X2:] \subseteq [:Y1,Y2:],$$

(120) $$[:X \cup Y,Z:] = [:X, Z:] \cup [:Y, Z:] \;\&\; [:Z, X \cup Y:] = [:Z, X:] \cup [:Z, Y:],$$

(121) $$[:X1 \cup X2,Y1 \cup Y2:] = [:X1,Y1:] \cup [:X1,Y2:] \cup [:X2,Y1:] \cup [:X2,Y2:],$$

(122) $$[:X \cap Y,Z:] = [:X, Z:] \cap [:Y, Z:] \;\&\; [:Z, X \cap Y:] = [:Z, X:] \cap [:Z, Y:],$$

(123) $$[:X1 \cap X2,Y1 \cap Y2:] = [:X1,Y1:] \cap [:X2,Y2:],$$

(124) $$A \subseteq X \;\&\; B \subseteq Y \textbf{ implies } [:A, Y:] \cap [:X, B:] = [:A, B:],$$

(125) $$[:X \setminus Y,Z:] = [:X, Z:] \setminus [:Y, Z:] \;\&\; [:Z, X \setminus Y:] = [:Z, X:] \setminus [:Z, Y:],$$

(126) $$[:X1,X2:] \setminus [:Y1,Y2:] = [:X1 \setminus Y1,X2:] \cup [:X1,X2 \setminus Y2:],$$

(127) $$X1 \cap X2 = \emptyset \textbf{ or } Y1 \cap Y2 = \emptyset \textbf{ implies } [:X1,Y1:] \cap [:X2,Y2:] = \emptyset,$$

(128) $$\langle x, y\rangle \in [:\{z\}, Y:] \textbf{ iff } x = z \ \&\ y \in Y,$$

(129) $$\langle x, y\rangle \in [:X, \{z\}:] \textbf{ iff } x \in X \ \&\ y = z,$$

(130) $$X \neq \emptyset \textbf{ implies } [:\{x\}, X:] \neq \emptyset \ \&\ [:X, \{x\}:] \neq \emptyset,$$

(131) $$x \neq y \textbf{ implies } [:\{x\}, X:] \cap [:\{y\}, Y:] = \emptyset \ \&\ [:X, \{x\}:] \cap [:Y, \{y\}:] = \emptyset,$$

(132) $$[:\{x, y\}, X:] = [:\{x\}, X:] \cup [:\{y\}, X:] \ \&\ [:X, \{x, y\}:] = [:X, \{x\}:] \cup [:X, \{y\}:],$$

(133) $$Z = [:X, Y:] \textbf{ iff for } z \textbf{ holds } z \in Z \textbf{ iff ex } x, y \textbf{ st } x \in X \ \&\ y \in Y \ \&\ z = \langle x, y\rangle,$$

(134) $$X1 \neq \emptyset \ \&\ Y1 \neq \emptyset \ \&\ [:X1, Y1:] = [:X2, Y2:] \textbf{ implies } X1 = X2 \ \&\ Y1 = Y2,$$

(135) $$X \subseteq [:X, Y:] \textbf{ or } X \subseteq [:Y, X:] \textbf{ implies } X = \emptyset.$$

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Functions and Their Basic Properties

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Summary. The definitions of the mode Function and the graph of a function are introduced. The graph of a function is defined to be identical with the function. The following concepts are also defined: the domain of a function, the range of a function, the identity function, the composition of functions, the 1-1 function, the inverse function, the restriction of a function, the image and the inverse image. Certain basic facts about functions and the notions defined in the article are proved.

The notation and terminology used here are introduced in the papers [1] and [2]. For simplicity we adopt the following convention: X, $X1$, $X2$, Y, $Y1$, $Y2$ have the type set; p, x, $x1$, $x2$, y, $y1$, $y2$, z have the type Any. The mode

$$\mathrm{Function}\,,$$

which widens to the type Any, is defined by

$$\begin{gathered}\mathbf{ex}\,F\ \mathbf{being}\ \mathrm{set}\ \mathbf{st}\ \mathbf{it} = F\ \&\ (\mathbf{for}\,p\ \mathbf{st}\ p \in F\ \mathbf{ex}\,x{,}y\ \mathbf{st}\ \langle x, y\rangle = p)\\ \&\ \mathbf{for}\,x{,}y1{,}y2\ \mathbf{st}\ \langle x, y1\rangle \in F\ \&\ \langle x, y2\rangle \in F\ \mathbf{holds}\ y1 = y2.\end{gathered}$$

In the sequel f, g, h will have the type Function. Let us consider f. The functor

$$\mathrm{graph}\,f,$$

yields the type set and is defined by

$$f = \mathbf{it}\,.$$

Next we state several propositions:

(1) $$\mathrm{graph}\,f = f,$$

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(2) $$\textbf{for } F \textbf{ being } \text{set } \textbf{st} \ (\textbf{for } p \textbf{ st } p \in F \textbf{ ex } x,y \textbf{ st } \langle x, y\rangle = p) \ \& \ \textbf{for } x,y1,y2 \textbf{ st } \langle x, y1\rangle \in F \ \& \ \langle x, y2\rangle \in F \textbf{ holds } y1 = y2 \ \textbf{ex } f \textbf{ being } \text{Function } \textbf{st } \text{graph} f = F,$$

(3) $$p \in \text{graph} f \textbf{ implies ex } x,y \textbf{ st } \langle x, y\rangle = p,$$

(4) $$\langle x, y1\rangle \in \text{graph} f \ \& \ \langle x, y2\rangle \in \text{graph} f \textbf{ implies } y1 = y2,$$

(5) $$\text{graph} f = \text{graph} g \textbf{ implies } f = g.$$

The scheme *GraphFunc* concerns a constant $\mathcal{A}$ that has the type set and a binary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex } f \textbf{ st for } x,y \textbf{ holds } \langle x, y\rangle \in \text{graph} f \textbf{ iff } x \in \mathcal{A} \ \& \ \mathcal{P}[x, y]$$

provided the parameters satisfy the following condition:

- $$\textbf{for } x,y1,y2 \textbf{ st } \mathcal{P}[x, y1] \ \& \ \mathcal{P}[x, y2] \textbf{ holds } y1 = y2.$$

Let us consider f. The functor

$$\text{dom} f,$$

yields the type set and is defined by

$$\textbf{for } x \textbf{ holds } x \in \textbf{it iff ex } y \textbf{ st } \langle x, y\rangle \in \text{graph} f.$$

One can prove the following proposition

(6) $$X = \text{dom} f \textbf{ iff for } x \textbf{ holds } x \in X \textbf{ iff ex } y \textbf{ st } \langle x, y\rangle \in \text{graph} f.$$

Let us consider f, x. Assume that the following holds

$$x \in \text{dom} f.$$

The functor

$$f.x,$$

yields the type Any and is defined by

$$\langle x, \textbf{it}\rangle \in \text{graph} f.$$

The following three propositions are true:

(7) $$x \in \text{dom} f \textbf{ implies } (y = f.x \textbf{ iff } \langle x, y\rangle \in \text{graph} f),$$

(8) $$\langle x, y\rangle \in \text{graph} f \textbf{ iff } x \in \text{dom} f \ \& \ y = f.x,$$

(9) $X = \text{dom} f \ \& \ X = \text{dom} g \ \& \ (\textbf{for } x \textbf{ st } x \in X \textbf{ holds } f.x = g.x) \textbf{ implies } f = g.$

Let us consider f. The functor

$$\operatorname{rng} f,$$

with values of the type set, is defined by

$$\textbf{for}\, y \,\textbf{holds}\, y \in \textbf{it} \,\textbf{iff}\, \textbf{ex}\, x \,\textbf{st}\, x \in \operatorname{dom} f \mathbin{\&} y = f\,.x.$$

One can prove the following propositions:

(10) $$Y = \operatorname{rng} f \,\textbf{iff}\, \textbf{for}\, y \,\textbf{holds}\, y \in Y \,\textbf{iff}\, \textbf{ex}\, x \,\textbf{st}\, x \in \operatorname{dom} f \mathbin{\&} y = f\,.x,$$

(11) $$y \in \operatorname{rng} f \,\textbf{iff}\, \textbf{ex}\, x \,\textbf{st}\, x \in \operatorname{dom} f \mathbin{\&} y = f\,.x,$$

(12) $$x \in \operatorname{dom} f \,\textbf{implies}\, f\,.x \in \operatorname{rng} f,$$

(13) $$\operatorname{dom} f = \emptyset \,\textbf{iff}\, \operatorname{rng} f = \emptyset,$$

(14) $$\operatorname{dom} f = \{x\} \,\textbf{implies}\, \operatorname{rng} f = \{f\,.x\}.$$

Now we present two schemes. The scheme *FuncEx* concerns a constant $\mathcal{A}$ that has the type set and a binary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex}\, f \,\textbf{st}\, \operatorname{dom} f = \mathcal{A} \mathbin{\&} \textbf{for}\, x \,\textbf{st}\, x \in \mathcal{A} \,\textbf{holds}\, \mathcal{P}[x, f\,.x]$$

provided the parameters satisfy the following conditions:

- $$\textbf{for}\, x,y1,y2 \,\textbf{st}\, x \in \mathcal{A} \mathbin{\&} \mathcal{P}[x, y1] \mathbin{\&} \mathcal{P}[x, y2] \,\textbf{holds}\, y1 = y2,$$
- $$\textbf{for}\, x \,\textbf{st}\, x \in \mathcal{A} \,\textbf{ex}\, y \,\textbf{st}\, \mathcal{P}[x, y].$$

The scheme *Lambda* concerns a constant $\mathcal{A}$ that has the type set and a unary functor $\mathcal{F}$ and states that the following holds

$$\textbf{ex}\, f \,\textbf{being}\, \text{Function} \,\textbf{st}\, \operatorname{dom} f = \mathcal{A} \mathbin{\&} \textbf{for}\, x \,\textbf{st}\, x \in \mathcal{A} \,\textbf{holds}\, f\,.x = \mathcal{F}(x)$$

for all values of the parameters.

Next we state several propositions:

(15) $$X \neq \emptyset \,\textbf{implies}\, \textbf{for}\, y \,\textbf{ex}\, f \,\textbf{st}\, \operatorname{dom} f = X \mathbin{\&} \operatorname{rng} f = \{y\},$$

(16) $$(\textbf{for}\, f,g \,\textbf{st}\, \operatorname{dom} f = X \mathbin{\&} \operatorname{dom} g = X \,\textbf{holds}\, f = g) \,\textbf{implies}\, X = \emptyset,$$

(17) $$\operatorname{dom} f = \operatorname{dom} g \mathbin{\&} \operatorname{rng} f = \{y\} \mathbin{\&} \operatorname{rng} g = \{y\} \,\textbf{implies}\, f = g,$$

(18) $$Y \neq \emptyset \,\textbf{or}\, X = \emptyset \,\textbf{implies}\, \textbf{ex}\, f \,\textbf{st}\, X = \operatorname{dom} f \mathbin{\&} \operatorname{rng} f \subseteq Y,$$

(19) $$(\textbf{for}\, y \,\textbf{st}\, y \in Y \,\textbf{ex}\, x \,\textbf{st}\, x \in \operatorname{dom} f \mathbin{\&} y = f\,.x) \,\textbf{implies}\, Y \subseteq \operatorname{rng} f.$$

Let us consider f, g. The functor

$$g \cdot f,$$

yields the type Function and is defined by

$$(\textbf{for}\ x\ \textbf{holds}\ x \in \mathrm{dom}\,\textbf{it}\ \textbf{iff}\ x \in \mathrm{dom}\, f\ \&\ f\,.x \in \mathrm{dom}\, g)$$
$$\&\ \textbf{for}\ x\ \textbf{st}\ x \in \mathrm{dom}\,\textbf{it}\ \textbf{holds}\ \textbf{it}.x = g.(f\,.x).$$

The following propositions are true:

(20) $$h = g \cdot f\ \textbf{iff}\ (\textbf{for}\ x\ \textbf{holds}\ x \in \mathrm{dom}\, h\ \textbf{iff}\ x \in \mathrm{dom}\, f\ \&\ f\,.x \in \mathrm{dom}\, g)$$
$$\&\ \textbf{for}\ x\ \textbf{st}\ x \in \mathrm{dom}\, h\ \textbf{holds}\ h.x = g.(f\,.x),$$

(21) $$x \in \mathrm{dom}\,(g \cdot f)\ \textbf{iff}\ x \in \mathrm{dom}\, f\ \&\ f\,.x \in \mathrm{dom}\, g,$$

(22) $$x \in \mathrm{dom}\,(g \cdot f)\ \textbf{implies}\ (g \cdot f).x = g.(f\,.x),$$

(23) $$x \in \mathrm{dom}\, f\ \&\ f\,.x \in \mathrm{dom}\, g\ \textbf{implies}\ (g \cdot f).x = g.(f\,.x),$$

(24) $$\mathrm{dom}\,(g \cdot f) \subseteq \mathrm{dom}\, f,$$

(25) $$z \in \mathrm{rng}\,(g \cdot f)\ \textbf{implies}\ z \in \mathrm{rng}\, g,$$

(26) $$\mathrm{rng}\,(g \cdot f) \subseteq \mathrm{rng}\, g,$$

(27) $$\mathrm{rng}\, f \subseteq \mathrm{dom}\, g\ \textbf{iff}\ \mathrm{dom}\,(g \cdot f) = \mathrm{dom}\, f,$$

(28) $$\mathrm{dom}\, g \subseteq \mathrm{rng}\, f\ \textbf{implies}\ \mathrm{rng}\,(g \cdot f) = \mathrm{rng}\, g,$$

(29) $$\mathrm{rng}\, f = \mathrm{dom}\, g\ \textbf{implies}\ \mathrm{dom}\,(g \cdot f) = \mathrm{dom}\, f\ \&\ \mathrm{rng}\,(g \cdot f) = \mathrm{rng}\, g,$$

(30) $$h \cdot (g \cdot f) = (h \cdot g) \cdot f,$$

(31) $$\mathrm{rng}\, f \subseteq \mathrm{dom}\, g\ \&\ x \in \mathrm{dom}\, f\ \textbf{implies}\ (g \cdot f).x = g.(f\,.x),$$

(32) $$\mathrm{rng}\, f = \mathrm{dom}\, g\ \&\ x \in \mathrm{dom}\, f\ \textbf{implies}\ (g \cdot f).x = g.(f\,.x),$$

(33) $$\mathrm{rng}\, f \subseteq Y\ \&\ (\textbf{for}\ g,h\ \textbf{st}\ \mathrm{dom}\, g = Y\ \&\ \mathrm{dom}\, h = Y\ \&\ g \cdot f = h \cdot f\ \textbf{holds}\ g = h)$$
$$\textbf{implies}\ Y = \mathrm{rng}\, f.$$

Let us consider X. The functor

$$\mathrm{id}\, X,$$

with values of the type Function, is defined by

$$\mathrm{dom}\,\textbf{it} = X\ \&\ \textbf{for}\ x\ \textbf{st}\ x \in X\ \textbf{holds}\ \textbf{it}.x = x.$$

Next we state a number of propositions:

(34) $$f = \mathrm{id}\, X\ \textbf{iff}\ \mathrm{dom}\, f = X\ \&\ \textbf{for}\ x\ \textbf{st}\ x \in X\ \textbf{holds}\ f\,.x = x,$$

(35) $$x \in X\ \textbf{implies}\ (\mathrm{id}\, X).x = x,$$

(36) $$\operatorname{dom}\operatorname{id}X = X \ \&\ \operatorname{rng}\operatorname{id}X = X,$$

(37) $$\operatorname{dom}(f \cdot (\operatorname{id}X)) = \operatorname{dom}f \cap X,$$

(38) $$x \in \operatorname{dom}f \cap X \textbf{ implies } f\,.x = (f \cdot (\operatorname{id}X))\,.x,$$

(39) $$\operatorname{dom}f \subseteq X \textbf{ implies } f \cdot (\operatorname{id}X) = f,$$

(40) $$x \in \operatorname{dom}((\operatorname{id}Y) \cdot f) \textbf{ iff } x \in \operatorname{dom}f \ \&\ f\,.x \in Y,$$

(41) $$\operatorname{rng}f \subseteq Y \textbf{ implies } (\operatorname{id}Y) \cdot f = f,$$

(42) $$f \cdot (\operatorname{id}\operatorname{dom}f) = f \ \&\ (\operatorname{id}\operatorname{rng}f) \cdot f = f,$$

(43) $$(\operatorname{id}X) \cdot (\operatorname{id}Y) = \operatorname{id}(X \cap Y),$$

(44) $$\operatorname{dom}f = X \ \&\ \operatorname{rng}f = X \ \&\ \operatorname{dom}g = X \ \&\ g \cdot f = f \textbf{ implies } g = \operatorname{id}X.$$

Let us consider f. The predicate

$$f \text{ is\_one-to-one}$$

is defined by

$$\textbf{for } x1,x2 \textbf{ st } x1 \in \operatorname{dom}f \ \&\ x2 \in \operatorname{dom}f \ \&\ f\,.x1 = f\,.x2 \textbf{ holds } x1 = x2.$$

One can prove the following propositions:

(45) $$f \text{ is\_one-to-one} \textbf{ iff for } x1,x2 \textbf{ st } x1 \in \operatorname{dom}f \ \&\ x2 \in \operatorname{dom}f \ \&\ f\,.x1 = f\,.x2 \textbf{ holds } x1 = x2,$$

(46) $$f \text{ is\_one-to-one} \ \&\ g \text{ is\_one-to-one} \textbf{ implies } g \cdot f \text{ is\_one-to-one},$$

(47) $$g \cdot f \text{ is\_one-to-one} \ \&\ \operatorname{rng}f \subseteq \operatorname{dom}g \textbf{ implies } f \text{ is\_one-to-one},$$

(48) $$g \cdot f \text{ is\_one-to-one} \ \&\ \operatorname{rng}f = \operatorname{dom}g \textbf{ implies } f \text{ is\_one-to-one} \ \&\ g \text{ is\_one-to-one},$$

(49) $$f \text{ is\_one-to-one} \textbf{ iff for } g,h \textbf{ st } \operatorname{rng}g \subseteq \operatorname{dom}f \ \&\ \operatorname{rng}h \subseteq \operatorname{dom}f \ \&\ \operatorname{dom}g = \operatorname{dom}h \ \&\ f \cdot g = f \cdot h \textbf{ holds } g = h,$$

(50) $$\operatorname{dom}f = X \ \&\ \operatorname{dom}g = X \ \&\ \operatorname{rng}g \subseteq X \ \&\ f \text{ is\_one-to-one} \ \&\ f \cdot g = f \textbf{ implies } g = \operatorname{id}X,$$

(51) $$\operatorname{rng}(g \cdot f) = \operatorname{rng}g \ \&\ g \text{ is\_one-to-one} \textbf{ implies } \operatorname{dom}g \subseteq \operatorname{rng}f,$$

(52) $$\operatorname{id}X \text{ is\_one-to-one},$$

(53) $$(\textbf{ex } g \textbf{ st } g \cdot f = \operatorname{id}\operatorname{dom}f) \textbf{ implies } f \text{ is\_one-to-one}.$$

Let us consider f. Assume that the following holds

$$f \text{ is\_one-to-one}.$$

The functor

$$f^{-1},$$

with values of the type Function, is defined by

$$\operatorname{dom} \textbf{it} = \operatorname{rng} f \;\&\; \textbf{for } y,x \textbf{ holds } y \in \operatorname{rng} f \;\&\; x = \textbf{it}.y \textbf{ iff } x \in \operatorname{dom} f \;\&\; y = f.x.$$

We now state a number of propositions:

$$f \text{ is\_one-to-one} \textbf{ implies } (g = f^{-1} \textbf{ iff } \operatorname{dom} g = \operatorname{rng} f \;\&\; \textbf{for } y,x \textbf{ holds } y \in \operatorname{rng} f \;\&\; x = g.y \textbf{ iff } x \in \operatorname{dom} f \;\&\; y = f.x), \tag{54}$$

$$f \text{ is\_one-to-one} \textbf{ implies } \operatorname{rng} f = \operatorname{dom}(f^{-1}) \;\&\; \operatorname{dom} f = \operatorname{rng}(f^{-1}), \tag{55}$$

$$f \text{ is\_one-to-one} \;\&\; x \in \operatorname{dom} f \textbf{ implies } x = (f^{-1}).(f.x) \;\&\; x = (f^{-1} \cdot f).x, \tag{56}$$

$$f \text{ is\_one-to-one} \;\&\; y \in \operatorname{rng} f \textbf{ implies } y = f.((f^{-1}).y) \;\&\; y = (f \cdot f^{-1}).y, \tag{57}$$

$$f \text{ is\_one-to-one} \textbf{ implies } \operatorname{dom}(f^{-1} \cdot f) = \operatorname{dom} f \;\&\; \operatorname{rng}(f^{-1} \cdot f) = \operatorname{dom} f, \tag{58}$$

$$f \text{ is\_one-to-one} \textbf{ implies } \operatorname{dom}(f \cdot f^{-1}) = \operatorname{rng} f \;\&\; \operatorname{rng}(f \cdot f^{-1}) = \operatorname{rng} f, \tag{59}$$

$$f \text{ is\_one-to-one} \;\&\; \operatorname{dom} f = \operatorname{rng} g \;\&\; \operatorname{rng} f = \operatorname{dom} g \;\&\; (\textbf{for } x,y \textbf{ st } x \in \operatorname{dom} f \;\&\; y \in \operatorname{dom} g \textbf{ holds } f.x = y \textbf{ iff } g.y = x) \textbf{ implies } g = f^{-1}, \tag{60}$$

$$f \text{ is\_one-to-one} \textbf{ implies } f^{-1} \cdot f = \operatorname{id} \operatorname{dom} f \;\&\; f \cdot f^{-1} = \operatorname{id} \operatorname{rng} f, \tag{61}$$

$$f \text{ is\_one-to-one} \textbf{ implies } f^{-1} \text{ is\_one-to-one}, \tag{62}$$

$$f \text{ is\_one-to-one} \;\&\; \operatorname{rng} f = \operatorname{dom} g \;\&\; g \cdot f = \operatorname{id} \operatorname{dom} f \textbf{ implies } g = f^{-1}, \tag{63}$$

$$f \text{ is\_one-to-one} \;\&\; \operatorname{rng} g = \operatorname{dom} f \;\&\; f \cdot g = \operatorname{id} \operatorname{rng} f \textbf{ implies } g = f^{-1}, \tag{64}$$

$$f \text{ is\_one-to-one} \textbf{ implies } (f^{-1})^{-1} = f, \tag{65}$$

$$f \text{ is\_one-to-one} \;\&\; g \text{ is\_one-to-one} \textbf{ implies } (g \cdot f)^{-1} = f^{-1} \cdot g^{-1}, \tag{66}$$

$$(\operatorname{id} X)^{-1} = (\operatorname{id} X). \tag{67}$$

Let us consider f, X. The functor

$$f \mid X,$$

yields the type Function and is defined by

$$\operatorname{dom} \textbf{it} = \operatorname{dom} f \cap X \;\&\; \textbf{for } x \textbf{ st } x \in \operatorname{dom} \textbf{it} \textbf{ holds } \textbf{it}.x = f.x.$$

We now state a number of propositions:

(68) $g = f \mid X$ **iff** $\operatorname{dom} g = \operatorname{dom} f \cap X$ & **for** x **st** $x \in \operatorname{dom} g$ **holds** $g.x = f.x$,

(69) $$\operatorname{dom}(f \mid X) = \operatorname{dom} f \cap X,$$

(70) $$x \in \operatorname{dom}(f \mid X) \textbf{ implies } (f \mid X).x = f.x,$$

(71) $$x \in \operatorname{dom} f \cap X \textbf{ implies } (f \mid X).x = f.x,$$

(72) $$x \in \operatorname{dom} f \;\&\; x \in X \textbf{ implies } (f \mid X).x = f.x,$$

(73) $$x \in \operatorname{dom} f \;\&\; x \in X \textbf{ implies } f.x \in \operatorname{rng}(f \mid X),$$

(74) $$X \subseteq \operatorname{dom} f \textbf{ implies } \operatorname{dom}(f \mid X) = X,$$

(75) $$\operatorname{dom}(f \mid X) \subseteq X,$$

(76) $$\operatorname{dom}(f \mid X) \subseteq \operatorname{dom} f \;\&\; \operatorname{rng}(f \mid X) \subseteq \operatorname{rng} f,$$

(77) $$f \mid X = f \cdot (\operatorname{id} X),$$

(78) $$\operatorname{dom} f \subseteq X \textbf{ implies } f \mid X = f,$$

(79) $$f \mid (\operatorname{dom} f) = f,$$

(80) $$(f \mid X) \mid Y = f \mid (X \cap Y),$$

(81) $$(f \mid X) \mid X = f \mid X,$$

(82) $$X \subseteq Y \textbf{ implies } (f \mid X) \mid Y = f \mid X \;\&\; (f \mid Y) \mid X = f \mid X,$$

(83) $$(g \cdot f) \mid X = g \cdot (f \mid X),$$

(84) $$f \text{ is\_one-to-one} \textbf{ implies } f \mid X \text{ is\_one-to-one}.$$

Let us consider Y, f. The functor

$$Y \mid f,$$

with values of the type Function, is defined by

$$(\textbf{for } x \textbf{ holds } x \in \operatorname{dom} \textbf{it} \textbf{ iff } x \in \operatorname{dom} f \;\&\; f.x \in Y)$$
$$\& \textbf{ for } x \textbf{ st } x \in \operatorname{dom} \textbf{it} \textbf{ holds } \textbf{it}.x = f.x.$$

We now state a number of propositions:

(85) $$g = Y \mid f \textbf{ iff } (\textbf{for } x \textbf{ holds } x \in \operatorname{dom} g \textbf{ iff } x \in \operatorname{dom} f \;\&\; f.x \in Y)$$
$$\& \textbf{ for } x \textbf{ st } x \in \operatorname{dom} g \textbf{ holds } g.x = f.x,$$

$$x \in \operatorname{dom}(Y \mid f) \ \textbf{iff}\ x \in \operatorname{dom} f \ \&\ f\,.x \in Y, \tag{86}$$

$$x \in \operatorname{dom}(Y \mid f) \ \textbf{implies}\ (Y \mid f).x = f\,.x, \tag{87}$$

$$\operatorname{rng}(Y \mid f) \subseteq Y, \tag{88}$$

$$\operatorname{dom}(Y \mid f) \subseteq \operatorname{dom} f \ \&\ \operatorname{rng}(Y \mid f) \subseteq \operatorname{rng} f, \tag{89}$$

$$\operatorname{rng}(Y \mid f) = \operatorname{rng} f \cap Y, \tag{90}$$

$$Y \subseteq \operatorname{rng} f \ \textbf{implies}\ \operatorname{rng}(Y \mid f) = Y, \tag{91}$$

$$Y \mid f = (\operatorname{id} Y) \cdot f, \tag{92}$$

$$\operatorname{rng} f \subseteq Y \ \textbf{implies}\ Y \mid f = f, \tag{93}$$

$$(\operatorname{rng} f) \mid f = f, \tag{94}$$

$$Y \mid (X \mid f) = (Y \cap X) \mid f, \tag{95}$$

$$Y \mid (Y \mid f) = Y \mid f, \tag{96}$$

$$X \subseteq Y \ \textbf{implies}\ Y \mid (X \mid f) = X \mid f \ \&\ X \mid (Y \mid f) = X \mid f, \tag{97}$$

$$Y \mid (g \cdot f) = (Y \mid g) \cdot f, \tag{98}$$

$$f \text{ is\_one-to-one } \textbf{implies}\ Y \mid f \text{ is\_one-to-one}, \tag{99}$$

$$(Y \mid f) \mid X = Y \mid (f \mid X). \tag{100}$$

Let us consider f, X. The functor

$$f \,^\circ\, X,$$

yields the type set and is defined by

$$\textbf{for}\ y\ \textbf{holds}\ y \in \textbf{it}\ \textbf{iff}\ \textbf{ex}\ x\ \textbf{st}\ x \in \operatorname{dom} f \ \&\ x \in X \ \&\ y = f\,.x.$$

The following propositions are true:

$$Y = f \,^\circ\, X \ \textbf{iff}\ \textbf{for}\ y\ \textbf{holds}\ y \in Y \ \textbf{iff}\ \textbf{ex}\ x\ \textbf{st}\ x \in \operatorname{dom} f \ \&\ x \in X \ \&\ y = f\,.x, \tag{101}$$

$$y \in f \,^\circ\, X \ \textbf{iff}\ \textbf{ex}\ x\ \textbf{st}\ x \in \operatorname{dom} f \ \&\ x \in X \ \&\ y = f\,.x, \tag{102}$$

$$f \,^\circ\, X \subseteq \operatorname{rng} f, \tag{103}$$

$$f \,^\circ\, (X) = f \,^\circ\, (\operatorname{dom} f \cap X), \tag{104}$$

$$f \,^\circ\, (\operatorname{dom} f) = \operatorname{rng} f, \tag{105}$$

$$f \,^\circ\, X \subseteq f \,^\circ\, (\operatorname{dom} f), \tag{106}$$

$$\text{rng}\,(f \mid X) = f^{\circ}\, X, \tag{107}$$

$$f^{\circ}\, X = \emptyset \textbf{ iff } \text{dom}\, f \cap X = \emptyset, \tag{108}$$

$$f^{\circ}\, \emptyset = \emptyset, \tag{109}$$

$$X \neq \emptyset \;\&\; X \subseteq \text{dom}\, f \textbf{ implies } f^{\circ}\, X \neq \emptyset, \tag{110}$$

$$X1 \subseteq X2 \textbf{ implies } f^{\circ}\, X1 \subseteq f^{\circ}\, X2, \tag{111}$$

$$f^{\circ}\,(X1 \cup X2) = f^{\circ}\, X1 \cup f^{\circ}\, X2, \tag{112}$$

$$f^{\circ}\,(X1 \cap X2) \subseteq f^{\circ}\, X1 \cap f^{\circ}\, X2, \tag{113}$$

$$f^{\circ}\, X1 \setminus f^{\circ}\, X2 \subseteq f^{\circ}\,(X1 \setminus X2), \tag{114}$$

$$(g \cdot f)^{\circ}\, X = g^{\circ}\,(f^{\circ}\, X), \tag{115}$$

$$\text{rng}\,(g \cdot f) = g^{\circ}\,(\text{rng}\, f), \tag{116}$$

$$x \in \text{dom}\, f \textbf{ implies } f^{\circ}\, \{x\} = \{f\,.x\}, \tag{117}$$

$$x1 \in \text{dom}\, f \;\&\; x2 \in \text{dom}\, f \textbf{ implies } f^{\circ}\, \{x1,x2\} = \{f\,.x1,f\,.x2\}, \tag{118}$$

$$(f \mid Y)^{\circ}\, X \subseteq f^{\circ}\, X, \tag{119}$$

$$(Y \mid f)^{\circ}\, X \subseteq f^{\circ}\, X, \tag{120}$$

$$f \text{ is\_one-to-one } \textbf{implies } f^{\circ}\,(X1 \cap X2) = f^{\circ}\, X1 \cap f^{\circ}\, X2, \tag{121}$$

$$(\textbf{for } X1,X2 \textbf{ holds } f^{\circ}\,(X1 \cap X2) = f^{\circ}\, X1 \cap f^{\circ}\, X2) \\ \textbf{implies } f \text{ is\_one-to-one}\,, \tag{122}$$

$$f \text{ is\_one-to-one } \textbf{implies } f^{\circ}\,(X1 \setminus X2) = f^{\circ}\, X1 \setminus f^{\circ}\, X2, \tag{123}$$

$$(\textbf{for } X1,X2 \textbf{ holds } f^{\circ}\,(X1 \setminus X2) = f^{\circ}\, X1 \setminus f^{\circ}\, X2) \textbf{ implies } f \text{ is\_one-to-one}\,, \tag{124}$$

$$X \cap Y = \emptyset \;\&\; f \text{ is\_one-to-one } \textbf{implies } f^{\circ}\, X \cap f^{\circ}\, Y = \emptyset, \tag{125}$$

$$(Y \mid f)^{\circ}\, X = Y \cap f^{\circ}\, X. \tag{126}$$

Let us consider f, Y. The functor

$$f^{-1}\, Y,$$

yields the type set and is defined by

$$\textbf{for } x \textbf{ holds } x \in \textbf{it iff } x \in \text{dom}\, f \;\&\; f\,.x \in Y.$$

We now state a number of propositions:

(127) $$X = f^{-1}\, Y \textbf{ iff for } x \textbf{ holds } x \in X \textbf{ iff } x \in \operatorname{dom} f \ \&\ f.x \in Y,$$

(128) $$x \in f^{-1}\, Y \textbf{ iff } x \in \operatorname{dom} f \ \&\ f.x \in Y,$$

(129) $$f^{-1}\, Y \subseteq \operatorname{dom} f,$$

(130) $$f^{-1}\, Y = f^{-1}\, (\operatorname{rng} f \cap Y),$$

(131) $$f^{-1}\, (\operatorname{rng} f) = \operatorname{dom} f,$$

(132) $$f^{-1}\, \emptyset = \emptyset,$$

(133) $$f^{-1}\, Y = \emptyset \textbf{ iff } \operatorname{rng} f \cap Y = \emptyset,$$

(134) $$Y \subseteq \operatorname{rng} f \textbf{ implies } (f^{-1}\, Y = \emptyset \textbf{ iff } Y = \emptyset),$$

(135) $$Y1 \subseteq Y2 \textbf{ implies } f^{-1}\, Y1 \subseteq f^{-1}\, Y2,$$

(136) $$f^{-1}\, (Y1 \cup Y2) = f^{-1}\, Y1 \cup f^{-1}\, Y2,$$

(137) $$f^{-1}\, (Y1 \cap Y2) = f^{-1}\, Y1 \cap f^{-1}\, Y2,$$

(138) $$f^{-1}\, (Y1 \setminus Y2) = f^{-1}\, Y1 \setminus f^{-1}\, Y2,$$

(139) $$(f \mid X)^{-1}\, Y = X \cap (f^{-1}\, Y),$$

(140) $$(g \cdot f)^{-1}\, Y = f^{-1}\, (g^{-1}\, Y),$$

(141) $$\operatorname{dom}(g \cdot f) = f^{-1}\, (\operatorname{dom} g),$$

(142) $$y \in \operatorname{rng} f \textbf{ iff } f^{-1}\, \{y\} \neq \emptyset,$$

(143) $$(\textbf{for } y \textbf{ st } y \in Y \textbf{ holds } f^{-1}\, \{y\} \neq \emptyset) \textbf{ implies } Y \subseteq \operatorname{rng} f,$$

(144) $$(\textbf{for } y \textbf{ st } y \in \operatorname{rng} f \textbf{ ex } x \textbf{ st } f^{-1}\, \{y\} = \{x\}) \textbf{ iff } f \text{ is\_one-to-one},$$

(145) $$f^{\circ}\, (f^{-1}\, Y) \subseteq Y,$$

(146) $$X \subseteq \operatorname{dom} f \textbf{ implies } X \subseteq f^{-1}\, (f^{\circ}\, X),$$

(147) $$Y \subseteq \operatorname{rng} f \textbf{ implies } f^{\circ}\, (f^{-1}\, Y) = Y,$$

(148) $$f^{\circ}\, (f^{-1}\, Y) = Y \cap f^{\circ}\, (\operatorname{dom} f),$$

(149) $$f^{\circ}\, (X \cap f^{-1}\, Y) \subseteq (f^{\circ}\, X) \cap Y,$$

(150) $$f^{\circ}\, (X \cap f^{-1}\, Y) = (f^{\circ}\, X) \cap Y,$$

$$X \cap f^{-1} Y \subseteq f^{-1} (f^{\circ} X \cap Y), \tag{151}$$

$$f \text{ is\_one-to-one } \textbf{implies } f^{-1} (f^{\circ} X) \subseteq X, \tag{152}$$

$$(\textbf{for } X \textbf{ holds } f^{-1} (f^{\circ} X) \subseteq X) \textbf{ implies } f \text{ is\_one-to-one}, \tag{153}$$

$$f \text{ is\_one-to-one } \textbf{implies } f^{\circ} X = (f^{-1})^{-1} X, \tag{154}$$

$$f \text{ is\_one-to-one } \textbf{implies } f^{-1} Y = (f^{-1})^{\circ} Y, \tag{155}$$

$$Y = \operatorname{rng} f \ \& \ \operatorname{dom} g = Y \ \& \ \operatorname{dom} h = Y \ \& \ g \cdot f = h \cdot f \textbf{ implies } g = h, \tag{156}$$

$$f^{\circ} X1 \subseteq f^{\circ} X2 \ \& \ X1 \subseteq \operatorname{dom} f \ \& \ f \text{ is\_one-to-one } \textbf{implies } X1 \subseteq X2, \tag{157}$$

$$f^{-1} Y1 \subseteq f^{-1} Y2 \ \& \ Y1 \subseteq \operatorname{rng} f \textbf{ implies } Y1 \subseteq Y2, \tag{158}$$

$$f \text{ is\_one-to-one } \textbf{iff for } y \textbf{ ex } x \textbf{ st } f^{-1} \{y\} \subseteq \{x\}, \tag{159}$$

$$\operatorname{rng} f \subseteq \operatorname{dom} g \textbf{ implies } f^{-1} X \subseteq (g \cdot f)^{-1} (g^{\circ} X). \tag{160}$$

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[1] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1, 1990.

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Properties of Subsets

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Summary. The text includes theorems concerning properties of subsets, and some operations on sets. The functions yielding improper subsets of a set, i.e. the empty set and the set itself are introduced. Functions and predicates introduced for sets are redefined. Some theorems about enumerated sets are proved.

The articles [2], [3], and [1] provide the terminology and notation for this paper. In the sequel E, X denote objects of the type set; x denotes an object of the type Any. One can prove the following propositions:

$$E \neq \emptyset \textbf{ implies } (x \textbf{ is } \text{Element } \textbf{of } E \textbf{ iff } x \in E), \tag{1}$$

$$x \in E \textbf{ implies } x \textbf{ is } \text{Element } \textbf{of } E, \tag{2}$$

$$X \textbf{ is } \text{Subset } \textbf{of } E \textbf{ iff } X \subseteq E. \tag{3}$$

We now define two new functors. Let us consider E. The functor

$$\emptyset E,$$

yields the type Subset **of** E and is defined by

$$\textbf{it} = \emptyset.$$

The functor

$$\Omega\, E,$$

with values of the type Subset **of** E, is defined by

$$\textbf{it} = E.$$

We now state two propositions:

$$\emptyset \textbf{ is } \text{Subset } \textbf{of } X, \tag{4}$$

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$$X \textbf{ is } \text{Subset} \textbf{ of } X. \tag{5}$$

In the sequel A, B, C denote objects of the type Subset **of** E. Next we state several propositions:

$$x \in A \textbf{ implies } x \textbf{ is } \text{Element} \textbf{ of } E, \tag{6}$$

$$(\textbf{for } x \textbf{ being } \text{Element} \textbf{ of } E \textbf{ holds } x \in A \textbf{ implies } x \in B) \textbf{ implies } A \subseteq B, \tag{7}$$

$$(\textbf{for } x \textbf{ being } \text{Element} \textbf{ of } E \textbf{ holds } x \in A \textbf{ iff } x \in B) \textbf{ implies } A = B, \tag{8}$$

$$x \in A \textbf{ implies } x \in E, \tag{9}$$

$$A \neq \emptyset \textbf{ iff ex } x \textbf{ being } \text{Element} \textbf{ of } E \textbf{ st } x \in A. \tag{10}$$

Let us consider E, A. The functor

$$A^{\mathrm{c}},$$

yields the type Subset **of** E and is defined by

$$\textbf{it} = E \setminus A.$$

Let us consider B. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$\begin{array}{ccc} A \cup B & \text{is} & \text{Subset } \textbf{of } E, \\ A \cap B & \text{is} & \text{Subset } \textbf{of } E, \\ A \setminus B & \text{is} & \text{Subset } \textbf{of } E, \\ A \dot{-} B & \text{is} & \text{Subset } \textbf{of } E. \end{array}$$

One can prove the following propositions:

$$x \in A \cap B \textbf{ implies } x \textbf{ is } \text{Element} \textbf{ of } A \;\&\; x \textbf{ is } \text{Element} \textbf{ of } B, \tag{11}$$

$$x \in A \cup B \textbf{ implies } x \textbf{ is } \text{Element} \textbf{ of } A \textbf{ or } x \textbf{ is } \text{Element} \textbf{ of } B, \tag{12}$$

$$x \in A \setminus B \textbf{ implies } x \textbf{ is } \text{Element} \textbf{ of } A, \tag{13}$$

$$x \in A \dot{-} B \textbf{ implies } x \textbf{ is } \text{Element} \textbf{ of } A \textbf{ or } x \textbf{ is } \text{Element} \textbf{ of } B, \tag{14}$$

$$\begin{gathered}(\textbf{for } x \textbf{ being } \text{Element} \textbf{ of } E \textbf{ holds } x \in A \textbf{ iff } x \in B \textbf{ or } x \in C) \\ \textbf{implies } A = B \cup C,\end{gathered} \tag{15}$$

$$\begin{gathered}(\textbf{for } x \textbf{ being } \text{Element} \textbf{ of } E \textbf{ holds } x \in A \textbf{ iff } x \in B \;\&\; x \in C) \\ \textbf{implies } A = B \cap C,\end{gathered} \tag{16}$$

$$\begin{gathered}(\textbf{for } x \textbf{ being } \text{Element} \textbf{ of } E \textbf{ holds } x \in A \textbf{ iff } x \in B \;\&\; \textbf{not } x \in C) \\ \textbf{implies } A = B \setminus C,\end{gathered} \tag{17}$$

$$(\textbf{for}\ x\ \textbf{being}\ \text{Element}\ \textbf{of}\ E\ \textbf{holds}\ x \in A\ \textbf{iff}\ \textbf{not}\,(x \in B\ \textbf{iff}\ x \in C))\ \textbf{implies}\ A = B \dot{-} C, \tag{18}$$

$$\emptyset E = \emptyset, \tag{19}$$

$$\Omega\, E = E, \tag{20}$$

$$\emptyset E = (\Omega\, E)^{\,\mathrm{c}}, \tag{21}$$

$$\Omega\, E = (\emptyset E)^{\,\mathrm{c}}, \tag{22}$$

$$A^{\,\mathrm{c}} = E \setminus A, \tag{23}$$

$$A^{\,\mathrm{c}\,\mathrm{c}} = A, \tag{24}$$

$$A \cup A^{\,\mathrm{c}} = \Omega\, E\ \&\ A^{\,\mathrm{c}} \cup A = \Omega\, E, \tag{25}$$

$$A \cap A^{\,\mathrm{c}} = \emptyset E\ \&\ A^{\,\mathrm{c}} \cap A = \emptyset E, \tag{26}$$

$$A \cap \emptyset E = \emptyset E\ \&\ \emptyset E \cap A = \emptyset E, \tag{27}$$

$$A \cup \Omega\, E = \Omega\, E\ \&\ \Omega\, E \cup A = \Omega\, E, \tag{28}$$

$$(A \cup B)^{\,\mathrm{c}} = A^{\,\mathrm{c}} \cap B^{\,\mathrm{c}}, \tag{29}$$

$$(A \cap B)^{\,\mathrm{c}} = A^{\,\mathrm{c}} \cup B^{\,\mathrm{c}}, \tag{30}$$

$$A \subseteq B\ \textbf{iff}\ B^{\,\mathrm{c}} \subseteq A^{\,\mathrm{c}}, \tag{31}$$

$$A \setminus B = A \cap B^{\,\mathrm{c}}, \tag{32}$$

$$(A \setminus B)^{\,\mathrm{c}} = A^{\,\mathrm{c}} \cup B, \tag{33}$$

$$(A \dot{-} B)^{\,\mathrm{c}} = A \cap B \cup A^{\,\mathrm{c}} \cap B^{\,\mathrm{c}}, \tag{34}$$

$$A \subseteq B^{\,\mathrm{c}}\ \textbf{implies}\ B \subseteq A^{\,\mathrm{c}}, \tag{35}$$

$$A^{\,\mathrm{c}} \subseteq B\ \textbf{implies}\ B^{\,\mathrm{c}} \subseteq A, \tag{36}$$

$$\emptyset E \subseteq E, \tag{37}$$

$$A \subseteq A^{\,\mathrm{c}}\ \textbf{iff}\ A = \emptyset E, \tag{38}$$

$$A^{\,\mathrm{c}} \subseteq A\ \textbf{iff}\ A = \Omega\, E, \tag{39}$$

$$X \subseteq A\ \&\ X \subseteq A^{\,\mathrm{c}}\ \textbf{implies}\ X = \emptyset, \tag{40}$$

$$(A \cup B)^{\,\mathrm{c}} \subseteq A^{\,\mathrm{c}}\ \&\ (A \cup B)^{\,\mathrm{c}} \subseteq B^{\,\mathrm{c}}, \tag{41}$$

$$A^{\mathrm{c}} \subseteq (A \cap B)^{\mathrm{c}} \ \&\ B^{\mathrm{c}} \subseteq (A \cap B)^{\mathrm{c}}, \tag{42}$$

$$A \text{ misses } B \textbf{ iff } A \subseteq B^{\mathrm{c}}, \tag{43}$$

$$A \text{ misses } B^{\mathrm{c}} \textbf{ iff } A \subseteq B, \tag{44}$$

$$A \text{ misses } A^{\mathrm{c}}, \tag{45}$$

$$A \text{ misses } B \ \&\ A^{\mathrm{c}} \text{ misses } B^{\mathrm{c}} \textbf{ implies } A = B^{\mathrm{c}}, \tag{46}$$

$$A \subseteq B \ \&\ C \text{ misses } B \textbf{ implies } A \subseteq C^{\mathrm{c}}, \tag{47}$$

$$(\textbf{for } a \textbf{ being } \text{Element } \textbf{of } A \textbf{ holds } a \in B) \textbf{ implies } A \subseteq B, \tag{48}$$

$$(\textbf{for } x \textbf{ being } \text{Element } \textbf{of } E \textbf{ holds } x \in A) \textbf{ implies } E = A, \tag{49}$$

$$\begin{gathered} E \neq \emptyset \textbf{ implies for } A,B \\ \textbf{holds } A = B^{\mathrm{c}} \textbf{ iff for } x \textbf{ being } \text{Element } \textbf{of } E \textbf{ holds } x \in A \textbf{ iff not } x \in B, \end{gathered} \tag{50}$$

$$\begin{gathered} E \neq \emptyset \textbf{ implies for } A,B \\ \textbf{holds } A = B^{\mathrm{c}} \textbf{ iff for } x \textbf{ being } \text{Element } \textbf{of } E \textbf{ holds not } x \in A \textbf{ iff } x \in B, \end{gathered} \tag{51}$$

$$\begin{gathered} E \neq \emptyset \textbf{ implies for } A,B \\ \textbf{holds } A = B^{\mathrm{c}} \textbf{ iff for } x \textbf{ being } \text{Element } \textbf{of } E \textbf{ holds not } (x \in A \textbf{ iff } x \in B), \end{gathered} \tag{52}$$

$$x \in A^{\mathrm{c}} \textbf{ implies not } x \in A. \tag{53}$$

In the sequel $x1$, $x2$, $x3$, $x4$, $x5$, $x6$, $x7$, $x8$ will have the type Element **of** X. One can prove the following propositions:

$$X \neq \emptyset \textbf{ implies } \{x1\} \textbf{ is } \text{Subset } \textbf{of } X, \tag{54}$$

$$X \neq \emptyset \textbf{ implies } \{x1,x2\} \textbf{ is } \text{Subset } \textbf{of } X, \tag{55}$$

$$X \neq \emptyset \textbf{ implies } \{x1,x2,x3\} \textbf{ is } \text{Subset } \textbf{of } X, \tag{56}$$

$$X \neq \emptyset \textbf{ implies } \{x1,x2,x3,x4\} \textbf{ is } \text{Subset } \textbf{of } X, \tag{57}$$

$$X \neq \emptyset \textbf{ implies } \{x1,x2,x3,x4,x5\} \textbf{ is } \text{Subset } \textbf{of } X, \tag{58}$$

$$X \neq \emptyset \textbf{ implies } \{x1,x2,x3,x4,x5,x6\} \textbf{ is } \text{Subset } \textbf{of } X, \tag{59}$$

$$X \neq \emptyset \textbf{ implies } \{x1,x2,x3,x4,x5,x6,x7\} \textbf{ is } \text{Subset } \textbf{of } X, \tag{60}$$

$$X \neq \emptyset \textbf{ implies } \{x1,x2,x3,x4,x5,x6,x7,x8\} \textbf{ is } \text{Subset } \textbf{of } X. \tag{61}$$

In the sequel $x1$, $x2$, $x3$, $x4$, $x5$, $x6$, $x7$, $x8$ denote objects of the type Any. We now state several propositions:

$$x1 \in X \textbf{ implies } \{x1\} \textbf{ is } \text{Subset } \textbf{of } X, \tag{62}$$

$$x1 \in X \mathbin{\&} x2 \in X \textbf{ implies } \{x1,x2\} \textbf{ is } \text{Subset} \textbf{ of } X, \tag{63}$$

$$x1 \in X \mathbin{\&} x2 \in X \mathbin{\&} x3 \in X \textbf{ implies } \{x1,x2,x3\} \textbf{ is } \text{Subset} \textbf{ of } X, \tag{64}$$

$$x1 \in X \mathbin{\&} x2 \in X \mathbin{\&} x3 \in X \mathbin{\&} x4 \in X \textbf{ implies } \{x1,x2,x3,x4\} \textbf{ is } \text{Subset} \textbf{ of } X, \tag{65}$$

$$\begin{gathered} x1 \in X \mathbin{\&} x2 \in X \mathbin{\&} x3 \in X \mathbin{\&} x4 \in X \mathbin{\&} x5 \in X \\ \textbf{implies } \{x1,x2,x3,x4,x5\} \textbf{ is } \text{Subset} \textbf{ of } X, \end{gathered} \tag{66}$$

$$\begin{gathered} x1 \in X \mathbin{\&} x2 \in X \mathbin{\&} x3 \in X \mathbin{\&} x4 \in X \mathbin{\&} x5 \in X \mathbin{\&} x6 \in X \\ \textbf{implies } \{x1,x2,x3,x4,x5,x6\} \textbf{ is } \text{Subset} \textbf{ of } X, \end{gathered} \tag{67}$$

$$\begin{gathered} x1 \in X \mathbin{\&} x2 \in X \mathbin{\&} x3 \in X \mathbin{\&} x4 \in X \mathbin{\&} x5 \in X \mathbin{\&} x6 \in X \mathbin{\&} x7 \in X \\ \textbf{implies } \{x1,x2,x3,x4,x5,x6,x7\} \textbf{ is } \text{Subset} \textbf{ of } X, \end{gathered} \tag{68}$$

$$\begin{gathered} x1 \in X \\ \mathbin{\&} x2 \in X \mathbin{\&} x3 \in X \mathbin{\&} x4 \in X \mathbin{\&} x5 \in X \mathbin{\&} x6 \in X \mathbin{\&} x7 \in X \mathbin{\&} x8 \in X \\ \textbf{implies } \{x1,x2,x3,x4,x5,x6,x7,x8\} \textbf{ is } \text{Subset} \textbf{ of } X. \end{gathered} \tag{69}$$

The scheme *Subset_Ex* concerns a constant $\mathcal{A}$ that has the type set and a unary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex } X \textbf{ being } \text{Subset} \textbf{ of } \mathcal{A} \textbf{ st for } x \textbf{ holds } x \in X \textbf{ iff } x \in \mathcal{A} \mathbin{\&} \mathcal{P}[x]$$

for all values of the parameters.

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[1] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1, 1990.

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Relations and Their Basic Properties

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Summary. We define here: mode Relation as a set of pairs, the domain, the codomain, and the field of relation, the empty and the identity relations, the composition of relations, the image and the inverse image of a set under a relation. Two predicates = and ⊆ , and three functions ∪, ∩ and \ are redefined. Basic facts about the above mentioned notions are presented.

The terminology and notation used in this paper have been introduced in the articles [1] and [2]. For simplicity we adopt the following convention: A, B, X, Y, $Y1$, $Y2$ denote objects of the type set; a, b, c, d, x, y, z denote objects of the type Any. The mode

$$\mathrm{Relation}\,,$$

which widens to the type set, is defined by

$$x \in \textbf{it implies ex}\, y{,}z \textbf{ st } x = \langle y, z\rangle.$$

One can prove the following proposition

(1) $\textbf{for } R \textbf{ being } \mathrm{set} \textbf{ st for } x \textbf{ st } x \in R \textbf{ ex } y{,}z \textbf{ st } x = \langle y, z\rangle \textbf{ holds } R \textbf{ is } \mathrm{Relation}\,.$

In the sequel P, $P1$, $P2$, Q, R, S will have the type Relation. Next we state several propositions:

(2) $$x \in R \textbf{ implies ex}\, y{,}z \textbf{ st } x = \langle y, z\rangle,$$

(3) $$A \subseteq R \textbf{ implies } A \textbf{ is } \mathrm{Relation}\,,$$

(4) $$\{\langle x, y\rangle\} \textbf{ is } \mathrm{Relation}\,,$$

(5) $$\{\langle a, b\rangle, \langle c, d\rangle\} \textbf{ is } \mathrm{Relation}\,,$$

(6) $$[\!:X, Y:\!] \textbf{ is } \mathrm{Relation}\,.$$

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The scheme *Rel_Existence* deals with a constant $\mathcal{A}$ that has the type set, a constant $\mathcal{B}$ that has the type set and a binary predicate $\mathcal{P}$ and states that the following holds

$$\mathbf{ex}\, R\ \mathbf{being}\ \text{Relation}\ \mathbf{st}\ \mathbf{for}\, x{,}y\ \mathbf{holds}\ \langle x, y\rangle \in R\ \mathbf{iff}\ x \in \mathcal{A}\ \&\ y \in \mathcal{B}\ \&\ \mathcal{P}[x, y]$$

for all values of the parameters.

Let us consider P, R. Let us note that one can characterize the predicate

$$P = R$$

by the following (equivalent) condition:

$$\mathbf{for}\, a{,}b\ \mathbf{holds}\ \langle a, b\rangle \in P\ \mathbf{iff}\ \langle a, b\rangle \in R.$$

The following proposition is true

$$P = R\ \mathbf{iff}\ \mathbf{for}\, a{,}b\ \mathbf{holds}\ \langle a, b\rangle \in P\ \mathbf{iff}\ \langle a, b\rangle \in R. \tag{7}$$

For convenience we may adopt another formulas defining notions considered in the paper. From now on we shall treat them as new definitions.

Let us consider P, R. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$P \cap R \quad \text{is} \quad \text{Relation}\,,$$

$$P \cup R \quad \text{is} \quad \text{Relation}\,,$$

$$P \setminus R \quad \text{is} \quad \text{Relation}\,.$$

Let us note that one can characterize the predicate

$$P \subseteq R$$

by the following (equivalent) condition:

$$\mathbf{for}\, a{,}b\ \mathbf{holds}\ \langle a, b\rangle \in P\ \mathbf{implies}\ \langle a, b\rangle \in R.$$

The following three propositions are true:

$$P \subseteq R\ \mathbf{iff}\ \mathbf{for}\, a{,}b\ \mathbf{holds}\ \langle a, b\rangle \in P\ \mathbf{implies}\ \langle a, b\rangle \in R, \tag{8}$$

$$X \cap R\ \mathbf{is}\ \text{Relation}\ \&\ R \cap X\ \mathbf{is}\ \text{Relation}\,, \tag{9}$$

$$R \setminus X\ \mathbf{is}\ \text{Relation}\,. \tag{10}$$

Let us consider R. The functor

$$\operatorname{dom} R,$$

with values of the type set, is defined by

$$x \in \mathbf{it}\ \mathbf{iff}\ \mathbf{ex}\, y\ \mathbf{st}\ \langle x, y\rangle \in R.$$

We now state several propositions:

(11) $$X = \operatorname{dom} R \textbf{ iff for } x \textbf{ holds } x \in X \textbf{ iff ex } y \textbf{ st } \langle x, y\rangle \in R,$$

(12) $$x \in \operatorname{dom} R \textbf{ iff ex } y \textbf{ st } \langle x, y\rangle \in R,$$

(13) $$\operatorname{dom}(P \cup R) = \operatorname{dom} P \cup \operatorname{dom} R,$$

(14) $$\operatorname{dom}(P \cap R) \subseteq \operatorname{dom} P \cap \operatorname{dom} R,$$

(15) $$\operatorname{dom} P \setminus \operatorname{dom} R \subseteq \operatorname{dom}(P \setminus R).$$

Let us consider R. The functor

$$\operatorname{rng} R,$$

yields the type set and is defined by

$$y \in \textbf{it iff ex } x \textbf{ st } \langle x, y\rangle \in R.$$

One can prove the following propositions:

(16) $$X = \operatorname{rng} R \textbf{ iff for } x \textbf{ holds } x \in X \textbf{ iff ex } y \textbf{ st } \langle y, x\rangle \in R,$$

(17) $$x \in \operatorname{rng} R \textbf{ iff ex } y \textbf{ st } \langle y, x\rangle \in R,$$

(18) $$x \in \operatorname{dom} R \textbf{ implies ex } y \textbf{ st } y \in \operatorname{rng} R,$$

(19) $$y \in \operatorname{rng} R \textbf{ implies ex } x \textbf{ st } x \in \operatorname{dom} R,$$

(20) $$\langle x, y\rangle \in R \textbf{ implies } x \in \operatorname{dom} R \ \&\ y \in \operatorname{rng} R,$$

(21) $$R \subseteq [\!:\operatorname{dom} R, \operatorname{rng} R:\!],$$

(22) $$R \cap [\!:\operatorname{dom} R, \operatorname{rng} R:\!] = R,$$

(23) $$R = \{\langle x, y\rangle\} \textbf{ implies } \operatorname{dom} R = \{x\} \ \&\ \operatorname{rng} R = \{y\},$$

(24) $$R = \{\langle a, b\rangle, \langle x, y\rangle\} \textbf{ implies } \operatorname{dom} R = \{a, x\} \ \&\ \operatorname{rng} R = \{b, y\},$$

(25) $$P \subseteq R \textbf{ implies } \operatorname{dom} P \subseteq \operatorname{dom} R \ \&\ \operatorname{rng} P \subseteq \operatorname{rng} R,$$

(26) $$\operatorname{rng}(P \cup R) = \operatorname{rng} P \cup \operatorname{rng} R,$$

(27) $$\operatorname{rng}(P \cap R) \subseteq \operatorname{rng} P \cap \operatorname{rng} R,$$

(28) $$\operatorname{rng} P \setminus \operatorname{rng} R \subseteq \operatorname{rng}(P \setminus R).$$

Let us consider R. The functor

$$\operatorname{field} R,$$

yields the type set and is defined by

$$\mathbf{it} = \mathrm{dom}\, R \cup \mathrm{rng}\, R.$$

We now state several propositions:

$$\text{(29)} \qquad \mathrm{field}\, R = \mathrm{dom}\, R \cup \mathrm{rng}\, R,$$

$$\text{(30)} \qquad \langle a, b\rangle \in R \textbf{ implies } a \in \mathrm{field}\, R \;\&\; b \in \mathrm{field}\, R,$$

$$\text{(31)} \qquad P \subseteq R \textbf{ implies } \mathrm{field}\, P \subseteq \mathrm{field}\, R,$$

$$\text{(32)} \qquad R = \{\langle x, y\rangle\} \textbf{ implies } \mathrm{field}\, R = \{x, y\},$$

$$\text{(33)} \qquad \mathrm{field}\,(P \cup R) = \mathrm{field}\, P \cup \mathrm{field}\, R,$$

$$\text{(34)} \qquad \mathrm{field}\,(P \cap R) \subseteq \mathrm{field}\, P \cap \mathrm{field}\, R.$$

Let us consider R. The functor

$$R\breve{}\,,$$

yields the type Relation and is defined by

$$\langle x, y\rangle \in \mathbf{it} \textbf{ iff } \langle y, x\rangle \in R.$$

One can prove the following propositions:

$$\text{(35)} \qquad R = P\breve{}\; \textbf{iff for } x{,}y \textbf{ holds } \langle x, y\rangle \in R \textbf{ iff } \langle y, x\rangle \in P,$$

$$\text{(36)} \qquad \langle x, y\rangle \in P\breve{}\; \textbf{iff } \langle y, x\rangle \in P,$$

$$\text{(37)} \qquad (R\breve{}\,)\breve{} = R,$$

$$\text{(38)} \qquad \mathrm{field}\, R = \mathrm{field}\,(R\breve{}\,),$$

$$\text{(39)} \qquad (P \cap R)\breve{} = P\breve{} \cap R\breve{}\,,$$

$$\text{(40)} \qquad (P \cup R)\breve{} = P\breve{} \cup R\breve{}\,,$$

$$\text{(41)} \qquad (P \setminus R)\breve{} = P\breve{} \setminus R\breve{}\,.$$

Let us consider P, R. The functor

$$P \cdot R,$$

with values of the type Relation, is defined by

$$\langle x, y\rangle \in \mathbf{it} \textbf{ iff ex } z \textbf{ st } \langle x, z\rangle \in P \;\&\; \langle z, y\rangle \in R.$$

We now state a number of propositions:

$$\text{(42)} \qquad Q = P \cdot R \textbf{ iff for } x{,}y \textbf{ holds } \langle x, y\rangle \in Q \textbf{ iff ex } z \textbf{ st } \langle x, z\rangle \in P \;\&\; \langle z, y\rangle \in R,$$

(43) $$\langle x, y\rangle \in P \cdot R \textbf{ iff ex } z \textbf{ st } \langle x, z\rangle \in P \,\&\, \langle z, y\rangle \in R,$$

(44) $$\operatorname{dom}(P \cdot R) \subseteq \operatorname{dom} P,$$

(45) $$\operatorname{rng}(P \cdot R) \subseteq \operatorname{rng} R,$$

(46) $$\operatorname{rng} R \subseteq \operatorname{dom} P \textbf{ implies } \operatorname{dom}(R \cdot P) = \operatorname{dom} R,$$

(47) $$\operatorname{dom} P \subseteq \operatorname{rng} R \textbf{ implies } \operatorname{rng}(R \cdot P) = \operatorname{rng} P,$$

(48) $$P \subseteq R \textbf{ implies } Q \cdot P \subseteq Q \cdot R,$$

(49) $$P \subseteq Q \textbf{ implies } P \cdot R \subseteq Q \cdot R,$$

(50) $$P \subseteq R \,\&\, Q \subseteq S \textbf{ implies } P \cdot Q \subseteq R \cdot S,$$

(51) $$P \cdot (R \cup Q) = (P \cdot R) \cup (P \cdot Q),$$

(52) $$P \cdot (R \cap Q) \subseteq (P \cdot R) \cap (P \cdot Q),$$

(53) $$(P \cdot R) \setminus (P \cdot Q) \subseteq P \cdot (R \setminus Q),$$

(54) $$(P \cdot R)\tilde{} = R\tilde{} \cdot P\tilde{},$$

(55) $$(P \cdot R) \cdot Q = P \cdot (R \cdot Q).$$

The constant Ø has the type Relation, and is defined by

$$\textbf{not } \langle x, y\rangle \in \textbf{it}\,.$$

One can prove the following propositions:

(56) $$R = \varnothing \textbf{ iff for } x{,}y \textbf{ holds not } \langle x, y\rangle \in R,$$

(57) $$\textbf{not } \langle x, y\rangle \in \varnothing\,,$$

(58) $$\varnothing \subseteq [:A, B:],$$

(59) $$\varnothing \subseteq R,$$

(60) $$\operatorname{dom} \varnothing = \emptyset \,\&\, \operatorname{rng} \varnothing = \emptyset,$$

(61) $$\varnothing \cap R = \varnothing \,\&\, \varnothing \cup R = R,$$

(62) $$\varnothing \cdot R = \varnothing \,\&\, R \cdot \varnothing = \varnothing\,,$$

(63) $$R \cdot \varnothing = \varnothing \cdot R,$$

(64) $$\operatorname{dom} R = \emptyset \textbf{ or } \operatorname{rng} R = \emptyset \textbf{ implies } R = \varnothing\,,$$

(65) $$\operatorname{dom} R = \emptyset \textbf{ iff } \operatorname{rng} R = \emptyset,$$

(66) $$\varnothing\tilde{} = \varnothing\,,$$

(67) $$\operatorname{rng} R \cap \operatorname{dom} P = \emptyset \textbf{ implies } R \cdot P = \varnothing\,.$$

Let us consider X. The functor

$$\triangle\, X,$$

with values of the type Relation, is defined by

$$\langle x, y\rangle \in \textbf{it} \textbf{ iff } x \in X \;\&\; x = y.$$

The following propositions are true:

(68) $$P = \triangle\, X \textbf{ iff for } x,y \textbf{ holds } \langle x, y\rangle \in P \textbf{ iff } x \in X \;\&\; x = y,$$

(69) $$\langle x, y\rangle \in \triangle\, X \textbf{ iff } x \in X \;\&\; x = y,$$

(70) $$x \in X \textbf{ iff } \langle x, x\rangle \in \triangle\, X,$$

(71) $$\operatorname{dom} \triangle\, X = X \;\&\; \operatorname{rng} \triangle\, X = X,$$

(72) $$(\triangle\, X)\tilde{} = \triangle\, X,$$

(73) $$(\textbf{for } x \textbf{ st } x \in X \textbf{ holds } \langle x, x\rangle \in R) \textbf{ implies } \triangle\, X \subseteq R,$$

(74) $$\langle x, y\rangle \in (\triangle\, X) \cdot R \textbf{ iff } x \in X \;\&\; \langle x, y\rangle \in R,$$

(75) $$\langle x, y\rangle \in R \cdot \triangle\, Y \textbf{ iff } y \in Y \;\&\; \langle x, y\rangle \in R,$$

(76) $$R \cdot (\triangle\, X) \subseteq R \;\&\; (\triangle\, X) \cdot R \subseteq R,$$

(77) $$\operatorname{dom} R \subseteq X \textbf{ implies } (\triangle\, X) \cdot R = R,$$

(78) $$(\triangle \operatorname{dom} R) \cdot R = R,$$

(79) $$\operatorname{rng} R \subseteq Y \textbf{ implies } R \cdot (\triangle\, Y) = R,$$

(80) $$R \cdot (\triangle \operatorname{rng} R) = R,$$

(81) $$\triangle\, \emptyset = \varnothing\,,$$

(82) $$\operatorname{dom} R = X \;\&\; \operatorname{rng} P2 \subseteq X \;\&\; P2 \cdot R = \triangle (\operatorname{dom} P1) \;\&\; R \cdot P1 = \triangle\, X \textbf{ implies } P1 = P2,$$

(83) $$\operatorname{dom} R = X \;\&\; \operatorname{rng} P2 = X \;\&\; P2 \cdot R = \triangle (\operatorname{dom} P1) \;\&\; R \cdot P1 = \triangle\, X \textbf{ implies } P1 = P2.$$

Let us consider R, X. The functor

$$R \mid X,$$

with values of the type Relation, is defined by

$$\langle x,y\rangle \in \textbf{it} \ \textbf{iff} \ x \in X \ \& \ \langle x,y\rangle \in R.$$

We now state a number of propositions:

$$P = R \mid X \ \textbf{iff} \ \textbf{for} \ x,y \ \textbf{holds} \ \langle x,y\rangle \in P \ \textbf{iff} \ x \in X \ \& \ \langle x,y\rangle \in R, \tag{84}$$

$$\langle x,y\rangle \in R \mid X \ \textbf{iff} \ x \in X \ \& \ \langle x,y\rangle \in R, \tag{85}$$

$$x \in \operatorname{dom}(R \mid X) \ \textbf{iff} \ x \in X \ \& \ x \in \operatorname{dom} R, \tag{86}$$

$$\operatorname{dom}(R \mid X) \subseteq X, \tag{87}$$

$$R \mid X \subseteq R, \tag{88}$$

$$\operatorname{dom}(R \mid X) \subseteq \operatorname{dom} R, \tag{89}$$

$$\operatorname{dom}(R \mid X) = \operatorname{dom} R \cap X, \tag{90}$$

$$X \subseteq \operatorname{dom} R \ \textbf{implies} \ \operatorname{dom}(R \mid X) = X, \tag{91}$$

$$(R \mid X) \cdot P \subseteq R \cdot P, \tag{92}$$

$$P \cdot (R \mid X) \subseteq P \cdot R, \tag{93}$$

$$R \mid X = (\triangle X) \cdot R, \tag{94}$$

$$R \mid X = \emptyset \ \textbf{iff} \ (\operatorname{dom} R) \cap X = \emptyset, \tag{95}$$

$$R \mid X = R \cap [\!:X, \operatorname{rng} R:\!], \tag{96}$$

$$\operatorname{dom} R \subseteq X \ \textbf{implies} \ R \mid X = R, \tag{97}$$

$$R \mid \operatorname{dom} R = R, \tag{98}$$

$$\operatorname{rng}(R \mid X) \subseteq \operatorname{rng} R, \tag{99}$$

$$(R \mid X) \mid Y = R \mid (X \cap Y), \tag{100}$$

$$(R \mid X) \mid X = R \mid X, \tag{101}$$

$$X \subseteq Y \ \textbf{implies} \ (R \mid X) \mid Y = R \mid X, \tag{102}$$

$$Y \subseteq X \ \textbf{implies} \ (R \mid X) \mid Y = R \mid Y, \tag{103}$$

$$X \subseteq Y \ \textbf{implies} \ R \mid X \subseteq R \mid Y, \tag{104}$$

$$P \subseteq R \textbf{ implies } P \mid X \subseteq R \mid X, \tag{105}$$

$$P \subseteq R \mathbin{\&} X \subseteq Y \textbf{ implies } P \mid X \subseteq R \mid Y, \tag{106}$$

$$R \mid (X \cup Y) = (R \mid X) \cup (R \mid Y), \tag{107}$$

$$R \mid (X \cap Y) = (R \mid X) \cap (R \mid Y), \tag{108}$$

$$R \mid (X \setminus Y) = R \mid X \setminus R \mid Y, \tag{109}$$

$$R \mid \emptyset = \varnothing\,, \tag{110}$$

$$\varnothing \mid X = \varnothing\,, \tag{111}$$

$$(P \cdot R) \mid X = (P \mid X) \cdot R. \tag{112}$$

Let us consider Y, R. The functor

$$Y \mid R,$$

yields the type Relation and is defined by

$$\langle x, y \rangle \in \textbf{it iff } y \in Y \mathbin{\&} \langle x, y \rangle \in R.$$

The following propositions are true:

$$P = Y \mid R \textbf{ iff for } x{,}y \textbf{ holds } \langle x, y \rangle \in P \textbf{ iff } y \in Y \mathbin{\&} \langle x, y \rangle \in R, \tag{113}$$

$$\langle x, y \rangle \in Y \mid R \textbf{ iff } y \in Y \mathbin{\&} \langle x, y \rangle \in R, \tag{114}$$

$$y \in \operatorname{rng}(Y \mid R) \textbf{ iff } y \in Y \mathbin{\&} y \in \operatorname{rng} R, \tag{115}$$

$$\operatorname{rng}(Y \mid R) \subseteq Y, \tag{116}$$

$$Y \mid R \subseteq R, \tag{117}$$

$$\operatorname{rng}(Y \mid R) \subseteq \operatorname{rng} R, \tag{118}$$

$$\operatorname{rng}(Y \mid R) = \operatorname{rng} R \cap Y, \tag{119}$$

$$Y \subseteq \operatorname{rng} R \textbf{ implies } \operatorname{rng}(Y \mid R) = Y, \tag{120}$$

$$(Y \mid R) \cdot P \subseteq R \cdot P, \tag{121}$$

$$P \cdot (Y \mid R) \subseteq P \cdot R, \tag{122}$$

$$Y \mid R = R \cdot (\triangle\, Y), \tag{123}$$

$$Y \mid R = R \cap [\!:\operatorname{dom} R{,}Y:\!], \tag{124}$$

$$\operatorname{rng} R \subseteq Y \textbf{ implies } Y \mid R = R, \tag{125}$$

$$\operatorname{rng} R \mid R = R, \tag{126}$$

$$Y \mid (X \mid R) = (Y \cap X) \mid R, \tag{127}$$

$$Y \mid (Y \mid R) = Y \mid R, \tag{128}$$

$$X \subseteq Y \textbf{ implies } Y \mid (X \mid R) = X \mid R, \tag{129}$$

$$Y \subseteq X \textbf{ implies } Y \mid (X \mid R) = Y \mid R, \tag{130}$$

$$X \subseteq Y \textbf{ implies } X \mid R \subseteq Y \mid R, \tag{131}$$

$$P1 \subseteq P2 \textbf{ implies } Y \mid P1 \subseteq Y \mid P2, \tag{132}$$

$$P1 \subseteq P2 \ \&\ Y1 \subseteq Y2 \textbf{ implies } Y1 \mid P1 \subseteq Y2 \mid P2, \tag{133}$$

$$(X \cup Y) \mid R = (X \mid R) \cup (Y \mid R), \tag{134}$$

$$(X \cap Y) \mid R = X \mid R \cap Y \mid R, \tag{135}$$

$$(X \setminus Y) \mid R = X \mid R \setminus Y \mid R, \tag{136}$$

$$\emptyset \mid R = \emptyset\,, \tag{137}$$

$$Y \mid \emptyset = \emptyset\,, \tag{138}$$

$$Y \mid (P \cdot R) = P \cdot (Y \mid R), \tag{139}$$

$$(Y \mid R) \mid X = Y \mid (R \mid X). \tag{140}$$

Let us consider R, X. The functor

$$R^{\circ}\, X,$$

yields the type set and is defined by

$$y \in \textbf{it iff ex}\, x \textbf{ st } \langle x, y\rangle \in R \ \&\ x \in X.$$

One can prove the following propositions:

$$Y = R^{\circ}\, X \textbf{ iff for } y \textbf{ holds } y \in Y \textbf{ iff ex}\, x \textbf{ st } \langle x, y\rangle \in R \ \&\ x \in X, \tag{141}$$

$$y \in R^{\circ}\, X \textbf{ iff ex}\, x \textbf{ st } \langle x, y\rangle \in R \ \&\ x \in X, \tag{142}$$

$$y \in R^{\circ}\, X \textbf{ iff ex}\, x \textbf{ st } x \in \operatorname{dom} R \ \&\ \langle x, y\rangle \in R \ \&\ x \in X, \tag{143}$$

$$R^{\circ}\, X \subseteq \operatorname{rng} R, \tag{144}$$

$$R^{\circ}\, X = R^{\circ}\, (\operatorname{dom} R \cap X), \tag{145}$$

$$R^{\circ} \operatorname{dom} R = \operatorname{rng} R, \tag{146}$$

(147) $$R^{\circ} X \subseteq R^{\circ} (\operatorname{dom} R),$$

(148) $$\operatorname{rng}(R \mid X) = R^{\circ} X,$$

(149) $$R^{\circ} \emptyset = \emptyset,$$

(150) $$\varnothing^{\circ} X = \emptyset,$$

(151) $$R^{\circ} X = \emptyset \textbf{ iff } \operatorname{dom} R \cap X = \emptyset,$$

(152) $$X \neq \emptyset \;\&\; X \subseteq \operatorname{dom} R \textbf{ implies } R^{\circ} X \neq \emptyset,$$

(153) $$R^{\circ} (X \cup Y) = R^{\circ} X \cup R^{\circ} Y,$$

(154) $$R^{\circ} (X \cap Y) \subseteq R^{\circ} X \cap R^{\circ} Y,$$

(155) $$R^{\circ} X \setminus R^{\circ} Y \subseteq R^{\circ} (X \setminus Y),$$

(156) $$X \subseteq Y \textbf{ implies } R^{\circ} X \subseteq R^{\circ} Y,$$

(157) $$P \subseteq R \textbf{ implies } P^{\circ} X \subseteq R^{\circ} X,$$

(158) $$P \subseteq R \;\&\; X \subseteq Y \textbf{ implies } P^{\circ} X \subseteq R^{\circ} Y,$$

(159) $$(P \cdot R)^{\circ} X = R^{\circ} (P^{\circ} X),$$

(160) $$\operatorname{rng}(P \cdot R) = R^{\circ} (\operatorname{rng} P),$$

(161) $$(R \mid X)^{\circ} Y \subseteq R^{\circ} Y,$$

(162) $$R \mid X = \varnothing \textbf{ iff } (\operatorname{dom} R) \cap X = \emptyset,$$

(163) $$(\operatorname{dom} R) \cap X \subseteq (R\,\check{})^{\circ} (R^{\circ} X).$$

Let us consider R, Y. The functor

$$R^{-1}\, Y,$$

with values of the type set, is defined by

$$x \in \textbf{it iff ex}\, y \textbf{ st } \langle x, y \rangle \in R \;\&\; y \in Y.$$

Next we state a number of propositions:

(164) $$X = R^{-1}\, Y \textbf{ iff for } x \textbf{ holds } x \in X \textbf{ iff ex}\, y \textbf{ st } \langle x, y \rangle \in R \;\&\; y \in Y,$$

(165) $$x \in R^{-1}\, Y \textbf{ iff ex}\, y \textbf{ st } \langle x, y \rangle \in R \;\&\; y \in Y,$$

(166) $$x \in R^{-1}\, Y \textbf{ iff ex}\, y \textbf{ st } y \in \operatorname{rng} R \;\&\; \langle x, y \rangle \in R \;\&\; y \in Y,$$

(167) $$R^{-1}\, Y \subseteq \operatorname{dom} R,$$

(168) $$R^{-1}\,Y = R^{-1}\,(\operatorname{rng} R \cap Y),$$

(169) $$R^{-1}\,\operatorname{rng} R = \operatorname{dom} R,$$

(170) $$R^{-1}\,Y \subseteq R^{-1}\,\operatorname{rng} R,$$

(171) $$R^{-1}\,\emptyset = \emptyset,$$

(172) $$\emptyset^{-1}\,Y = \emptyset,$$

(173) $$R^{-1}\,Y = \emptyset \textbf{ iff } \operatorname{rng} R \cap Y = \emptyset,$$

(174) $$Y \neq \emptyset \;\&\; Y \subseteq \operatorname{rng} R \textbf{ implies } R^{-1}\,Y \neq \emptyset,$$

(175) $$R^{-1}\,(X \cup Y) = R^{-1}\,X \cup R^{-1}\,Y,$$

(176) $$R^{-1}\,(X \cap Y) \subseteq R^{-1}\,Y \cap R^{-1}\,Y,$$

(177) $$R^{-1}\,X \setminus R^{-1}\,Y \subseteq R^{-1}\,(X \setminus Y),$$

(178) $$X \subseteq Y \textbf{ implies } R^{-1}\,X \subseteq R^{-1}\,Y,$$

(179) $$P \subseteq R \textbf{ implies } P^{-1}\,Y \subseteq R^{-1}\,Y,$$

(180) $$P \subseteq R \;\&\; X \subseteq Y \textbf{ implies } P^{-1}\,X \subseteq R^{-1}\,Y,$$

(181) $$(P \cdot R)^{-1}\,Y = P^{-1}\,(R^{-1}\,Y),$$

(182) $$\operatorname{dom}(P \cdot R) = P^{-1}\,(\operatorname{dom} R),$$

(183) $$(\operatorname{rng} R) \cap Y \subseteq (R\check{})^{-1}\,(R^{-1}\,Y).$$

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Properties of Binary Relations

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Summary. The paper contains definitions of some properties of binary relations: reflexivity, irreflexivity, symmetry, asymmetry, antisymmetry, connectedness, strong connectedness, and transitivity. Basic theorems relating the above mentioned notions are given.

The terminology and notation used here have been introduced in the following articles: [1], [2], and [3]. For simplicity we adopt the following convention: X will have the type set; x, y, z will have the type Any; P, R will have the type Relation. We now define several new predicates. Let us consider R, X. The predicate

R is_reflexive_in X is defined by $x \in X$ **implies** $\langle x, x\rangle \in R$.

The predicate

R is_irreflexive_in X is defined by $x \in X$ **implies not** $\langle x, x\rangle \in R$.

The predicate

R is_symmetric_in X

is defined by

$x \in X$ & $y \in X$ & $\langle x, y\rangle \in R$ **implies** $\langle y, x\rangle \in R$.

The predicate

R is_antisymmetric_in X

is defined by

$x \in X$ & $y \in X$ & $\langle x, y\rangle \in R$ & $\langle y, x\rangle \in R$ **implies** $x = y$.

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The predicate

$$R \text{ is\_asymmetric\_in } X$$

is defined by

$$x \in X \;\&\; y \in X \;\&\; \langle x, y\rangle \in R \textbf{ implies not } \langle y, x\rangle \in R.$$

The predicate

$$R \text{ is\_connected\_in } X$$

is defined by

$$x \in X \;\&\; y \in X \;\&\; x \neq y \textbf{ implies } \langle x, y\rangle \in R \textbf{ or } \langle y, x\rangle \in R.$$

The predicate

$$R \text{ is\_strongly\_connected\_in } X$$

is defined by

$$x \in X \;\&\; y \in X \textbf{ implies } \langle x, y\rangle \in R \textbf{ or } \langle y, x\rangle \in R.$$

The predicate

$$R \text{ is\_transitive\_in } X$$

is defined by

$$x \in X \;\&\; y \in X \;\&\; z \in X \;\&\; \langle x, y\rangle \in R \;\&\; \langle y, z\rangle \in R \textbf{ implies } \langle x, z\rangle \in R.$$

We now state several propositions:

$$R \text{ is\_reflexive\_in } X \textbf{ iff for } x \textbf{ st } x \in X \textbf{ holds } \langle x, x\rangle \in R, \tag{1}$$

$$R \text{ is\_irreflexive\_in } X \textbf{ iff for } x \textbf{ st } x \in X \textbf{ holds not } \langle x, x\rangle \in R, \tag{2}$$

$$\begin{gathered} R \text{ is\_symmetric\_in } X \\ \textbf{iff for } x{,}y \textbf{ st } x \in X \;\&\; y \in X \;\&\; \langle x, y\rangle \in R \textbf{ holds } \langle y, x\rangle \in R, \end{gathered} \tag{3}$$

$$\begin{gathered} R \text{ is\_antisymmetric\_in } X \\ \textbf{iff for } x{,}y \textbf{ st } x \in X \;\&\; y \in X \;\&\; \langle x, y\rangle \in R \;\&\; \langle y, x\rangle \in R \textbf{ holds } x = y, \end{gathered} \tag{4}$$

$$\begin{gathered} R \text{ is\_asymmetric\_in } X \\ \textbf{iff for } x{,}y \textbf{ st } x \in X \;\&\; y \in X \;\&\; \langle x, y\rangle \in R \textbf{ holds not } \langle y, x\rangle \in R, \end{gathered} \tag{5}$$

$$\begin{gathered} R \text{ is\_connected\_in } X \\ \textbf{iff for } x{,}y \textbf{ st } x \in X \;\&\; y \in X \;\&\; x \neq y \textbf{ holds } \langle x, y\rangle \in R \textbf{ or } \langle y, x\rangle \in R, \end{gathered} \tag{6}$$

$$\begin{gathered} R \text{ is\_strongly\_connected\_in } X \\ \textbf{iff for } x{,}y \textbf{ st } x \in X \;\&\; y \in X \textbf{ holds } \langle x, y\rangle \in R \textbf{ or } \langle y, x\rangle \in R, \end{gathered} \tag{7}$$

(8) $$R \text{ is\_transitive\_in } X \textbf{ iff for } x,y,z \\ \textbf{st } x \in X \;\&\; y \in X \;\&\; z \in X \;\&\; \langle x, y\rangle \in R \;\&\; \langle y, z\rangle \in R \textbf{ holds } \langle x, z\rangle \in R.$$

We now define several new predicates. Let us consider R. The predicate

R is_reflexive is defined by R is_reflexive_in field R.

The predicate

R is_irreflexive is defined by R is_irreflexive_in field R.

The predicate

R is_symmetric is defined by R is_symmetric_in field R.

The predicate

R is_antisymmetric is defined by R is_antisymmetric_in field R.

The predicate

R is_asymmetric is defined by R is_asymmetric_in field R.

The predicate

R is_connected is defined by R is_connected_in field R.

The predicate

R is_strongly_connected is defined by R is_strongly_connected_in field R.

The predicate

R is_transitive is defined by R is_transitive_in field R.

We now state a number of propositions:

(9) R is_reflexive **iff** R is_reflexive_in field R,

(10) R is_irreflexive **iff** R is_irreflexive_in field R,

(11) R is_symmetric **iff** R is_symmetric_in field R,

(12) R is_antisymmetric **iff** R is_antisymmetric_in field R,

(13) R is_asymmetric **iff** R is_asymmetric_in field R,

(14) R is_connected **iff** R is_connected_in field R,

(15) R is_strongly_connected **iff** R is_strongly_connected_in field R,

(16) R is_transitive **iff** R is_transitive_in field R,

$$R \,\text{is\_reflexive}\ \textbf{iff}\ \triangle\,\text{field}\,R \subseteq R, \tag{17}$$

$$R \,\text{is\_irreflexive}\ \textbf{iff}\ \triangle\,(\text{field}\,R) \cap R = \emptyset\,, \tag{18}$$

$$R\ \text{is\_antisymmetric\_in}\ X\ \textbf{iff}\ R \setminus \triangle\,X\ \text{is\_asymmetric\_in}\ X, \tag{19}$$

$$R\ \text{is\_asymmetric\_in}\ X\ \textbf{implies}\ R \cup \triangle\,X\ \text{is\_antisymmetric\_in}\ X, \tag{20}$$

$$R\ \text{is\_antisymmetric\_in}\ X\ \textbf{implies}\ R \setminus \triangle\,X\ \text{is\_asymmetric\_in}\ X, \tag{21}$$

$$R \,\text{is\_symmetric}\ \&\ R \,\text{is\_transitive}\ \textbf{implies}\ R \,\text{is\_reflexive}\,, \tag{22}$$

$$\triangle\,X\ \text{is\_symmetric}\ \&\ \triangle\,X\ \text{is\_transitive}\,, \tag{23}$$

$$\triangle\,X\ \text{is\_antisymmetric}\ \&\ \triangle\,X\ \text{is\_reflexive}\,, \tag{24}$$

$$R \,\text{is\_irreflexive}\ \&\ R \,\text{is\_transitive}\ \textbf{implies}\ R \,\text{is\_asymmetric}\,, \tag{25}$$

$$R \,\text{is\_asymmetric}\ \textbf{implies}\ R \,\text{is\_irreflexive}\ \&\ R \,\text{is\_antisymmetric}\,, \tag{26}$$

$$R \,\text{is\_reflexive}\ \textbf{implies}\ R\tilde{}\ \text{is\_reflexive}\,, \tag{27}$$

$$R \,\text{is\_irreflexive}\ \textbf{implies}\ R\tilde{}\ \text{is\_irreflexive}\,, \tag{28}$$

$$R \,\text{is\_reflexive}\ \textbf{implies}\ \text{dom}\,R = \text{dom}\,(R\tilde{})\ \&\ \text{rng}\,R = \text{rng}\,(R\tilde{}), \tag{29}$$

$$R \,\text{is\_symmetric}\ \textbf{iff}\ R = R\tilde{}, \tag{30}$$

$$P \,\text{is\_reflexive}\ \&\ R \,\text{is\_reflexive}\ \textbf{implies}\ P \cup R \,\text{is\_reflexive}\ \&\ P \cap R \,\text{is\_reflexive}\,, \tag{31}$$

$$\begin{gathered} P \,\text{is\_irreflexive}\ \&\ R \,\text{is\_irreflexive} \\ \textbf{implies}\ P \cup R \,\text{is\_irreflexive}\ \&\ P \cap R \,\text{is\_irreflexive}\,, \end{gathered} \tag{32}$$

$$P \,\text{is\_irreflexive}\ \textbf{implies}\ P \setminus R \,\text{is\_irreflexive}\,, \tag{33}$$

$$R \,\text{is\_symmetric}\ \textbf{implies}\ R\tilde{}\ \text{is\_symmetric}\,, \tag{34}$$

$$\begin{gathered} P \,\text{is\_symmetric}\ \&\ R \,\text{is\_symmetric} \\ \textbf{implies}\ P \cup R \,\text{is\_symmetric}\ \&\ P \cap R \,\text{is\_symmetric}\ \&\ P \setminus R \,\text{is\_symmetric}\,, \end{gathered} \tag{35}$$

$$R \,\text{is\_asymmetric}\ \textbf{implies}\ R\tilde{}\ \text{is\_asymmetric}\,, \tag{36}$$

$$P \,\text{is\_asymmetric}\ \&\ R \,\text{is\_asymmetric}\ \textbf{implies}\ P \cap R \,\text{is\_asymmetric}\,, \tag{37}$$

$$P \,\text{is\_asymmetric}\ \textbf{implies}\ P \setminus R \,\text{is\_asymmetric}\,, \tag{38}$$

$$R \,\text{is\_antisymmetric}\ \textbf{iff}\ R \cap (R\tilde{}) \subseteq \triangle\,(\text{dom}\,R), \tag{39}$$

$$R \,\text{is\_antisymmetric}\ \textbf{implies}\ R\tilde{}\ \text{is\_antisymmetric}\,, \tag{40}$$

$$P \text{ is\_antisymmetric} \\ \textbf{implies } P \cap R \text{ is\_antisymmetric} \ \& \ P \setminus R \text{ is\_antisymmetric}, \tag{41}$$

$$R \text{ is\_transitive } \textbf{implies } R\check{} \text{ is\_transitive}, \tag{42}$$

$$P \text{ is\_transitive} \ \& \ R \text{ is\_transitive } \textbf{implies } P \cap R \text{ is\_transitive}, \tag{43}$$

$$R \text{ is\_transitive } \textbf{iff } R \cdot R \subseteq R, \tag{44}$$

$$R \text{ is\_connected } \textbf{iff } [:\text{field } R, \text{field } R:] \setminus \triangle(\text{field } R) \subseteq R \cup R\check{}, \tag{45}$$

$$R \text{ is\_strongly\_connected } \textbf{implies } R \text{ is\_connected} \ \& \ R \text{ is\_reflexive}, \tag{46}$$

$$R \text{ is\_strongly\_connected } \textbf{iff } [:\text{field } R, \text{field } R:] = R \cup R\check{}. \tag{47}$$

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The Ordinal Numbers

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Summary. In the beginning of article we show some consequences of the regularity axiom. In the second part we introduce the successor of a set and the notions of transitivity and connectedness wrt membership relation. Then we define ordinal numbers as transitive and connected sets, and we prove some theorems of them and of their sets. Lastly we introduce the concept of a transfinite sequence and we show transfinite induction and schemes of defining by transfinite induction.

The notation and terminology used in this paper have been introduced in the following articles: [2], [3], and [1]. For simplicity we adopt the following convention: X, Y, Z, A, B, $X1$, $X2$, $X3$, $X4$, $X5$, $X6$ will denote objects of the type set; x will denote an object of the type Any. Next we state several propositions:

$$\textbf{not}\, X \in X, \tag{1}$$

$$\textbf{not}\, (X \in Y \,\&\, Y \in X), \tag{2}$$

$$\textbf{not}\, (X \in Y \,\&\, Y \in Z \,\&\, Z \in X), \tag{3}$$

$$\textbf{not}\, (X1 \in X2 \,\&\, X2 \in X3 \,\&\, X3 \in X4 \,\&\, X4 \in X1), \tag{4}$$

$$\textbf{not}\, (X1 \in X2 \,\&\, X2 \in X3 \,\&\, X3 \in X4 \,\&\, X4 \in X5 \,\&\, X5 \in X1), \tag{5}$$

$$\textbf{not}\, (X1 \in X2 \,\&\, X2 \in X3 \,\&\, X3 \in X4 \,\&\, X4 \in X5 \,\&\, X5 \in X6 \,\&\, X6 \in X1), \tag{6}$$

$$Y \in X \textbf{ implies not } X \subseteq Y. \tag{7}$$

The scheme *Comprehension* deals with a constant $\mathcal{A}$ that has the type set and a unary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex}\, B \textbf{ st for } Z \textbf{ being } \mathrm{set} \textbf{ holds } Z \in B \textbf{ iff } Z \in \mathcal{A} \,\&\, \mathcal{P}[Z]$$

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for all values of the parameters.

One can prove the following proposition

$$(\mathbf{for}\ X\ \mathbf{holds}\ X \in A\ \mathbf{iff}\ X \in B)\ \mathbf{implies}\ A = B. \tag{8}$$

Let us consider X. The functor

$$\operatorname{succ} X,$$

with values of the type set, is defined by

$$\mathbf{it} = X \cup \{X\}.$$

Next we state several propositions:

$$\operatorname{succ} X = X \cup \{X\}, \tag{9}$$

$$X \in \operatorname{succ} X, \tag{10}$$

$$\operatorname{succ} X \neq \emptyset, \tag{11}$$

$$\operatorname{succ} X = \operatorname{succ} Y\ \mathbf{implies}\ X = Y, \tag{12}$$

$$x \in \operatorname{succ} X\ \mathbf{iff}\ x \in X\ \mathbf{or}\ x = X, \tag{13}$$

$$X \neq \operatorname{succ} X. \tag{14}$$

For simplicity we adopt the following convention: a has the type Any; X, Y, Z, x, y have the type set. We now define two new predicates. Let us consider X. The predicate

$$X \text{ is\_}\in\text{-transitive} \qquad \text{is defined by} \qquad \mathbf{for}\ x\ \mathbf{st}\ x \in X\ \mathbf{holds}\ x \subseteq X.$$

The predicate

$$X \text{ is\_}\in\text{-connected}$$

is defined by

$$\mathbf{for}\ x,y\ \mathbf{st}\ x \in X\ \&\ y \in X\ \mathbf{holds}\ x \in y\ \mathbf{or}\ x = y\ \mathbf{or}\ y \in x.$$

One can prove the following two propositions:

$$X \text{ is\_}\in\text{-transitive}\ \mathbf{iff}\ \mathbf{for}\ x\ \mathbf{st}\ x \in X\ \mathbf{holds}\ x \subseteq X, \tag{15}$$

$$X \text{ is\_}\in\text{-connected}\ \mathbf{iff}\ \mathbf{for}\ x,y\ \mathbf{st}\ x \in X\ \&\ y \in X\ \mathbf{holds}\ x \in y\ \mathbf{or}\ x = y\ \mathbf{or}\ y \in x. \tag{16}$$

The mode

$$\text{Ordinal},$$

which widens to the type set, is defined by

$$\mathbf{it} \text{ is\_}\in\text{-transitive}\ \&\ \mathbf{it} \text{ is\_}\in\text{-connected}.$$

In the sequel A, B, C will have the type Ordinal. The following propositions are true:

(17) $$X \textbf{ is } \mathrm{Ordinal} \textbf{ iff } X \text{ is\_}{\in}\text{-transitive} \ \& \ X \text{ is\_}{\in}\text{-connected}\,,$$

(18) $$x \in A \textbf{ implies } x \subseteq A,$$

(19) $$A \in B \ \& \ B \in C \textbf{ implies } A \in C,$$

(20) $$x \in A \ \& \ y \in A \textbf{ implies } x \in y \textbf{ or } x = y \textbf{ or } y \in x,$$

(21) $$\textbf{for } x,A \textbf{ being } \mathrm{Ordinal} \textbf{ st } x \subseteq A \ \& \ x \neq A \textbf{ holds } x \in A,$$

(22) $$A \subseteq B \ \& \ B \in C \textbf{ implies } A \in C,$$

(23) $$a \in A \textbf{ implies } a \textbf{ is } \mathrm{Ordinal}\,,$$

(24) $$A \in B \textbf{ or } A = B \textbf{ or } B \in A,$$

(25) $$A \subseteq B \textbf{ or } B \subseteq A,$$

(26) $$A \subseteq B \textbf{ or } B \in A,$$

(27) $$\emptyset \textbf{ is } \mathrm{Ordinal}\,.$$

The constant $\mathbf{0}$ has the type Ordinal, and is defined by

$$\textbf{it} = \emptyset.$$

Next we state three propositions:

(28) $$\mathbf{0} = \emptyset,$$

(29) $$x \textbf{ is } \mathrm{Ordinal} \textbf{ implies } \mathrm{succ}\, x \textbf{ is } \mathrm{Ordinal}\,,$$

(30) $$x \textbf{ is } \mathrm{Ordinal} \textbf{ implies } \bigcup x \textbf{ is } \mathrm{Ordinal}\,.$$

Let us consider A. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$\begin{array}{ccc} \mathrm{succ}\, A & \text{is} & \mathrm{Ordinal}\,, \\ \bigcup A & \text{is} & \mathrm{Ordinal}\,. \end{array}$$

One can prove the following propositions:

(31) $$(\textbf{for } x \textbf{ st } x \in X \textbf{ holds } x \textbf{ is } \mathrm{Ordinal} \ \& \ x \subseteq X) \textbf{ implies } X \textbf{ is } \mathrm{Ordinal}\,,$$

(32) $$X \subseteq A \ \& \ X \neq \emptyset \textbf{ implies ex } C \textbf{ st } C \in X \ \& \textbf{ for } B \textbf{ st } B \in X \textbf{ holds } C \subseteq B,$$

(33) $$A \in B \textbf{ iff } \mathrm{succ}\, A \subseteq B,$$

$$A \in \operatorname{succ} C \textbf{ iff } A \subseteq C. \tag{34}$$

Now we present two schemes. The scheme *Ordinal_Min* concerns a unary predicate $\mathcal{P}$ states that the following holds

$$\textbf{ex}\, A \textbf{ st } \mathcal{P}[A] \ \&\ \textbf{for}\, B \textbf{ st } \mathcal{P}[B] \textbf{ holds } A \subseteq B$$

provided the parameter satisfies the following condition:

- $$\textbf{ex}\, A \textbf{ st } \mathcal{P}[A].$$

The scheme *Transfinite_Ind* concerns a unary predicate $\mathcal{P}$ states that the following holds

$$\textbf{for}\, A \textbf{ holds } \mathcal{P}[A]$$

provided the parameter satisfies the following condition:

- $$\textbf{for}\, A \textbf{ st } \textbf{for}\, C \textbf{ st } C \in A \textbf{ holds } \mathcal{P}[C] \textbf{ holds } \mathcal{P}[A].$$

One can prove the following propositions:

$$\textbf{for}\, X \textbf{ st } \textbf{for}\, a \textbf{ st } a \in X \textbf{ holds } a \textbf{ is } \text{Ordinal} \textbf{ holds } \bigcup X \textbf{ is } \text{Ordinal}, \tag{35}$$

$$\textbf{for}\, X \textbf{ st } \textbf{for}\, a \textbf{ st } a \in X \textbf{ holds } a \textbf{ is } \text{Ordinal}\, \textbf{ex}\, A \textbf{ st } X \subseteq A, \tag{36}$$

$$\textbf{not}\, \textbf{ex}\, X \textbf{ st } \textbf{for}\, x \textbf{ holds } x \in X \textbf{ iff } x \textbf{ is } \text{Ordinal}, \tag{37}$$

$$\textbf{not}\, \textbf{ex}\, X \textbf{ st } \textbf{for}\, A \textbf{ holds } A \in X, \tag{38}$$

$$\textbf{for}\, X\, \textbf{ex}\, A \textbf{ st } \textbf{not}\, A \in X \ \&\ \textbf{for}\, B \textbf{ st } \textbf{not}\, B \in X \textbf{ holds } A \subseteq B. \tag{39}$$

Let us consider A. The predicate

$$A \,\text{is\_limit\_ordinal} \qquad \text{is defined by} \qquad A = \bigcup A.$$

One can prove the following three propositions:

$$A \,\text{is\_limit\_ordinal} \textbf{ iff } A = \bigcup A, \tag{40}$$

$$\textbf{for}\, A \textbf{ holds } A \,\text{is\_limit\_ordinal} \textbf{ iff } \textbf{for}\, C \textbf{ st } C \in A \textbf{ holds } \operatorname{succ} C \in A, \tag{41}$$

$$\textbf{not}\, A \,\text{is\_limit\_ordinal} \textbf{ iff } \textbf{ex}\, B \textbf{ st } A = \operatorname{succ} B. \tag{42}$$

In the sequel F denotes an object of the type Function. The mode

$$\text{Transfinite-Sequence},$$

which widens to the type Function, is defined by

$$\textbf{ex}\, A \textbf{ st } \operatorname{dom} \textbf{it} = A.$$

Let us consider Z. The mode

$$\text{Transfinite-Sequence } \mathbf{of}\ Z,$$

which widens to the type Transfinite-Sequence, is defined by

$$\operatorname{rng} \mathbf{it} \subseteq Z.$$

The following propositions are true:

(43) $$F\ \mathbf{is}\ \text{Transfinite-Sequence}\ \mathbf{iff}\ \mathbf{ex}\ A\ \mathbf{st}\ \operatorname{dom} F = A,$$

(44) $$F\ \mathbf{is}\ \text{Transfinite-Sequence}\ \mathbf{of}\ Z\ \mathbf{iff}\ F\ \mathbf{is}\ \text{Transfinite-Sequence}\ \&\ \operatorname{rng} F \subseteq Z,$$

(45) $$\emptyset\ \mathbf{is}\ \text{Transfinite-Sequence}\ \mathbf{of}\ Z.$$

In the sequel L, $L1$, $L2$ will have the type Transfinite-Sequence. The following proposition is true

(46) $$\operatorname{dom} F\ \mathbf{is}\ \text{Ordinal}\ \mathbf{implies}\ F\ \mathbf{is}\ \text{Transfinite-Sequence}\ \mathbf{of}\ \operatorname{rng} F.$$

Let us consider L. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\operatorname{dom} L \qquad \text{is} \qquad \text{Ordinal}\,.$$

We now state a proposition

(47) $$X \subseteq Y\ \mathbf{implies}$$
$$\mathbf{for}\ L\ \mathbf{being}\ \text{Transfinite-Sequence}\ \mathbf{of}\ X\ \mathbf{holds}\ L\ \mathbf{is}\ \text{Transfinite-Sequence}\ \mathbf{of}\ Y.$$

Let us consider L, A. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$L \mid A \qquad \text{is} \qquad \text{Transfinite-Sequence}\ \mathbf{of}\ \operatorname{rng} L.$$

The following two propositions are true:

(48) $$\mathbf{for}\ L\ \mathbf{being}\ \text{Transfinite-Sequence}\ \mathbf{of}\ X$$
$$\mathbf{for}\ A\ \mathbf{holds}\ L \mid A\ \mathbf{is}\ \text{Transfinite-Sequence}\ \mathbf{of}\ X,$$

(49) $$(\mathbf{for}\ a\ \mathbf{st}\ a \in X\ \mathbf{holds}\ a\ \mathbf{is}\ \text{Transfinite-Sequence})\ \&\ (\mathbf{for}\ L1,L2$$
$$\mathbf{st}\ L1 \in X\ \&\ L2 \in X\ \mathbf{holds}\ \operatorname{graph} L1 \subseteq \operatorname{graph} L2\ \mathbf{or}\ \operatorname{graph} L2 \subseteq \operatorname{graph} L1)$$
$$\mathbf{implies} \bigcup X\ \mathbf{is}\ \text{Transfinite-Sequence}\,.$$

Now we present three schemes. The scheme *TS_Uniq* deals with a constant $\mathcal{A}$ that has the type Ordinal, a unary functor $\mathcal{F}$, a constant $\mathcal{B}$ that has the type Transfinite-Sequence and a constant $\mathcal{C}$ that has the type Transfinite-Sequence, and states that the following holds

$$\mathcal{B} = \mathcal{C}$$

provided the parameters satisfy the following conditions:

- $\mathrm{dom}\,\mathcal{B} = \mathcal{A}$ & **for** B,L **st** $B \in \mathcal{A}$ & $L = \mathcal{B} \mid B$ **holds** $\mathcal{B}.B = \mathcal{F}(L)$,
- $\mathrm{dom}\,\mathcal{C} = \mathcal{A}$ & **for** B,L **st** $B \in \mathcal{A}$ & $L = \mathcal{C} \mid B$ **holds** $\mathcal{C}.B = \mathcal{F}(L)$.

The scheme *TS_Exist* deals with a constant $\mathcal{A}$ that has the type Ordinal and a unary functor $\mathcal{F}$ and states that the following holds

ex L **st** $\mathrm{dom}\,L = \mathcal{A}$ & **for** B,$L1$ **st** $B \in \mathcal{A}$ & $L1 = L \mid B$ **holds** $L.B = \mathcal{F}(L1)$

for all values of the parameters.

The scheme *Func_TS* concerns a constant $\mathcal{A}$ that has the type Transfinite-Sequence, a unary functor $\mathcal{F}$ and a unary functor $\mathcal{G}$ and states that the following holds

for B **st** $B \in \mathrm{dom}\,\mathcal{A}$ **holds** $\mathcal{A}.B = \mathcal{G}(\mathcal{A} \mid B)$

provided the parameters satisfy the following conditions:

- **for** A,a **holds** $a = \mathcal{F}(A)$
  iff **ex** L **st** $a = \mathcal{G}(L)$ & $\mathrm{dom}\,L = A$ & **for** B **st** $B \in A$ **holds** $L.B = \mathcal{G}(L \mid B)$,
- **for** A **st** $A \in \mathrm{dom}\,\mathcal{A}$ **holds** $\mathcal{A}.A = \mathcal{F}(A)$.

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Tuples, Projections and Cartesian Products

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Summary. The purpose of this article is to define projections of ordered pairs, and to introduce triples and quadruples, and their projections. The theorems in this paper may be roughly divided into two groups: theorems describing basic properties of introduced concepts and theorems related to the regularity, analogous to those proved for ordered pairs by Cz. Byliński [1]. Cartesian products of subsets are redefined as subsets of Cartesian products.

The notation and terminology used here are introduced in the following papers: [3], [4], and [2]. For simplicity we adopt the following convention: v, x, $x1$, $x2$, $x3$, $x4$, y, $y1$, $y2$, $y3$, $y4$, z denote objects of the type Any; X, $X1$, $X2$, $X3$, $X4$, Y, $Y1$, $Y2$, $Y3$, $Y4$, $Y5$, Z denote objects of the type set. One can prove the following propositions:

$$X \neq \emptyset \textbf{ implies } \textbf{ex}\, Y \textbf{ st } Y \in X \mathbin{\&} Y \text{ misses } X, \tag{1}$$

$$X \neq \emptyset \textbf{ implies } \textbf{ex}\, Y \textbf{ st } Y \in X \mathbin{\&} \textbf{for}\, Y1 \textbf{ st } Y1 \in Y \textbf{ holds } Y1 \text{ misses } X, \tag{2}$$

$$X \neq \emptyset \textbf{ implies } \textbf{ex}\, Y \textbf{ st } Y \in X \mathbin{\&} \textbf{for}\, Y1,Y2 \textbf{ st } Y1 \in Y2 \mathbin{\&} Y2 \in Y \textbf{ holds } Y1 \text{ misses } X, \tag{3}$$

$$X \neq \emptyset \textbf{ implies } \textbf{ex}\, Y \textbf{ st } Y \in X \mathbin{\&} \textbf{for}\, Y1,Y2,Y3 \textbf{ st } Y1 \in Y2 \mathbin{\&} Y2 \in Y3 \mathbin{\&} Y3 \in Y \textbf{ holds } Y1 \text{ misses } X, \tag{4}$$

$$X \neq \emptyset \textbf{ implies } \textbf{ex}\, Y \textbf{ st } Y \in X \mathbin{\&} \textbf{for}\, Y1,Y2,Y3,Y4 \textbf{ st } Y1 \in Y2 \mathbin{\&} Y2 \in Y3 \mathbin{\&} Y3 \in Y4 \mathbin{\&} Y4 \in Y \textbf{ holds } Y1 \text{ misses } X, \tag{5}$$

$$X \neq \emptyset \textbf{ implies } \textbf{ex}\, Y \textbf{ st } Y \in X \mathbin{\&} \textbf{for}\, Y1,Y2,Y3,Y4,Y5 \textbf{ st } Y1 \in Y2 \mathbin{\&} Y2 \in Y3 \mathbin{\&} Y3 \in Y4 \mathbin{\&} Y4 \in Y5 \mathbin{\&} Y5 \in Y \textbf{ holds } Y1 \text{ misses } X. \tag{6}$$

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We now define two new functors. Let us consider x. Assume there exist $x1$, $x2$, of the type Any such that

$$x = \langle x1, x2 \rangle.$$

The functor

$$x_{\mathbf{1}},$$

is defined by

$$x = \langle y1, y2 \rangle \textbf{ implies it} = y1.$$

The functor

$$x_{\mathbf{2}},$$

is defined by

$$x = \langle y1, y2 \rangle \textbf{ implies it} = y2.$$

We now state a number of propositions:

$$\langle x, y \rangle_{\mathbf{1}} = x \;\&\; \langle x, y \rangle_{\mathbf{2}} = y, \tag{7}$$

$$(\textbf{ex}\, x,y \textbf{ st } z = \langle x, y \rangle) \textbf{ implies } \langle z_{\mathbf{1}}, z_{\mathbf{2}} \rangle = z, \tag{8}$$

$$X \neq \emptyset \textbf{ implies ex}\, v \textbf{ st } v \in X \;\&\; \textbf{not ex}\, x,y \textbf{ st } (x \in X \textbf{ or } y \in X) \;\&\; v = \langle x, y \rangle, \tag{9}$$

$$z \in [:X, Y:] \textbf{ implies } z_{\mathbf{1}} \in X \;\&\; z_{\mathbf{2}} \in Y, \tag{10}$$

$$(\textbf{ex}\, x,y \textbf{ st } z = \langle x, y \rangle) \;\&\; z_{\mathbf{1}} \in X \;\&\; z_{\mathbf{2}} \in Y \textbf{ implies } z \in [:X, Y:], \tag{11}$$

$$z \in [:\{x\}, Y:] \textbf{ implies } z_{\mathbf{1}} = x \;\&\; z_{\mathbf{2}} \in Y, \tag{12}$$

$$z \in [:X, \{y\}:] \textbf{ implies } z_{\mathbf{1}} \in X \;\&\; z_{\mathbf{2}} = y, \tag{13}$$

$$z \in [:\{x\}, \{y\}:] \textbf{ implies } z_{\mathbf{1}} = x \;\&\; z_{\mathbf{2}} = y, \tag{14}$$

$$z \in [:\{x1, x2\}, Y:] \textbf{ implies } (z_{\mathbf{1}} = x1 \textbf{ or } z_{\mathbf{1}} = x2) \;\&\; z_{\mathbf{2}} \in Y, \tag{15}$$

$$z \in [:X, \{y1, y2\}:] \textbf{ implies } z_{\mathbf{1}} \in X \;\&\; (z_{\mathbf{2}} = y1 \textbf{ or } z_{\mathbf{2}} = y2), \tag{16}$$

$$z \in [:\{x1, x2\}, \{y\}:] \textbf{ implies } (z_{\mathbf{1}} = x1 \textbf{ or } z_{\mathbf{1}} = x2) \;\&\; z_{\mathbf{2}} = y, \tag{17}$$

$$z \in [:\{x\}, \{y1, y2\}:] \textbf{ implies } z_{\mathbf{1}} = x \;\&\; (z_{\mathbf{2}} = y1 \textbf{ or } z_{\mathbf{2}} = y2), \tag{18}$$

$$\begin{gathered} z \in [:\{x1, x2\}, \{y1, y2\}:] \\ \textbf{implies } (z_{\mathbf{1}} = x1 \textbf{ or } z_{\mathbf{1}} = x2) \;\&\; (z_{\mathbf{2}} = y1 \textbf{ or } z_{\mathbf{2}} = y2), \end{gathered} \tag{19}$$

$$(\textbf{ex}\, y,z \textbf{ st } x = \langle y, z \rangle) \textbf{ implies } x \neq x_{\mathbf{1}} \;\&\; x \neq x_{\mathbf{2}}. \tag{20}$$

In the sequel xx will have the type Element **of** X; yy will have the type Element **of** Y. One can prove the following propositions:

$$X \neq \emptyset \mathbin{\&} Y \neq \emptyset \textbf{ implies } \langle xx{,}yy \rangle \in [\!: X, Y :\!], \tag{21}$$

$$X \neq \emptyset \mathbin{\&} Y \neq \emptyset \textbf{ implies } \langle xx{,}yy \rangle \textbf{ is } \text{Element } \textbf{of } [\!: X, Y :\!], \tag{22}$$

$$x \in [\!: X, Y :\!] \textbf{ implies } x = \langle x_{\mathbf{1}}, x_{\mathbf{2}} \rangle, \tag{23}$$

$$X \neq \emptyset \mathbin{\&} Y \neq \emptyset \textbf{ implies for } x \textbf{ being } \text{Element } \textbf{of } [\!: X, Y :\!] \textbf{ holds } x = \langle x_{\mathbf{1}}, x_{\mathbf{2}} \rangle, \tag{24}$$

$$[\!: \{x1{,}x2\}, \{y1{,}y2\} :\!] = \{\langle x1{,}y1 \rangle, \langle x1{,}y2 \rangle, \langle x2{,}y1 \rangle, \langle x2{,}y2 \rangle\}, \tag{25}$$

$$X \neq \emptyset \mathbin{\&} Y \neq \emptyset \tag{26}$$
$$\textbf{implies for } x \textbf{ being } \text{Element } \textbf{of } [\!: X, Y :\!] \textbf{ holds } x \neq x_{\mathbf{1}} \mathbin{\&} x \neq x_{\mathbf{2}}.$$

Let us consider $x1$, $x2$, $x3$. The functor

$$\langle x1{,}x2{,}x3 \rangle,$$

is defined by

$$\textbf{it} = \langle \langle x1{,}x2 \rangle{,}x3 \rangle.$$

One can prove the following three propositions:

$$\langle x1{,}x2{,}x3 \rangle = \langle \langle x1{,}x2 \rangle{,}x3 \rangle, \tag{27}$$

$$\langle x1{,}x2{,}x3 \rangle = \langle y1{,}y2{,}y3 \rangle \textbf{ implies } x1 = y1 \mathbin{\&} x2 = y2 \mathbin{\&} x3 = y3, \tag{28}$$

$$X \neq \emptyset \tag{29}$$
$$\textbf{implies ex } v \textbf{ st } v \in X \mathbin{\&} \textbf{not ex } x{,}y{,}z \textbf{ st } (x \in X \textbf{ or } y \in X) \mathbin{\&} v = \langle x, y, z \rangle.$$

Let us consider $x1$, $x2$, $x3$, $x4$. The functor

$$\langle x1{,}x2{,}x3{,}x4 \rangle,$$

is defined by

$$\textbf{it} = \langle \langle x1{,}x2{,}x3 \rangle{,}x4 \rangle.$$

The following propositions are true:

$$\langle x1{,}x2{,}x3{,}x4 \rangle = \langle \langle x1{,}x2{,}x3 \rangle{,}x4 \rangle, \tag{30}$$

$$\langle x1{,}x2{,}x3{,}x4 \rangle = \langle \langle \langle x1{,}x2 \rangle{,}x3 \rangle{,}x4 \rangle, \tag{31}$$

$$\langle x1{,}x2{,}x3{,}x4 \rangle = \langle \langle x1{,}x2 \rangle{,}x3{,}x4 \rangle, \tag{32}$$

$$\langle x1{,}x2{,}x3{,}x4 \rangle = \langle y1{,}y2{,}y3{,}y4 \rangle \tag{33}$$
$$\textbf{implies } x1 = y1 \mathbin{\&} x2 = y2 \mathbin{\&} x3 = y3 \mathbin{\&} x4 = y4,$$

$$X \neq \emptyset \ \textbf{implies} \ \textbf{ex} \ v \ \textbf{st} \ v \in X \ \& \ \textbf{not} \ \textbf{ex} \ x1,x2,x3,x4 \ \textbf{st} \ (x1 \in X \ \textbf{or} \ x2 \in X) \ \& \ v = \langle x1,x2,x3,x4 \rangle, \tag{34}$$

$$X1 \neq \emptyset \ \& \ X2 \neq \emptyset \ \& \ X3 \neq \emptyset \ \textbf{iff} \ [:X1,X2,X3:] \neq \emptyset. \tag{35}$$

In the sequel $xx1$ has the type Element **of** $X1$; $xx2$ has the type Element **of** $X2$; $xx3$ has the type Element **of** $X3$. One can prove the following propositions:

$$X1 \neq \emptyset \ \& \ X2 \neq \emptyset \ \& \ X3 \neq \emptyset \ \textbf{implies} \ ([:X1,X2,X3:] = [:Y1,Y2,Y3:] \ \textbf{implies} \ X1 = Y1 \ \& \ X2 = Y2 \ \& \ X3 = Y3), \tag{36}$$

$$[:X1,X2,X3:] \neq \emptyset \ \& \ [:X1,X2,X3:] = [:Y1,Y2,Y3:] \ \textbf{implies} \ X1 = Y1 \ \& \ X2 = Y2 \ \& \ X3 = Y3, \tag{37}$$

$$[:X, X, X:] = [:Y, Y, Y:] \ \textbf{implies} \ X = Y, \tag{38}$$

$$[:\{x1\},\{x2\},\{x3\}:] = \{\langle x1,x2,x3 \rangle\}, \tag{39}$$

$$[:\{x1,y1\},\{x2\},\{x3\}:] = \{\langle x1,x2,x3 \rangle,\langle y1,x2,x3 \rangle\}, \tag{40}$$

$$[:\{x1\},\{x2,y2\},\{x3\}:] = \{\langle x1,x2,x3 \rangle,\langle x1,y2,x3 \rangle\}, \tag{41}$$

$$[:\{x1\},\{x2\},\{x3,y3\}:] = \{\langle x1,x2,x3 \rangle,\langle x1,x2,y3 \rangle\}, \tag{42}$$

$$[:\{x1,y1\},\{x2,y2\},\{x3\}:] = \{\langle x1,x2,x3 \rangle,\langle y1,x2,x3 \rangle,\langle x1,y2,x3 \rangle,\langle y1,y2,x3 \rangle\}, \tag{43}$$

$$[:\{x1,y1\},\{x2\},\{x3,y3\}:] = \{\langle x1,x2,x3 \rangle,\langle y1,x2,x3 \rangle,\langle x1,x2,y3 \rangle,\langle y1,x2,y3 \rangle\}, \tag{44}$$

$$[:\{x1\},\{x2,y2\},\{x3,y3\}:] = \{\langle x1,x2,x3 \rangle,\langle x1,y2,x3 \rangle,\langle x1,x2,y3 \rangle,\langle x1,y2,y3 \rangle\}, \tag{45}$$

$$[:\{x1,y1\},\{x2,y2\},\{x3,y3\}:] = \{\langle x1,x2,x3 \rangle, \langle x1,y2,x3 \rangle,\langle x1,x2,y3 \rangle,\langle x1,y2,y3 \rangle,\langle y1,x2,x3 \rangle,\langle y1,y2,x3 \rangle,\langle y1,x2,y3 \rangle,\langle y1,y2,y3 \rangle\}. \tag{46}$$

We now define three new functors. Let us consider $X1$, $X2$, $X3$. Assume that the following holds

$$X1 \neq \emptyset \ \& \ X2 \neq \emptyset \ \& \ X3 \neq \emptyset.$$

Let x have the type Element **of** $[:X1,X2,X3:]$. The functor

$$x_{\mathbf{1}}\,,$$

with values of the type Element **of** $X1$, is defined by

$$x = \langle x1,x2,x3 \rangle \ \textbf{implies} \ \textbf{it} = x1.$$

The functor

$$x_{\mathbf{2}}\,,$$

yields the type Element **of** $X2$ and is defined by

$$x = \langle x1,x2,x3\rangle \textbf{ implies it} = x2.$$

The functor

$$x_{\mathbf{3}},$$

with values of the type Element **of** $X3$, is defined by

$$x = \langle x1,x2,x3\rangle \textbf{ implies it} = x3.$$

One can prove the following propositions:

(47) $X1 \neq \emptyset$ & $X2 \neq \emptyset$ & $X3 \neq \emptyset$ **implies for** x **being** Element **of** $[:X1,X2,X3:]$ **for** $x1,x2,x3$ **st** $x = \langle x1,x2,x3\rangle$ **holds** $x_{\mathbf{1}} = x1$ & $x_{\mathbf{2}} = x2$ & $x_{\mathbf{3}} = x3$,

(48) $X1 \neq \emptyset$ & $X2 \neq \emptyset$ & $X3 \neq \emptyset$ **implies for** x **being** Element **of** $[:X1,X2,X3:]$ **holds** $x = \langle x_{\mathbf{1}},x_{\mathbf{2}},x_{\mathbf{3}}\rangle$,

(49) $X \subseteq [:X,Y,Z:]$ **or** $X \subseteq [:Y,Z,X:]$ **or** $X \subseteq [:Z,X,Y:]$ **implies** $X = \emptyset$,

(50) $X1 \neq \emptyset$ & $X2 \neq \emptyset$ & $X3 \neq \emptyset$ **implies for** x **being** Element **of** $[:X1,X2,X3:]$ **holds** $x_{\mathbf{1}} = (x$ **qua** Any$)_{\mathbf{1}\,\mathbf{1}}$ & $x_{\mathbf{2}} = (x$ **qua** Any$)_{\mathbf{1}\,\mathbf{2}}$ & $x_{\mathbf{3}} = (x$ **qua** Any$)_{\mathbf{2}}$,

(51) $X1 \neq \emptyset$ & $X2 \neq \emptyset$ & $X3 \neq \emptyset$ **implies for** x **being** Element **of** $[:X1,X2,X3:]$ **holds** $x \neq x_{\mathbf{1}}$ & $x \neq x_{\mathbf{2}}$ & $x \neq x_{\mathbf{3}}$,

(52) $[:X1,X2,X3:]$ meets $[:Y1,Y2,Y3:]$ **implies** $X1$ meets $Y1$ & $X2$ meets $Y2$ & $X3$ meets $Y3$,

(53) $[:X1,X2,X3,X4:] = [:[:[:X1,X2:],X3:],X4:]$,

(54) $[:[:X1,X2:],X3,X4:] = [:X1,X2,X3,X4:]$,

(55) $X1 \neq \emptyset$ & $X2 \neq \emptyset$ & $X3 \neq \emptyset$ & $X4 \neq \emptyset$ **iff** $[:X1,X2,X3,X4:] \neq \emptyset$,

(56) $X1 \neq \emptyset$ & $X2 \neq \emptyset$ & $X3 \neq \emptyset$ & $X4 \neq \emptyset$ **implies** $([:X1,X2,X3,X4:] = [:Y1,Y2,Y3,Y4:]$ **implies** $X1 = Y1$ & $X2 = Y2$ & $X3 = Y3$ & $X4 = Y4)$,

(57) $[:X1,X2,X3,X4:] \neq \emptyset$ & $[:X1,X2,X3,X4:] = [:Y1,Y2,Y3,Y4:]$ **implies** $X1 = Y1$ & $X2 = Y2$ & $X3 = Y3$ & $X4 = Y4$,

(58) $[:X,X,X,X:] = [:Y,Y,Y,Y:]$ **implies** $X = Y$.

In the sequel $xx4$ will have the type Element **of** $X4$. We now define four new functors. Let us consider $X1$, $X2$, $X3$, $X4$. Assume that the following holds

$$X1 \neq \emptyset \;\&\; X2 \neq \emptyset \;\&\; X3 \neq \emptyset \;\&\; X4 \neq \emptyset.$$

Let x have the type Element **of** $[:X1,X2,X3,X4:]$. The functor

$$x_{\mathbf{1}},$$

yields the type Element **of** $X1$ and is defined by

$$x = \langle x1,x2,x3,x4\rangle \ \textbf{implies it} = x1.$$

The functor

$$x_{\mathbf{2}},$$

with values of the type Element **of** $X2$, is defined by

$$x = \langle x1,x2,x3,x4\rangle \ \textbf{implies it} = x2.$$

The functor

$$x_{\mathbf{3}},$$

yields the type Element **of** $X3$ and is defined by

$$x = \langle x1,x2,x3,x4\rangle \ \textbf{implies it} = x3.$$

The functor

$$x_{\mathbf{4}},$$

with values of the type Element **of** $X4$, is defined by

$$x = \langle x1,x2,x3,x4\rangle \ \textbf{implies it} = x4.$$

Next we state several propositions:

(59)
$$X1 \neq \emptyset \ \& \ X2 \neq \emptyset \ \& \ X3 \neq \emptyset \ \& \ X4 \neq \emptyset \ \textbf{implies}$$
$$\textbf{for}\ x\ \textbf{being}\ \text{Element}\ \textbf{of}\ [:X1,X2,X3,X4:]\ \textbf{for}\ x1,x2,x3,x4$$
$$\textbf{st}\ x = \langle x1,x2,x3,x4\rangle\ \textbf{holds}\ x_{\mathbf{1}} = x1 \ \& \ x_{\mathbf{2}} = x2 \ \& \ x_{\mathbf{3}} = x3 \ \& \ x_{\mathbf{4}} = x4,$$

(60)
$$X1 \neq \emptyset \ \& \ X2 \neq \emptyset \ \& \ X3 \neq \emptyset \ \& \ X4 \neq \emptyset$$
$$\textbf{implies for}\ x\ \textbf{being}\ \text{Element}\ \textbf{of}\ [:X1,X2,X3,X4:]\ \textbf{holds}\ x = \langle x_{\mathbf{1}},x_{\mathbf{2}},x_{\mathbf{3}},x_{\mathbf{4}}\rangle,$$

(61)
$$X1 \neq \emptyset \ \& \ X2 \neq \emptyset \ \& \ X3 \neq \emptyset \ \& \ X4 \neq \emptyset \ \textbf{implies}$$
$$\textbf{for}\ x\ \textbf{being}\ \text{Element}\ \textbf{of}\ [:X1,X2,X3,X4:]\ \textbf{holds}\ x_{\mathbf{1}} = (x\ \textbf{qua}\ \text{Any})_{\mathbf{1}\,\mathbf{1}\,\mathbf{1}}$$
$$\&\ x_{\mathbf{2}} = (x\ \textbf{qua}\ \text{Any})_{\mathbf{1}\,\mathbf{1}\,\mathbf{2}}\ \&\ x_{\mathbf{3}} = (x\ \textbf{qua}\ \text{Any})_{\mathbf{1}\,\mathbf{2}}\ \&\ x_{\mathbf{4}} = (x\ \textbf{qua}\ \text{Any})_{\mathbf{2}},$$

(62)
$$X1 \neq \emptyset \ \& \ X2 \neq \emptyset \ \& \ X3 \neq \emptyset \ \& \ X4 \neq \emptyset \ \textbf{implies}$$
$$\textbf{for}\ x\ \textbf{being}\ \text{Element}\ \textbf{of}\ [:X1,X2,X3,X4:]$$
$$\textbf{holds}\ x \neq x_{\mathbf{1}}\ \&\ x \neq x_{\mathbf{2}}\ \&\ x \neq x_{\mathbf{3}}\ \&\ x \neq x_{\mathbf{4}},$$

(63)
$$X1 \subseteq [:X1,X2,X3,X4:]\ \textbf{or}$$
$$X1 \subseteq [:X2,X3,X4,X1:]\ \textbf{or}\ X1 \subseteq [:X3,X4,X1,X2:]\ \textbf{or}\ X1 \subseteq [:X4,X1,X2,X3:]$$
$$\textbf{implies}\ X1 = \emptyset,$$

(64) $[:X1,X2,X3,X4:]$ meets $[:Y1,Y2,Y3,Y4:]$
implies $X1$ meets $Y1$ & $X2$ meets $Y2$ & $X3$ meets $Y3$ & $X4$ meets $Y4$,

(65) $[:\{x1\},\{x2\},\{x3\},\{x4\}:] = \{\langle x1,x2,x3,x4\rangle\}$,

(66) $[:X,Y:] \neq \emptyset$ **implies for** x **being** Element **of** $[:X,Y:]$ **holds** $x \neq x_{\mathbf{1}}$ & $x \neq x_{\mathbf{2}}$,

(67) $x \in [:X,Y:]$ **implies** $x \neq x_{\mathbf{1}}$ & $x \neq x_{\mathbf{2}}$.

For simplicity we adopt the following convention: $A1$ will denote an object of the type Subset **of** $X1$; $A2$ will denote an object of the type Subset **of** $X2$; $A3$ will denote an object of the type Subset **of** $X3$; $A4$ will denote an object of the type Subset **of** $X4$; x will denote an object of the type Element **of** $[:X1,X2,X3:]$. We now state a number of propositions:

(68) $X1 \neq \emptyset$ & $X2 \neq \emptyset$ & $X3 \neq \emptyset$ **implies**
for $x1,x2,x3$ **st** $x = \langle x1,x2,x3\rangle$ **holds** $x_{\mathbf{1}} = x1$ & $x_{\mathbf{2}} = x2$ & $x_{\mathbf{3}} = x3$,

(69) $X1 \neq \emptyset$ &
$X2 \neq \emptyset$ & $X3 \neq \emptyset$ & (**for** $xx1,xx2,xx3$ **st** $x = \langle xx1,xx2,xx3\rangle$ **holds** $y1 = xx1$)
implies $y1 = x_{\mathbf{1}}$,

(70) $X1 \neq \emptyset$ &
$X2 \neq \emptyset$ & $X3 \neq \emptyset$ & (**for** $xx1,xx2,xx3$ **st** $x = \langle xx1,xx2,xx3\rangle$ **holds** $y2 = xx2$)
implies $y2 = x_{\mathbf{2}}$,

(71) $X1 \neq \emptyset$ &
$X2 \neq \emptyset$ & $X3 \neq \emptyset$ & (**for** $xx1,xx2,xx3$ **st** $x = \langle xx1,xx2,xx3\rangle$ **holds** $y3 = xx3$)
implies $y3 = x_{\mathbf{3}}$,

(72) $z \in [:X1,X2,X3:]$
implies ex $x1,x2,x3$ **st** $x1 \in X1$ & $x2 \in X2$ & $x3 \in X3$ & $z = \langle x1,x2,x3\rangle$,

(73) $\langle x1,x2,x3\rangle \in [:X1,X2,X3:]$ **iff** $x1 \in X1$ & $x2 \in X2$ & $x3 \in X3$,

(74) (**for** z **holds**
$z \in Z$ **iff ex** $x1,x2,x3$ **st** $x1 \in X1$ & $x2 \in X2$ & $x3 \in X3$ & $z = \langle x1,x2,x3\rangle$)
implies $Z = [:X1,X2,X3:]$,

(75) $X1 \neq \emptyset$ & $X2 \neq \emptyset$ & $X3 \neq \emptyset$ & $Y1 \neq \emptyset$ & $Y2 \neq \emptyset$ & $Y3 \neq \emptyset$ **implies**
for x **being** Element **of** $[:X1,X2,X3:]$, y **being** Element **of** $[:Y1,Y2,Y3:]$
holds $x = y$ **implies** $x_{\mathbf{1}} = y_{\mathbf{1}}$ & $x_{\mathbf{2}} = y_{\mathbf{2}}$ & $x_{\mathbf{3}} = y_{\mathbf{3}}$,

$$\textbf{for}\ x\ \textbf{being}\ \text{Element}\ \textbf{of}\ [:X1,X2,X3:]\ \textbf{st}\ x \in [:A1,A2,A3:]\ \textbf{holds}\ x_{\mathbf{1}} \in A1\ \&\ x_{\mathbf{2}} \in A2\ \&\ x_{\mathbf{3}} \in A3, \tag{76}$$

$$X1 \subseteq Y1\ \&\ X2 \subseteq Y2\ \&\ X3 \subseteq Y3\ \textbf{implies}\ [:X1,X2,X3:] \subseteq [:Y1,Y2,Y3:]. \tag{77}$$

In the sequel x has the type Element **of** $[:X1,X2,X3,X4:]$. We now state a number of propositions:

$$X1 \neq \emptyset\ \&\ X2 \neq \emptyset\ \&\ X3 \neq \emptyset\ \&\ X4 \neq \emptyset\ \textbf{implies for}\ x1,x2,x3,x4\ \textbf{st}\ x = \langle x1,x2,x3,x4\rangle\ \textbf{holds}\ x_{\mathbf{1}} = x1\ \&\ x_{\mathbf{2}} = x2\ \&\ x_{\mathbf{3}} = x3\ \&\ x_{\mathbf{4}} = x4, \tag{78}$$

$$X1 \neq \emptyset\ \&\ X2 \neq \emptyset\ \&\ X3 \neq \emptyset\ \&\ X4 \neq \emptyset\ \&\ (\textbf{for}\ xx1,xx2,xx3,xx4\ \textbf{st}\ x = \langle xx1,xx2,xx3,xx4\rangle\ \textbf{holds}\ y1 = xx1)\ \textbf{implies}\ y1 = x_{\mathbf{1}}, \tag{79}$$

$$X1 \neq \emptyset\ \&\ X2 \neq \emptyset\ \&\ X3 \neq \emptyset\ \&\ X4 \neq \emptyset\ \&\ (\textbf{for}\ xx1,xx2,xx3,xx4\ \textbf{st}\ x = \langle xx1,xx2,xx3,xx4\rangle\ \textbf{holds}\ y2 = xx2)\ \textbf{implies}\ y2 = x_{\mathbf{2}}, \tag{80}$$

$$X1 \neq \emptyset\ \&\ X2 \neq \emptyset\ \&\ X3 \neq \emptyset\ \&\ X4 \neq \emptyset\ \&\ (\textbf{for}\ xx1,xx2,xx3,xx4\ \textbf{st}\ x = \langle xx1,xx2,xx3,xx4\rangle\ \textbf{holds}\ y3 = xx3)\ \textbf{implies}\ y3 = x_{\mathbf{3}}, \tag{81}$$

$$X1 \neq \emptyset\ \&\ X2 \neq \emptyset\ \&\ X3 \neq \emptyset\ \&\ X4 \neq \emptyset\ \&\ (\textbf{for}\ xx1,xx2,xx3,xx4\ \textbf{st}\ x = \langle xx1,xx2,xx3,xx4\rangle\ \textbf{holds}\ y4 = xx4)\ \textbf{implies}\ y4 = x_{\mathbf{4}}, \tag{82}$$

$$z \in [:X1,X2,X3,X4:]\ \textbf{implies ex}\ x1,x2,x3,x4\ \textbf{st}\ x1 \in X1\ \&\ x2 \in X2\ \&\ x3 \in X3\ \&\ x4 \in X4\ \&\ z = \langle x1,x2,x3,x4\rangle, \tag{83}$$

$$\langle x1,x2,x3,x4\rangle \in [:X1,X2,X3,X4:]\ \textbf{iff}\ x1 \in X1\ \&\ x2 \in X2\ \&\ x3 \in X3\ \&\ x4 \in X4, \tag{84}$$

$$(\textbf{for}\ z\ \textbf{holds}\ z \in Z\ \textbf{iff ex}\ x1,x2,x3,x4\ \textbf{st}\ x1 \in X1\ \&\ x2 \in X2\ \&\ x3 \in X3\ \&\ x4 \in X4\ \&\ z = \langle x1,x2,x3,x4\rangle)\ \textbf{implies}\ Z = [:X1,X2,X3,X4:], \tag{85}$$

$$X1 \neq \emptyset\ \&\ X2 \neq \emptyset\ \&\ X3 \neq \emptyset\ \&\ X4 \neq \emptyset\ \&\ Y1 \neq \emptyset\ \&\ Y2 \neq \emptyset\ \&\ Y3 \neq \emptyset\ \&\ Y4 \neq \emptyset\ \textbf{implies for}\ x\ \textbf{being}\ \text{Element}\ \textbf{of}\ [:X1,X2,X3,X4:], y\ \textbf{being}\ \text{Element}\ \textbf{of}\ [:Y1,Y2,Y3,Y4:]\ \textbf{holds}\ x = y\ \textbf{implies}\ x_{\mathbf{1}} = y_{\mathbf{1}}\ \&\ x_{\mathbf{2}} = y_{\mathbf{2}}\ \&\ x_{\mathbf{3}} = y_{\mathbf{3}}\ \&\ x_{\mathbf{4}} = y_{\mathbf{4}}, \tag{86}$$

(87) $$\textbf{for}\ x\ \textbf{being}\ \text{Element}\ \textbf{of}\ [:X1,X2,X3,X4:]$$
$$\textbf{st}\ x \in [:A1,A2,A3,A4:]\ \textbf{holds}\ x_{\mathbf{1}} \in A1\ \&\ x_{\mathbf{2}} \in A2\ \&\ x_{\mathbf{3}} \in A3\ \&\ x_{\mathbf{4}} \in A4,$$

(88) $$X1 \subseteq Y1\ \&\ X2 \subseteq Y2\ \&\ X3 \subseteq Y3\ \&\ X4 \subseteq Y4$$
$$\textbf{implies}\ [:X1,X2,X3,X4:] \subseteq [:Y1,Y2,Y3,Y4:].$$

Let us consider $X1$, $X2$, $A1$, $A2$. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$[:A1,A2:] \quad \text{is} \quad \text{Subset}\ \textbf{of}\ [:X1,X2:].$$

Let us consider $X1$, $X2$, $X3$, $A1$, $A2$, $A3$. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$[:A1,A2,A3:] \quad \text{is} \quad \text{Subset}\ \textbf{of}\ [:X1,X2,X3:].$$

Let us consider $X1$, $X2$, $X3$, $X4$, $A1$, $A2$, $A3$, $A4$. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$[:A1,A2,A3,A4:] \quad \text{is} \quad \text{Subset}\ \textbf{of}\ [:X1,X2,X3,X4:].$$

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Segments of Natural Numbers and Finite Sequences

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Summary. We define the notion of an initial segment of natural numbers and prove a number of their properties. Using this notion we introduce finite sequences, subsequences, the empty sequence, a sequence of a domain, and the operation of concatenation of two sequences.

The papers [4], [5], [2], [3], and [1] provide the notation and terminology for this paper. For simplicity we adopt the following convention: k, l, m, n, $k1$, $k2$ denote objects of the type Nat; X denotes an object of the type set; x, y, z, $y1$, $y2$ denote objects of the type Any; f denotes an object of the type Function. Let us consider n. The functor

$$\mathrm{Seg}\, n,$$

with values of the type set, is defined by

$$\mathbf{it} = \{\, k : 1 \leq k \ \&\ k \leq n \,\}.$$

Let us consider n. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\mathrm{Seg}\, n \qquad \text{is} \qquad \mathbf{set\ of}\ \mathrm{Nat}\,.$$

One can prove the following propositions:

(1) $$\mathrm{Seg}\, n = \{\, k : 1 \leq k \ \&\ k \leq n \,\},$$

(2) $$x \in \mathrm{Seg}\, n \ \mathbf{implies}\ x \ \mathbf{is}\ \mathrm{Nat}\,,$$

(3) $$k \in \mathrm{Seg}\, n \ \mathbf{iff}\ 1 \leq k \ \&\ k \leq n,$$

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$$\text{Seg}\,0 = \emptyset \mathrel{\&} \text{Seg}\,1 = \{1\} \mathrel{\&} \text{Seg}\,2 = \{1,2\}, \tag{4}$$

$$n = 0 \textbf{ or } n \in \text{Seg}\,n, \tag{5}$$

$$n + 1 \in \text{Seg}\,(n + 1), \tag{6}$$

$$n \le m \textbf{ iff } \text{Seg}\,n \subseteq \text{Seg}\,m, \tag{7}$$

$$\text{Seg}\,n = \text{Seg}\,m \textbf{ implies } n = m, \tag{8}$$

$$k \le n \textbf{ implies } \text{Seg}\,k = \text{Seg}\,k \cap \text{Seg}\,n \mathrel{\&} \text{Seg}\,k = \text{Seg}\,n \cap \text{Seg}\,k, \tag{9}$$

$$\text{Seg}\,k = \text{Seg}\,k \cap \text{Seg}\,n \textbf{ or } \text{Seg}\,k = \text{Seg}\,n \cap \text{Seg}\,k \textbf{ implies } k \le n, \tag{10}$$

$$\text{Seg}\,n \cup \{n + 1\} = \text{Seg}\,(n + 1). \tag{11}$$

The mode

$$\text{FinSequence}\,,$$

which widens to the type Function, is defined by

$$\textbf{ex}\,n \textbf{ st } \text{dom}\,\textbf{it} = \text{Seg}\,n.$$

In the sequel p, q, r denote objects of the type FinSequence. Let us consider p. The functor

$$\text{len}\,p,$$

with values of the type Nat, is defined by

$$\text{Seg}\,\textbf{it} = \text{dom}\,p.$$

Next we state four propositions:

$$\textbf{for } f \textbf{ being } \text{Function } \textbf{holds } f \textbf{ is } \text{FinSequence } \textbf{iff } \textbf{ex}\,n \textbf{ st } \text{dom}\,f = \text{Seg}\,n, \tag{12}$$

$$k = \text{len}\,p \textbf{ iff } \text{Seg}\,k = \text{dom}\,p, \tag{13}$$

$$\emptyset \textbf{ is } \text{FinSequence}\,, \tag{14}$$

$$(\textbf{ex}\,k \textbf{ st } \text{dom}\,f \subseteq \text{Seg}\,k) \textbf{ implies } \textbf{ex}\,p \textbf{ st } \text{graph}\,f \subseteq \text{graph}\,p. \tag{15}$$

In the article we present several logical schemes. The scheme *SeqEx* concerns a constant $\mathcal{A}$ that has the type Nat and a binary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex}\,p \textbf{ st } \text{dom}\,p = \text{Seg}\,\mathcal{A} \mathrel{\&} \textbf{for } k \textbf{ st } k \in \text{Seg}\,\mathcal{A} \textbf{ holds } \mathcal{P}[k, p.k]$$

provided the parameters satisfy the following conditions:

- $\textbf{for } k,y1,y2 \textbf{ st } k \in \text{Seg}\,\mathcal{A} \mathrel{\&} \mathcal{P}[k, y1] \mathrel{\&} \mathcal{P}[k, y2] \textbf{ holds } y1 = y2,$

- $$\textbf{for}\, k\ \textbf{st}\ k \in \operatorname{Seg} \mathcal{A}\ \textbf{ex}\, x\ \textbf{st}\ \mathcal{P}[k, x].$$

The scheme *SeqLambda* deals with a constant $\mathcal{A}$ that has the type Nat and a unary functor $\mathcal{F}$ and states that the following holds

$$\textbf{ex}\, p\ \textbf{being}\ \text{FinSequence}\ \textbf{st}\ \operatorname{len} p = \mathcal{A}\ \&\ \textbf{for}\, k\ \textbf{st}\ k \in \operatorname{Seg} \mathcal{A}\ \textbf{holds}\ p.k = \mathcal{F}(k)$$

for all values of the parameters.

We now state several propositions:

(16) $$z \in \operatorname{graph} p\ \textbf{implies}\ \textbf{ex}\, k\ \textbf{st}\ k \in \operatorname{dom} p\ \&\ z = \langle k, p.k \rangle,$$

(17) $$X = \operatorname{dom} p\ \&\ X = \operatorname{dom} q\ \&\ (\textbf{for}\, k\ \textbf{st}\ k \in X\ \textbf{holds}\ p.k = q.k)\ \textbf{implies}\ p = q,$$

(18) $$\textbf{for}\, p, q$$
$$\textbf{st}\ \operatorname{len} p = \operatorname{len} q\ \&\ \textbf{for}\, k\ \textbf{st}\ 1 \le k\ \&\ k \le \operatorname{len} p\ \textbf{holds}\ p.k = q.k\ \textbf{holds}\ p = q,$$

(19) $$p \mid (\operatorname{Seg} n)\ \textbf{is}\ \text{FinSequence}\,,$$

(20) $$\operatorname{rng} p \subseteq \operatorname{dom} f\ \textbf{implies}\ f \cdot p\ \textbf{is}\ \text{FinSequence}\,,$$

(21) $$k \le \operatorname{len} p\ \&\ q = p \mid (\operatorname{Seg} k)\ \textbf{implies}\ \operatorname{len} q = k\ \&\ \operatorname{dom} q = \operatorname{Seg} k.$$

Let D have the type DOMAIN. The mode

$$\text{FinSequence}\ \textbf{of}\ D,$$

which widens to the type FinSequence, is defined by

$$\operatorname{rng} \textbf{it} \subseteq D.$$

In the sequel D will have the type DOMAIN. The following three propositions are true:

(22) $$p\ \textbf{is}\ \text{FinSequence}\ \textbf{of}\ D\ \textbf{iff}\ \operatorname{rng} p \subseteq D,$$

(23) $$\textbf{for}\, D, k\ \textbf{for}\, p\ \textbf{being}\ \text{FinSequence}\ \textbf{of}\ D\ \textbf{holds}\ p \mid (\operatorname{Seg} k)\ \textbf{is}\ \text{FinSequence}\ \textbf{of}\ D,$$

(24) $$\textbf{ex}\, p\ \textbf{being}\ \text{FinSequence}\ \textbf{of}\ D\ \textbf{st}\ \operatorname{len} p = k.$$

The constant ε has the type FinSequence, and is defined by

$$\operatorname{len} \textbf{it} = 0.$$

The following propositions are true:

(25) $$p = \varepsilon\ \textbf{iff}\ \operatorname{len} p = 0,$$

(26) $$p = \varepsilon\ \textbf{iff}\ \operatorname{dom} p = \emptyset,$$

(27) $$p = \varepsilon\ \textbf{iff}\ \operatorname{rng} p = \emptyset,$$

(28) $$\mathrm{graph}\,\varepsilon = \emptyset,$$

(29) $$\textbf{for}\ D\ \textbf{holds}\ \varepsilon\ \textbf{is}\ \mathrm{FinSequence}\ \textbf{of}\ D.$$

Let D have the type DOMAIN. The functor

$$\varepsilon\, D,$$

yields the type FinSequence **of** D and is defined by

$$\textbf{it} = \varepsilon\,.$$

One can prove the following four propositions:

(30) $$p = \varepsilon\,(D)\ \textbf{iff}\ \mathrm{dom}\,p = \emptyset,$$

(31) $$\varepsilon\,(D) = \varepsilon\,,$$

(32) $$p = \varepsilon\,(D)\ \textbf{iff}\ \mathrm{len}\,p = 0,$$

(33) $$p = \varepsilon\,(D)\ \textbf{iff}\ \mathrm{rng}\,p = \emptyset.$$

Let us consider p, q. The functor

$$p \frown q,$$

with values of the type FinSequence, is defined by

$$\mathrm{dom}\,\textbf{it} = \mathrm{Seg}\,(\mathrm{len}\,p + \mathrm{len}\,q)\ \&$$
$$(\textbf{for}\ k\ \textbf{st}\ k \in \mathrm{dom}\,p\ \textbf{holds}\ \textbf{it}.k = p.k)\ \&\ \textbf{for}\ k\ \textbf{st}\ k \in \mathrm{dom}\,q\ \textbf{holds}\ \textbf{it}.(\mathrm{len}\,p + k) = q.k.$$

One can prove the following propositions:

(34) $$r = p \frown q\ \textbf{iff}\ \mathrm{dom}\,r = \mathrm{Seg}\,(\mathrm{len}\,p + \mathrm{len}\,q)\ \&$$
$$(\textbf{for}\ k\ \textbf{st}\ k \in \mathrm{dom}\,p\ \textbf{holds}\ r.k = p.k)$$
$$\&\ \textbf{for}\ k\ \textbf{st}\ k \in \mathrm{dom}\,q\ \textbf{holds}\ r.(\mathrm{len}\,p + k) = q.k,$$

(35) $$\mathrm{len}\,(p \frown q) = \mathrm{len}\,p + \mathrm{len}\,q,$$

(36) $$\textbf{for}\ k\ \textbf{st}\ \mathrm{len}\,p + 1 \le k\ \&\ k \le \mathrm{len}\,p + \mathrm{len}\,q\ \textbf{holds}\ (p \frown q).k = q.(k - \mathrm{len}\,p),$$

(37) $$\mathrm{len}\,p < k\ \&\ k \le \mathrm{len}\,(p \frown q)\ \textbf{implies}\ (p \frown q).k = q.(k - \mathrm{len}\,p),$$

(38) $$k \in \mathrm{dom}\,(p \frown q)\ \textbf{implies}\ k \in \mathrm{dom}\,p\ \textbf{or}\ \textbf{ex}\ n\ \textbf{st}\ n \in \mathrm{dom}\,q\ \&\ k = \mathrm{len}\,p + n,$$

(39) $$\mathrm{dom}\,p \subseteq \mathrm{dom}\,(p \frown q),$$

(40) $$x \in \mathrm{dom}\,q\ \textbf{implies}\ \textbf{ex}\ k\ \textbf{st}\ k = x\ \&\ \mathrm{len}\,p + k \in \mathrm{dom}\,(p \frown q),$$

(41) $$k \in \mathrm{dom}\,q\ \textbf{implies}\ \mathrm{len}\,p + k \in \mathrm{dom}\,(p \frown q),$$

(42) $$\operatorname{rng} p \subseteq \operatorname{rng}(p \frown q),$$

(43) $$\operatorname{rng} q \subseteq \operatorname{rng}(p \frown q),$$

(44) $$\operatorname{rng}(p \frown q) = \operatorname{rng} p \cup \operatorname{rng} q,$$

(45) $$p \frown q \frown r = p \frown (q \frown r),$$

(46) $$p \frown r = q \frown r \textbf{ or } r \frown p = r \frown q \textbf{ implies } p = q,$$

(47) $$p \frown \varepsilon = p \ \& \ \varepsilon \frown p = p,$$

(48) $$p \frown q = \varepsilon \textbf{ implies } p = \varepsilon \ \& \ q = \varepsilon\,.$$

The arguments of the notions defined below are the following: D which is an object of the type reserved above; p, q which are objects of the type FinSequence **of** D. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$p \frown q \qquad \text{is} \qquad \text{FinSequence } \textbf{of } D.$$

One can prove the following proposition

(49) $$\textbf{for } p,q \textbf{ being } \text{FinSequence } \textbf{of } D \textbf{ holds } p \frown q \textbf{ is } \text{FinSequence } \textbf{of } D.$$

Let us consider x. The functor

$$<x>,$$

with values of the type FinSequence, is defined by

$$\operatorname{dom} \textbf{it} = \operatorname{Seg} 1 \ \& \ \textbf{it}.1 = x.$$

The following proposition is true

(50) $$p \frown q \textbf{ is } \text{FinSequence } \textbf{of } D$$
$$\textbf{implies } p \textbf{ is } \text{FinSequence } \textbf{of } D \ \& \ q \textbf{ is } \text{FinSequence } \textbf{of } D.$$

We now define two new functors. Let us consider x, y. The functor

$$<x,y>,$$

with values of the type FinSequence, is defined by

$$\textbf{it} = <x> \frown <y>.$$

Let us consider z. The functor

$$<x,y,z>,$$

with values of the type FinSequence, is defined by

$$\textbf{it} = <x> \frown <y> \frown <z>.$$

Next we state a number of propositions:

(51) $$p = \mathord{<}x\mathord{>} \textbf{ iff } \operatorname{dom} p = \operatorname{Seg} 1 \ \&\ p.1 = x,$$

(52) $$\operatorname{graph} \mathord{<}x\mathord{>} = \{\langle 1,x\rangle\},$$

(53) $$\mathord{<}x, y\mathord{>} = \mathord{<}x\mathord{>} \frown \mathord{<}y\mathord{>},$$

(54) $$\mathord{<}x, y, z\mathord{>} = \mathord{<}x\mathord{>} \frown \mathord{<}y\mathord{>} \frown \mathord{<}z\mathord{>},$$

(55) $$p = \mathord{<}x\mathord{>} \textbf{ iff } \operatorname{dom} p = \operatorname{Seg} 1 \ \&\ \operatorname{rng} p = \{x\},$$

(56) $$p = \mathord{<}x\mathord{>} \textbf{ iff } \operatorname{len} p = 1 \ \&\ \operatorname{rng} p = \{x\},$$

(57) $$p = \mathord{<}x\mathord{>} \textbf{ iff } \operatorname{len} p = 1 \ \&\ p.1 = x,$$

(58) $$(\mathord{<}x\mathord{>} \frown p).1 = x,$$

(59) $$(p \frown \mathord{<}x\mathord{>}).(\operatorname{len} p + 1) = x,$$

(60) $$\mathord{<}x, y, z\mathord{>} = \mathord{<}x\mathord{>} \frown \mathord{<}y, z\mathord{>} \ \&\ \mathord{<}x, y, z\mathord{>} = \mathord{<}x, y\mathord{>} \frown \mathord{<}z\mathord{>},$$

(61) $$p = \mathord{<}x, y\mathord{>} \textbf{ iff } \operatorname{len} p = 2 \ \&\ p.1 = x \ \&\ p.2 = y,$$

(62) $$p = \mathord{<}x, y, z\mathord{>} \textbf{ iff } \operatorname{len} p = 3 \ \&\ p.1 = x \ \&\ p.2 = y \ \&\ p.3 = z,$$

(63) $$\textbf{for } p \textbf{ st } p \neq \varepsilon \textbf{ ex } q{,}x \textbf{ st } p = q \frown \mathord{<}x\mathord{>}.$$

The arguments of the notions defined below are the following: D which is an object of the type reserved above; x which is an object of the type Element **of** D. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\mathord{<}x\mathord{>} \qquad \text{is} \qquad \text{FinSequence } \textbf{of } D.$$

The arguments of the notions defined below are the following: D which is an object of the type reserved above; S which is an object of the type SUBDOMAIN **of** D; x which is an object of the type Element **of** S. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\mathord{<}x\mathord{>} \qquad \text{is} \qquad \text{FinSequence } \textbf{of } S.$$

The arguments of the notions defined below are the following: S which is an object of the type SUBDOMAIN **of** REAL; x which is an object of the type Element **of** S. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\mathord{<}x\mathord{>} \qquad \text{is} \qquad \text{FinSequence } \textbf{of } S.$$

The scheme *IndSeq* concerns a unary predicate $\mathcal{P}$ states that the following holds

$$\textbf{for } p \textbf{ holds } \mathcal{P}[p]$$

provided the parameter satisfies the following conditions:

- $$\mathcal{P}[\varepsilon],$$
- $$\textbf{for}\, p,x\ \textbf{st}\ \mathcal{P}[p]\ \textbf{holds}\ \mathcal{P}[p \frown <\!x\!>].$$

One can prove the following proposition

(64) $$\textbf{for}\, p,q,r,s\ \textbf{being}\ \mathrm{FinSequence}$$
$$\textbf{st}\ p \frown q = r \frown s\ \&\ \mathrm{len}\, p \le \mathrm{len}\, r\ \textbf{ex}\, t\ \textbf{being}\ \mathrm{FinSequence}\ \textbf{st}\ p \frown t = r.$$

Let us consider D. The functor
$$D^*,$$
yields the type DOMAIN and is defined by
$$x \in \textbf{it}\ \textbf{iff}\ x\ \textbf{is}\ \mathrm{FinSequence}\ \textbf{of}\ D.$$

One can prove the following propositions:

(65) $$x \in D^*\ \textbf{iff}\ x\ \textbf{is}\ \mathrm{FinSequence}\ \textbf{of}\ D,$$

(66) $$\varepsilon \in D^*.$$

The scheme *SepSeq* deals with a constant $\mathcal{A}$ that has the type DOMAIN and a unary predicate $\mathcal{P}$ and states that the following holds
$$\textbf{ex}\, X\ \textbf{st}\ \textbf{for}\, x\ \textbf{holds}\ x \in X\ \textbf{iff}\ \textbf{ex}\, p\ \textbf{st}\ p \in \mathcal{A}^*\ \&\ \mathcal{P}[p]\ \&\ x = p$$
for all values of the parameters.

The mode
$$\mathrm{FinSubsequence},$$
which widens to the type Function, is defined by
$$\textbf{ex}\, k\ \textbf{st}\ \mathrm{dom}\,\textbf{it} \subseteq \mathrm{Seg}\, k.$$

The following three propositions are true:

(67) $$f\ \textbf{is}\ \mathrm{FinSubsequence}\ \textbf{iff}\ \textbf{ex}\, k\ \textbf{st}\ \mathrm{dom}\, f \subseteq \mathrm{Seg}\, k,$$

(68) $$\textbf{for}\, p\ \textbf{being}\ \mathrm{FinSequence}\ \textbf{holds}\ p\ \textbf{is}\ \mathrm{FinSubsequence},$$

(69) $$\textbf{for}\, p,X\ \textbf{holds}\ p \mid X\ \textbf{is}\ \mathrm{FinSubsequence}\ \&\ X \mid p\ \textbf{is}\ \mathrm{FinSubsequence}.$$

In the sequel p' has the type FinSubsequence. Let us consider X. Assume there exists k, such that
$$X \subseteq \mathrm{Seg}\, k.$$
The functor
$$\mathrm{Sgm}\, X,$$

with values of the type FinSequence **of** NAT, is defined by

$$\operatorname{rng}\mathbf{it} = X \mathrel{\&}$$
$$\mathbf{for}\, l,m,k1,k2\ \mathbf{st}\ 1 \leq l \mathrel{\&} l < m \mathrel{\&} m \leq \operatorname{len}\mathbf{it} \mathrel{\&} k1 = \mathbf{it}.l \mathrel{\&} k2 = \mathbf{it}.m\ \mathbf{holds}\ k1 < k2.$$

One can prove the following propositions:

(70) $$(\mathbf{ex}\, k\ \mathbf{st}\ X \subseteq \operatorname{Seg} k)\ \mathbf{implies\ for}\, p\ \mathbf{being}\ \text{FinSequence}\ \mathbf{of}\ \text{NAT}\ \mathbf{holds}$$
$$p = \operatorname{Sgm} X\ \mathbf{iff}\ \operatorname{rng} p = X \mathrel{\&} \mathbf{for}\, l,m,k1,k2$$
$$\mathbf{st}\ 1 \leq l \mathrel{\&} l < m \mathrel{\&} m \leq \operatorname{len} p \mathrel{\&} k1 = p.l \mathrel{\&} k2 = p.m\ \mathbf{holds}\ k1 < k2,$$

(71) $$\operatorname{rng}\operatorname{Sgm}\operatorname{dom} p' = \operatorname{dom} p'.$$

Let us consider p'. The functor

$$\operatorname{Seq} p',$$

yields the type FinSequence and is defined by

$$\mathbf{it} = p' \cdot \operatorname{Sgm}(\operatorname{dom} p').$$

Next we state two propositions:

(72) $$\mathbf{for}\, X\ \mathbf{st}\ \mathbf{ex}\, k\ \mathbf{st}\ X \subseteq \operatorname{Seg} k\ \mathbf{holds}\ \operatorname{Sgm} X = \varepsilon\ \mathbf{iff}\ X = \emptyset,$$

(73) $$p = \operatorname{Seq} p'\ \mathbf{iff}\ p = p' \cdot \operatorname{Sgm}(\operatorname{dom} p').$$

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Domains and Their Cartesian Products

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Summary. The article includes: theorems related to domains, theorems related to Cartesian products presented earlier in various articles and simplified here by substituting domains for sets and omitting the assumption that the sets involved must not be empty. Several schemes and theorems related to Fraenkel operator are given. We also redefine subset yielding functions such as the pair of elements of a set and the union of two subsets of a set.

The terminology and notation used in this paper have been introduced in the following articles: [2], [5], [1], [4], and [3]. For simplicity we adopt the following convention: a, b, c, d will have the type Any; A, B will have the type set; D, $X1$, $X2$, $X3$, $X4$, $Y1$, $Y2$, $Y3$, $Y4$ will have the type DOMAIN; $x1$, $y1$, $z1$ will have the type Element **of** $X1$; $x2$ will have the type Element **of** $X2$; $x3$ will have the type Element **of** $X3$; $x4$ will have the type Element **of** $X4$. The following three propositions are true:

(1) $$A \textbf{ is } \text{DOMAIN} \textbf{ iff } A \neq \emptyset,$$

(2) $$D \neq \emptyset,$$

(3) $$a \textbf{ is } \text{Element } \textbf{of } D \textbf{ implies } a \in D.$$

In the sequel $A1$, $B1$ will denote objects of the type Subset **of** $X1$. One can prove the following propositions:

(4) $$A1 = B1^{\mathrm{c}} \textbf{ iff for } x1 \textbf{ holds } x1 \in A1 \textbf{ iff not } x1 \in B1,$$

(5) $$A1 = B1^{\mathrm{c}} \textbf{ iff for } x1 \textbf{ holds not } x1 \in A1 \textbf{ iff } x1 \in B1,$$

(6) $$A1 = B1^{\mathrm{c}} \textbf{ iff for } x1 \textbf{ holds not } (x1 \in A1 \textbf{ iff } x1 \in B1),$$

(7) $$\langle x1, x2 \rangle \in [\!: X1, X2 :\!],$$

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$$\langle x1,x2\rangle \ \textbf{is}\ \text{Element}\ \textbf{of}\ [:X1,X2:], \tag{8}$$

$$a \in [:X1,X2:]\ \textbf{implies}\ \textbf{ex}\ x1,x2\ \textbf{st}\ a = \langle x1,x2\rangle. \tag{9}$$

In the sequel x denotes an object of the type Element **of** $[:X1,X2:]$. One can prove the following propositions:

$$x = \langle x_{\mathbf{1}}, x_{\mathbf{2}}\rangle, \tag{10}$$

$$x \neq x_{\mathbf{1}}\ \&\ x \neq x_{\mathbf{2}}, \tag{11}$$

$$\textbf{for}\ x,y\ \textbf{being}\ \text{Element}\ \textbf{of}\ [:X1,X2:]\ \textbf{st}\ x_{\mathbf{1}} = y_{\mathbf{1}}\ \&\ x_{\mathbf{2}} = y_{\mathbf{2}}\ \textbf{holds}\ x = y, \tag{12}$$

$$[:A,D:] \subseteq [:B,D:]\ \textbf{or}\ [:D,A:] \subseteq [:D,B:]\ \textbf{implies}\ A \subseteq B, \tag{13}$$

$$[:X1,X2:] = [:A,B:]\ \textbf{implies}\ X1 = A\ \&\ X2 = B. \tag{14}$$

Let us consider $X1$, $X2$, $x1$, $x2$. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\langle x1,x2\rangle \qquad \text{is} \qquad \text{Element}\ \textbf{of}\ [:X1,X2:].$$

The arguments of the notions defined below are the following: $X1$, $X2$ which are objects of the type reserved above; x which is an object of the type Element **of** $[:X1,X2:]$. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$x_{\mathbf{1}} \qquad \text{is} \qquad \text{Element}\ \textbf{of}\ X1,$$

$$x_{\mathbf{2}} \qquad \text{is} \qquad \text{Element}\ \textbf{of}\ X2.$$

One can prove the following propositions:

$$a \in [:X1,X2,X3:]\ \textbf{iff}\ \textbf{ex}\ x1,x2,x3\ \textbf{st}\ a = \langle x1,x2,x3\rangle, \tag{15}$$

$$\begin{gathered}(\textbf{for}\ a\ \textbf{holds}\ a \in D\ \textbf{iff}\ \textbf{ex}\ x1,x2,x3\ \textbf{st}\ a = \langle x1,x2,x3\rangle) \\ \textbf{implies}\ D = [:X1,X2,X3:],\end{gathered} \tag{16}$$

$$D = [:X1,X2,X3:]\ \textbf{iff}\ \textbf{for}\ a\ \textbf{holds}\ a \in D\ \textbf{iff}\ \textbf{ex}\ x1,x2,x3\ \textbf{st}\ a = \langle x1,x2,x3\rangle, \tag{17}$$

$$[:X1,X2,X3:] = [:Y1,Y2,Y3:]\ \textbf{implies}\ X1 = Y1\ \&\ X2 = Y2\ \&\ X3 = Y3. \tag{18}$$

In the sequel x, y will have the type Element **of** $[:X1,X2,X3:]$. Next we state several propositions:

$$x = \langle a,b,c\rangle\ \textbf{implies}\ x_{\mathbf{1}} = a\ \&\ x_{\mathbf{2}} = b\ \&\ x_{\mathbf{3}} = c, \tag{19}$$

$$x = \langle x_{\mathbf{1}}, x_{\mathbf{2}}, x_{\mathbf{3}}\rangle, \tag{20}$$

$$x_{\mathbf{1}} = (x\ \textbf{qua}\ \text{Any})_{\mathbf{1}\,\mathbf{1}}\ \&\ x_{\mathbf{2}} = (x\ \textbf{qua}\ \text{Any})_{\mathbf{1}\,\mathbf{2}}\ \&\ x_{\mathbf{3}} = (x\ \textbf{qua}\ \text{Any})_{\mathbf{2}}, \tag{21}$$

$$x \neq x_{\mathbf{1}} \;\&\; x \neq x_{\mathbf{2}} \;\&\; x \neq x_{\mathbf{3}}\,, \tag{22}$$

$$\langle x1,x2,x3\rangle \in [:X1,X2,X3:]. \tag{23}$$

Let us consider $X1$, $X2$, $X3$, $x1$, $x2$, $x3$. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\langle x1,x2,x3\rangle \quad \text{is} \quad \text{Element } \mathbf{of}\ [:X1,X2,X3:].$$

The arguments of the notions defined below are the following: $X1$, $X2$, $X3$ which are objects of the type reserved above; x which is an object of the type Element **of** $[:X1,X2,X3:]$. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$x_{\mathbf{1}} \quad \text{is} \quad \text{Element } \mathbf{of}\ X1,$$

$$x_{\mathbf{2}} \quad \text{is} \quad \text{Element } \mathbf{of}\ X2,$$

$$x_{\mathbf{3}} \quad \text{is} \quad \text{Element } \mathbf{of}\ X3.$$

The following propositions are true:

$$a = x_{\mathbf{1}} \ \mathbf{iff\ for}\ x1,x2,x3\ \mathbf{st}\ x = \langle x1,x2,x3\rangle\ \mathbf{holds}\ a = x1, \tag{24}$$

$$b = x_{\mathbf{2}} \ \mathbf{iff\ for}\ x1,x2,x3\ \mathbf{st}\ x = \langle x1,x2,x3\rangle\ \mathbf{holds}\ b = x2, \tag{25}$$

$$c = x_{\mathbf{3}} \ \mathbf{iff\ for}\ x1,x2,x3\ \mathbf{st}\ x = \langle x1,x2,x3\rangle\ \mathbf{holds}\ c = x3, \tag{26}$$

$$\langle x_{\mathbf{1}}, x_{\mathbf{2}}, x_{\mathbf{3}}\rangle = x, \tag{27}$$

$$x_{\mathbf{1}} = y_{\mathbf{1}} \;\&\; x_{\mathbf{2}} = y_{\mathbf{2}} \;\&\; x_{\mathbf{3}} = y_{\mathbf{3}}\ \mathbf{implies}\ x = y, \tag{28}$$

$$\langle x1,x2,x3\rangle_{\mathbf{1}} = x1 \;\&\; \langle x1,x2,x3\rangle_{\mathbf{2}} = x2 \;\&\; \langle x1,x2,x3\rangle_{\mathbf{3}} = x3, \tag{29}$$

$$\begin{gathered}\mathbf{for}\ x\ \mathbf{being}\ \text{Element } \mathbf{of}\ [:X1,X2,X3:],\ y\ \mathbf{being}\ \text{Element } \mathbf{of}\ [:Y1,Y2,Y3:]\\ \mathbf{holds}\ x = y\ \mathbf{implies}\ x_{\mathbf{1}} = y_{\mathbf{1}} \;\&\; x_{\mathbf{2}} = y_{\mathbf{2}} \;\&\; x_{\mathbf{3}} = y_{\mathbf{3}}\,,\end{gathered} \tag{30}$$

$$a \in [:X1,X2,X3,X4:]\ \mathbf{iff\ ex}\ x1,x2,x3,x4\ \mathbf{st}\ a = \langle x1,x2,x3,x4\rangle, \tag{31}$$

$$\begin{gathered}(\mathbf{for}\ a\ \mathbf{holds}\ a \in D\ \mathbf{iff\ ex}\ x1,x2,x3,x4\ \mathbf{st}\ a = \langle x1,x2,x3,x4\rangle)\\ \mathbf{implies}\ D = [:X1,X2,X3,X4:],\end{gathered} \tag{32}$$

$$\begin{gathered}D = [:X1,X2,X3,X4:]\\ \mathbf{iff\ for}\ a\ \mathbf{holds}\ a \in D\ \mathbf{iff\ ex}\ x1,x2,x3,x4\ \mathbf{st}\ a = \langle x1,x2,x3,x4\rangle.\end{gathered} \tag{33}$$

In the sequel x denotes an object of the type Element **of** $[:X1,X2,X3,X4:]$. The following propositions are true:

$$\begin{gathered}[:X1,X2,X3,X4:] = [:Y1,Y2,Y3,Y4:]\\ \mathbf{implies}\ X1 = Y1 \;\&\; X2 = Y2 \;\&\; X3 = Y3 \;\&\; X4 = Y4,\end{gathered} \tag{34}$$

(35) $$x = \langle a, b, c, d\rangle \textbf{ implies } x_{\mathbf{1}} = a \ \&\ x_{\mathbf{2}} = b \ \&\ x_{\mathbf{3}} = c \ \&\ x_{\mathbf{4}} = d,$$

(36) $$x = \langle x_{\mathbf{1}}, x_{\mathbf{2}}, x_{\mathbf{3}}, x_{\mathbf{4}}\rangle,$$

(37) $$x_{\mathbf{1}} = (x \textbf{ qua } \mathrm{Any})_{\mathbf{1}\,\mathbf{1}\,\mathbf{1}} \ \&\ x_{\mathbf{2}} = (x \textbf{ qua } \mathrm{Any})_{\mathbf{1}\,\mathbf{1}\,\mathbf{2}} \ \&\ x_{\mathbf{3}} = (x \textbf{ qua } \mathrm{Any})_{\mathbf{1}\,\mathbf{2}} \ \&\ x_{\mathbf{4}} = (x \textbf{ qua } \mathrm{Any})_{\mathbf{2}},$$

(38) $$x \neq x_{\mathbf{1}} \ \&\ x \neq x_{\mathbf{2}} \ \&\ x \neq x_{\mathbf{3}} \ \&\ x \neq x_{\mathbf{4}},$$

(39) $$\langle x1, x2, x3, x4\rangle \in [:X1, X2, X3, X4:].$$

Let us consider $X1$, $X2$, $X3$, $X4$, $x1$, $x2$, $x3$, $x4$. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\langle x1, x2, x3, x4\rangle \quad \text{is} \quad \mathrm{Element}\ \textbf{of}\ [:X1, X2, X3, X4:].$$

The arguments of the notions defined below are the following: $X1$, $X2$, $X3$, $X4$ which are objects of the type reserved above; x which is an object of the type Element **of** $[:X1,X2,X3,X4:]$. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$x_{\mathbf{1}} \quad \text{is} \quad \mathrm{Element}\ \textbf{of}\ X1,$$

$$x_{\mathbf{2}} \quad \text{is} \quad \mathrm{Element}\ \textbf{of}\ X2,$$

$$x_{\mathbf{3}} \quad \text{is} \quad \mathrm{Element}\ \textbf{of}\ X3,$$

$$x_{\mathbf{4}} \quad \text{is} \quad \mathrm{Element}\ \textbf{of}\ X4.$$

The following propositions are true:

(40) $$a = x_{\mathbf{1}} \textbf{ iff for } x1, x2, x3, x4 \textbf{ st } x = \langle x1, x2, x3, x4\rangle \textbf{ holds } a = x1,$$

(41) $$b = x_{\mathbf{2}} \textbf{ iff for } x1, x2, x3, x4 \textbf{ st } x = \langle x1, x2, x3, x4\rangle \textbf{ holds } b = x2,$$

(42) $$c = x_{\mathbf{3}} \textbf{ iff for } x1, x2, x3, x4 \textbf{ st } x = \langle x1, x2, x3, x4\rangle \textbf{ holds } c = x3,$$

(43) $$d = x_{\mathbf{4}} \textbf{ iff for } x1, x2, x3, x4 \textbf{ st } x = \langle x1, x2, x3, x4\rangle \textbf{ holds } d = x4,$$

(44) $$\textbf{for } x \textbf{ being } \mathrm{Element}\ \textbf{of}\ [:X1, X2, X3, X4:] \textbf{ holds } \langle x_{\mathbf{1}}, x_{\mathbf{2}}, x_{\mathbf{3}}, x_{\mathbf{4}}\rangle = x,$$

(45) $$\textbf{for } x, y \textbf{ being } \mathrm{Element}\ \textbf{of}\ [:X1, X2, X3, X4:] \textbf{ st } x_{\mathbf{1}} = y_{\mathbf{1}} \ \&\ x_{\mathbf{2}} = y_{\mathbf{2}} \ \&\ x_{\mathbf{3}} = y_{\mathbf{3}} \ \&\ x_{\mathbf{4}} = y_{\mathbf{4}} \textbf{ holds } x = y,$$

(46) $$\langle x1, x2, x3, x4\rangle_{\mathbf{1}} = x1 \ \&\ \langle x1, x2, x3, x4\rangle_{\mathbf{2}} = x2 \ \&\ \langle x1, x2, x3, x4\rangle_{\mathbf{3}} = x3 \ \&\ \langle x1, x2, x3, x4\rangle_{\mathbf{4}} = x4,$$

(47) $$\textbf{for } x \textbf{ being } \mathrm{Element}\ \textbf{of}\ [:X1, X2, X3, X4:], y \textbf{ being } \mathrm{Element}\ \textbf{of}\ [:Y1, Y2, Y3, Y4:] \textbf{ holds } x = y \textbf{ implies } x_{\mathbf{1}} = y_{\mathbf{1}} \ \&\ x_{\mathbf{2}} = y_{\mathbf{2}} \ \&\ x_{\mathbf{3}} = y_{\mathbf{3}} \ \&\ x_{\mathbf{4}} = y_{\mathbf{4}}.$$

In the sequel $A2$ will denote an object of the type Subset **of** $X2$; $A3$ will denote an object of the type Subset **of** $X3$; $A4$ will denote an object of the type Subset **of** $X4$. In the article we present several logical schemes. The scheme *Fraenkel1* deals with a unary predicate $\mathcal{P}$ states that the following holds

$$\textbf{for } X1 \textbf{ holds } \{\, x1 : \mathcal{P}[x1] \,\} \textbf{ is } \text{Subset } \textbf{of } X1$$

for all values of the parameter.

The scheme *Fraenkel2* deals with a binary predicate $\mathcal{P}$ states that the following holds

$$\textbf{for } X1{,}X2 \textbf{ holds } \{\, \langle x1{,}x2 \rangle : \mathcal{P}[x1{,}x2] \,\} \textbf{ is } \text{Subset } \textbf{of } [\!: X1{,}X2 :\!]$$

for all values of the parameter.

The scheme *Fraenkel3* concerns a ternary predicate $\mathcal{P}$ states that the following holds

$$\textbf{for } X1{,}X2{,}X3 \textbf{ holds } \{\, \langle x1{,}x2{,}x3 \rangle : \mathcal{P}[x1{,}x2{,}x3] \,\} \textbf{ is } \text{Subset } \textbf{of } [\!: X1{,}X2{,}X3 :\!]$$

for all values of the parameter.

The scheme *Fraenkel4* deals with a 4-ary predicate $\mathcal{P}$ states that the following holds

$$\begin{gathered}\textbf{for } X1{,}X2{,}X3{,}X4 \\ \textbf{holds } \{\, \langle x1{,}x2{,}x3{,}x4 \rangle : \mathcal{P}[x1{,}x2{,}x3{,}x4] \,\} \textbf{ is } \text{Subset } \textbf{of } [\!: X1{,}X2{,}X3{,}X4 :\!]\end{gathered}$$

for all values of the parameter.

The scheme *Fraenkel5* concerns a unary predicate $\mathcal{P}$ and a unary predicate $\mathcal{Q}$ and states that the following holds

$$\textbf{for } X1 \textbf{ st for } x1 \textbf{ holds } \mathcal{P}[x1] \textbf{ implies } \mathcal{Q}[x1] \textbf{ holds } \{\, y1 : \mathcal{P}[y1] \,\} \subseteq \{\, z1 : \mathcal{Q}[z1] \,\}$$

for all values of the parameters.

The scheme *Fraenkel6* deals with a unary predicate $\mathcal{P}$ and a unary predicate $\mathcal{Q}$ and states that the following holds

$$\textbf{for } X1 \textbf{ st for } x1 \textbf{ holds } \mathcal{P}[x1] \textbf{ iff } \mathcal{Q}[x1] \textbf{ holds } \{\, y1 : \mathcal{P}[y1] \,\} = \{\, z1 : \mathcal{Q}[z1] \,\}$$

for all values of the parameters.

Next we state several propositions:

(48) $$X1 = \{\, x1 : \textbf{not contradiction} \,\},$$

(49) $$[\!: X1{,}X2 :\!] = \{\, \langle x1{,}x2 \rangle : \textbf{not contradiction} \,\},$$

(50) $$[\!: X1{,}X2{,}X3 :\!] = \{\, \langle x1{,}x2{,}x3 \rangle : \textbf{not contradiction} \,\},$$

(51) $$[\!: X1{,}X2{,}X3{,}X4 :\!] = \{\, \langle x1{,}x2{,}x3{,}x4 \rangle : \textbf{not contradiction} \,\},$$

(52) $$A1 = \{\, x1 : x1 \in A1 \,\}.$$

Let us consider $X1$, $X2$, $A1$, $A2$. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$[:A1,A2:] \quad \text{is} \quad \text{Subset } \mathbf{of}\ [:X1,X2:].$$

Next we state a proposition

$$[:A1,A2:] = \{ \langle x1,x2 \rangle : x1 \in A1 \ \&\ x2 \in A2 \}. \tag{53}$$

Let us consider $X1$, $X2$, $X3$, $A1$, $A2$, $A3$. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$[:A1,A2,A3:] \quad \text{is} \quad \text{Subset } \mathbf{of}\ [:X1,X2,X3:].$$

Next we state a proposition

$$[:A1,A2,A3:] = \{ \langle x1,x2,x3 \rangle : x1 \in A1 \ \&\ x2 \in A2 \ \&\ x3 \in A3 \}. \tag{54}$$

Let us consider $X1$, $X2$, $X3$, $X4$, $A1$, $A2$, $A3$, $A4$. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$[:A1,A2,A3,A4:] \quad \text{is} \quad \text{Subset } \mathbf{of}\ [:X1,X2,X3,X4:].$$

Next we state a number of propositions:

$$\begin{gathered}[:A1,A2,A3,A4:] \\ = \{ \langle x1,x2,x3,x4 \rangle : x1 \in A1 \ \&\ x2 \in A2 \ \&\ x3 \in A3 \ \&\ x4 \in A4 \},\end{gathered} \tag{55}$$

$$\emptyset X1 = \{ x1 : \mathbf{contradiction} \}, \tag{56}$$

$$A1^{\,\mathrm{c}} = \{ x1 : \mathbf{not}\, x1 \in A1 \}, \tag{57}$$

$$A1 \cap B1 = \{ x1 : x1 \in A1 \ \&\ x1 \in B1 \}, \tag{58}$$

$$A1 \cup B1 = \{ x1 : x1 \in A1 \ \mathbf{or}\ x1 \in B1 \}, \tag{59}$$

$$A1 \setminus B1 = \{ x1 : x1 \in A1 \ \&\ \mathbf{not}\, x1 \in B1 \}, \tag{60}$$

$$A1 \mathbin{\dot-} B1 = \{ x1 : x1 \in A1 \ \&\ \mathbf{not}\, x1 \in B1 \ \mathbf{or}\ \mathbf{not}\, x1 \in A1 \ \&\ x1 \in B1 \}, \tag{61}$$

$$A1 \mathbin{\dot-} B1 = \{ x1 : \mathbf{not}\, x1 \in A1 \ \mathbf{iff}\ x1 \in B1 \}, \tag{62}$$

$$A1 \mathbin{\dot-} B1 = \{ x1 : x1 \in A1 \ \mathbf{iff}\ \mathbf{not}\, x1 \in B1 \}, \tag{63}$$

$$A1 \mathbin{\dot-} B1 = \{ x1 : \mathbf{not}\, (x1 \in A1 \ \mathbf{iff}\ x1 \in B1) \}. \tag{64}$$

In the sequel $x1$, $x2$, $x3$, $x4$, $x5$, $x6$, $x7$, $x8$ will have the type Element **of** D. We now state several propositions:

$$\{x1\} \ \mathbf{is}\ \text{Subset } \mathbf{of}\ D, \tag{65}$$

(66) $$\{x1,x2\} \textbf{ is } \text{Subset} \textbf{ of } D,$$

(67) $$\{x1,x2,x3\} \textbf{ is } \text{Subset} \textbf{ of } D,$$

(68) $$\{x1,x2,x3,x4\} \textbf{ is } \text{Subset} \textbf{ of } D,$$

(69) $$\{x1,x2,x3,x4,x5\} \textbf{ is } \text{Subset} \textbf{ of } D,$$

(70) $$\{x1,x2,x3,x4,x5,x6\} \textbf{ is } \text{Subset} \textbf{ of } D,$$

(71) $$\{x1,x2,x3,x4,x5,x6,x7\} \textbf{ is } \text{Subset} \textbf{ of } D,$$

(72) $$\{x1,x2,x3,x4,x5,x6,x7,x8\} \textbf{ is } \text{Subset} \textbf{ of } D.$$

Let us consider D. Let $x1$ have the type Element **of** D. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\{x1\} \qquad \text{is} \qquad \text{Subset} \textbf{ of } D.$$

Let $x2$ have the type Element **of** D. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\{x1,x2\} \qquad \text{is} \qquad \text{Subset} \textbf{ of } D.$$

Let $x3$ have the type Element **of** D. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\{x1,x2,x3\} \qquad \text{is} \qquad \text{Subset} \textbf{ of } D.$$

Let $x4$ have the type Element **of** D. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\{x1,x2,x3,x4\} \qquad \text{is} \qquad \text{Subset} \textbf{ of } D.$$

Let $x5$ have the type Element **of** D. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\{x1,x2,x3,x4,x5\} \qquad \text{is} \qquad \text{Subset} \textbf{ of } D.$$

Let $x6$ have the type Element **of** D. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\{x1,x2,x3,x4,x5,x6\} \qquad \text{is} \qquad \text{Subset} \textbf{ of } D.$$

Let $x7$ have the type Element **of** D. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\{x1,x2,x3,x4,x5,x6,x7\} \qquad \text{is} \qquad \text{Subset} \textbf{ of } D.$$

Let $x8$ have the type Element **of** D. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\{x1,x2,x3,x4,x5,x6,x7,x8\} \quad \text{is} \quad \text{Subset } \textbf{of } D.$$

Let us consider $X1$, $A1$. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$A1^{\,\mathrm{c}} \quad \text{is} \quad \text{Subset } \textbf{of } X1.$$

Let us consider $B1$. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$A1 \cup B1 \quad \text{is} \quad \text{Subset } \textbf{of } X1,$$

$$A1 \cap B1 \quad \text{is} \quad \text{Subset } \textbf{of } X1,$$

$$A1 \setminus B1 \quad \text{is} \quad \text{Subset } \textbf{of } X1,$$

$$A1 \mathbin{\dot{-}} B1 \quad \text{is} \quad \text{Subset } \textbf{of } X1.$$

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The Well Ordering Relations

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Summary. Some theorems about well ordering relations are proved. The goal of the article is to prove that every two well ordering relations are either isomorphic or one of them is isomorphic to a segment of the other. The following concepts are defined: the segment of a relation induced by an element, well founded relations, well ordering relations, the restriction of a relation to a set, and the isomorphism of two relations. A number of simple facts is presented.

The terminology and notation used here are introduced in the following papers: [2], [3], [4], [5], and [1]. For simplicity we adopt the following convention: a, b, c, x denote objects of the type Any; X, Y, Z denote objects of the type set. The scheme *Extensionality* concerns a constant $\mathcal{A}$ that has the type set, a constant $\mathcal{B}$ that has the type set and a unary predicate $\mathcal{P}$ and states that the following holds

$$\mathcal{A} = \mathcal{B}$$

provided the parameters satisfy the following conditions:

- $\textbf{for } a \textbf{ holds } a \in \mathcal{A} \textbf{ iff } \mathcal{P}[a]$,
- $\textbf{for } a \textbf{ holds } a \in \mathcal{B} \textbf{ iff } \mathcal{P}[a]$.

In the sequel R, S, T will have the type Relation. Let us consider R, a. The functor

$$R\,{-}\mathrm{Seg}\,a,$$

with values of the type set, is defined by

$$x \in \textbf{it iff } x \neq a \;\&\; \langle x, a\rangle \in R.$$

One can prove the following propositions:

(1) $\textbf{for } R{,}Y{,}a \textbf{ holds } Y = R\,{-}\mathrm{Seg}\,(a) \textbf{ iff for } b \textbf{ holds } b \in Y \textbf{ iff } b \neq a \;\&\; \langle b, a\rangle \in R$,

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$$x \in \text{field}\, R \textbf{ or } R\,-\text{Seg}\,(x) = \emptyset. \tag{2}$$

We now define two new predicates. Let us consider R. The predicate

$$R\,\text{is\_well\_founded}$$

is defined by

$$\textbf{for}\, Y \textbf{ st } Y \subseteq \text{field}\, R \,\&\, Y \neq \emptyset\, \textbf{ex}\, a \textbf{ st } a \in Y \,\&\, R\,-\text{Seg}\,(a) \cap Y = \emptyset.$$

Let us consider X. The predicate

$$R \text{ is\_well\_founded\_in } X$$

is defined by

$$\textbf{for}\, Y \textbf{ st } Y \subseteq X \,\&\, Y \neq \emptyset\, \textbf{ex}\, a \textbf{ st } a \in Y \,\&\, R\,-\text{Seg}\,(a) \cap Y = \emptyset.$$

One can prove the following three propositions:

$$\begin{gathered} \textbf{for}\, R \textbf{ holds } R\,\text{is\_well\_founded} \\ \textbf{iff for}\, Y \textbf{ st } Y \subseteq \text{field}\, R \,\&\, Y \neq \emptyset\, \textbf{ex}\, a \textbf{ st } a \in Y \,\&\, R\,-\text{Seg}\,(a) \cap Y = \emptyset, \end{gathered} \tag{3}$$

$$\begin{gathered} \textbf{for}\, R{,}X \textbf{ holds } R \text{ is\_well\_founded\_in } X \\ \textbf{iff for}\, Y \textbf{ st } Y \subseteq X \,\&\, Y \neq \emptyset\, \textbf{ex}\, a \textbf{ st } a \in Y \,\&\, R\,-\text{Seg}\,(a) \cap Y = \emptyset, \end{gathered} \tag{4}$$

$$R\,\text{is\_well\_founded} \textbf{ iff } R \text{ is\_well\_founded\_in field}\, R. \tag{5}$$

We now define two new predicates. Let us consider R. The predicate

$$R\,\text{is\_well-ordering-relation}$$

is defined by

$$\begin{gathered} R\,\text{is\_reflexive} \\ \&\, R\,\text{is\_transitive} \,\&\, R\,\text{is\_antisymmetric} \,\&\, R\,\text{is\_connected} \,\&\, R\,\text{is\_well\_founded}\,. \end{gathered}$$

Let us consider X. The predicate

$$R \text{ well\_orders } X$$

is defined by

$$\begin{gathered} R \text{ is\_reflexive\_in } X \,\&\, R \text{ is\_transitive\_in } X \\ \&\, R \text{ is\_antisymmetric\_in } X \,\&\, R \text{ is\_connected\_in } X \,\&\, R \text{ is\_well\_founded\_in } X. \end{gathered}$$

The following propositions are true:

$$\begin{gathered} \textbf{for}\, R \textbf{ holds } R\,\text{is\_well-ordering-relation} \textbf{ iff } R\,\text{is\_reflexive} \\ \&\, R\,\text{is\_transitive} \,\&\, R\,\text{is\_antisymmetric} \,\&\, R\,\text{is\_connected} \,\&\, R\,\text{is\_well\_founded}\,, \end{gathered} \tag{6}$$

(7) $\mathbf{for}\ R,X\ \mathbf{holds}\ R\ \text{well\_orders}\ X\ \mathbf{iff}\ R\ \text{is\_reflexive\_in}\ X\ \&\ R\ \text{is\_transitive\_in}\ X$
$\&\ R\ \text{is\_antisymmetric\_in}\ X\ \&\ R\ \text{is\_connected\_in}\ X\ \&\ R\ \text{is\_well\_founded\_in}\ X,$

(8) $$R\ \text{well\_orders field}\ R\ \mathbf{iff}\ R\ \text{is\_well-ordering-relation},$$

(9) $$R\ \text{well\_orders}\ X\ \mathbf{implies}$$
$$\mathbf{for}\ Y\ \mathbf{st}\ Y \subseteq X\ \&\ Y \neq \emptyset\ \mathbf{ex}\ a\ \mathbf{st}\ a \in Y\ \&\ \mathbf{for}\ b\ \mathbf{st}\ b \in Y\ \mathbf{holds}\ \langle a, b\rangle \in R,$$

(10) $$R\ \text{is\_well-ordering-relation}\ \mathbf{implies}$$
$$\mathbf{for}\ Y\ \mathbf{st}\ Y \subseteq \text{field}\ R\ \&\ Y \neq \emptyset\ \mathbf{ex}\ a\ \mathbf{st}\ a \in Y\ \&\ \mathbf{for}\ b\ \mathbf{st}\ b \in Y\ \mathbf{holds}\ \langle a, b\rangle \in R,$$

(11) $$\mathbf{for}\ R\ \mathbf{st}\ R\ \text{is\_well-ordering-relation}\ \&\ \text{field}\ R \neq \emptyset$$
$$\mathbf{ex}\ a\ \mathbf{st}\ a \in \text{field}\ R\ \&\ \mathbf{for}\ b\ \mathbf{st}\ b \in \text{field}\ R\ \mathbf{holds}\ \langle a, b\rangle \in R,$$

(12) $$\mathbf{for}\ R\ \mathbf{st}\ R\ \text{is\_well-ordering-relation}\ \&\ \text{field}\ R \neq \emptyset\ \mathbf{for}\ a\ \mathbf{st}\ a \in \text{field}\ R\ \mathbf{holds}$$
$$(\mathbf{for}\ b\ \mathbf{st}\ b \in \text{field}\ R\ \mathbf{holds}\ \langle b, a\rangle \in R)\ \mathbf{or}\ \mathbf{ex}\ b\ \mathbf{st}\ b \in \text{field}\ R$$
$$\&\ \langle a, b\rangle \in R\ \&\ \mathbf{for}\ c\ \mathbf{st}\ c \in \text{field}\ R\ \&\ \langle a, c\rangle \in R\ \mathbf{holds}\ c = a\ \mathbf{or}\ \langle b, c\rangle \in R.$$

In the sequel F, G have the type Function. Next we state a proposition

(13) $$R\ -\text{Seg}\,(a) \subseteq \text{field}\ R.$$

Let us consider R, Y. The functor

$$R\,|^2\,Y,$$

yields the type Relation and is defined by

$$\mathbf{it} = R \cap [:Y, Y:].$$

We now state a number of propositions:

(14) $$R\,|^2\,Y = R \cap [:Y, Y:],$$

(15) $$R\,|^2\,X \subseteq R\ \&\ R\,|^2\,X \subseteq [:X, X:],$$

(16) $$x \in R\,|^2\,X\ \mathbf{iff}\ x \in R\ \&\ x \in [:X, X:],$$

(17) $$R\,|^2\,X = X \mid R \mid X,$$

(18) $$R\,|^2\,X = X \mid (R \mid X),$$

(19) $$x \in \text{field}\,(R\,|^2\,X)\ \mathbf{implies}\ x \in \text{field}\ R\ \&\ x \in X,$$

(20) $$\text{field}\,(R\,|^2\,X) \subseteq \text{field}\ R\ \&\ \text{field}\,(R\,|^2\,X) \subseteq X,$$

(21) $$(R\,|^2\,X)\ -\text{Seg}\,(a) \subseteq R\ -\text{Seg}\,(a),$$

$$R \,\text{is\_reflexive}\ \textbf{implies}\ R\,|^2\ X\ \text{is\_reflexive}\,, \tag{22}$$

$$R \,\text{is\_connected}\ \textbf{implies}\ R\,|^2\ Y\ \text{is\_connected}\,, \tag{23}$$

$$R \,\text{is\_transitive}\ \textbf{implies}\ R\,|^2\ Y\ \text{is\_transitive}\,, \tag{24}$$

$$R \,\text{is\_antisymmetric}\ \textbf{implies}\ R\,|^2\ Y\ \text{is\_antisymmetric}\,, \tag{25}$$

$$(R\,|^2\ X)\,|^2\ Y = R\,|^2\ (X \cap Y), \tag{26}$$

$$(R\,|^2\ X)\,|^2\ Y = (R\,|^2\ Y)\,|^2\ X, \tag{27}$$

$$(R\,|^2\ Y)\,|^2\ Y = R\,|^2\ Y, \tag{28}$$

$$Z \subseteq Y\ \textbf{implies}\ (R\,|^2\ Y)\,|^2\ Z = R\,|^2\ Z, \tag{29}$$

$$R\,|^2\ \text{field}\,R = R, \tag{30}$$

$$R \,\text{is\_well\_founded}\ \textbf{implies}\ R\,|^2\ X\ \text{is\_well\_founded}\,, \tag{31}$$

$$R \,\text{is\_well-ordering-relation}\ \textbf{implies}\ R\,|^2\ Y\ \text{is\_well-ordering-relation}\,, \tag{32}$$

$$\begin{gathered} R \,\text{is\_well-ordering-relation} \\ \textbf{implies}\ R\ -\text{Seg}\,(a) \subseteq R\ -\text{Seg}\,(b)\ \textbf{or}\ R\ -\text{Seg}\,(b) \subseteq R\ -\text{Seg}\,(a), \end{gathered} \tag{33}$$

$$R \,\text{is\_well-ordering-relation}\ \textbf{implies}\ R\,|^2\ (R\ -\text{Seg}\,(a))\ \text{is\_well-ordering-relation}\,, \tag{34}$$

$$\begin{gathered} R \,\text{is\_well-ordering-relation}\ \&\ a \in \text{field}\,R\ \&\ b \in R\ -\text{Seg}\,(a) \\ \textbf{implies}\ (R\,|^2\ (R\ -\text{Seg}\,(a)))\ -\text{Seg}\,(b) = R\ -\text{Seg}\,(b), \end{gathered} \tag{35}$$

$$\begin{gathered} R \,\text{is\_well-ordering-relation}\ \&\ Y \subseteq \text{field}\,R\ \textbf{implies} \\ (Y = \text{field}\,R\ \textbf{or}\ (\textbf{ex}\,a\ \textbf{st}\ a \in \text{field}\,R\ \&\ Y = R\ -\text{Seg}\,(a)) \\ \textbf{iff}\ \textbf{for}\,a\ \textbf{st}\ a \in Y\ \textbf{for}\,b\ \textbf{st}\ \langle b, a\rangle \in R\ \textbf{holds}\ b \in Y), \end{gathered} \tag{36}$$

$$\begin{gathered} R \,\text{is\_well-ordering-relation}\ \&\ a \in \text{field}\,R\ \&\ b \in \text{field}\,R \\ \textbf{implies}\ (\langle a, b\rangle \in R\ \textbf{iff}\ R\ -\text{Seg}\,(a) \subseteq R\ -\text{Seg}\,(b)), \end{gathered} \tag{37}$$

$$\begin{gathered} R \,\text{is\_well-ordering-relation}\ \&\ a \in \text{field}\,R\ \&\ b \in \text{field}\,R \\ \textbf{implies}\ (R\ -\text{Seg}\,(a) \subseteq R\ -\text{Seg}\,(b)\ \textbf{iff}\ a = b\ \textbf{or}\ a \in R\ -\text{Seg}\,(b)), \end{gathered} \tag{38}$$

$$R \,\text{is\_well-ordering-relation}\ \&\ X \subseteq \text{field}\,R\ \textbf{implies}\ \text{field}\,(R\,|^2\ X) = X, \tag{39}$$

$$R \,\text{is\_well-ordering-relation}\ \textbf{implies}\ \text{field}\,(R\,|^2\ R\ -\text{Seg}\,(a)) = R\ -\text{Seg}\,(a), \tag{40}$$

$$\begin{gathered} R \,\text{is\_well-ordering-relation}\ \textbf{implies} \\ \textbf{for}\ Z\ \textbf{st}\ \textbf{for}\,a\ \textbf{st}\ a \in \text{field}\,R\ \&\ R\ -\text{Seg}\,(a) \subseteq Z\ \textbf{holds}\ a \in Z\ \textbf{holds}\ \text{field}\,R \subseteq Z, \end{gathered} \tag{41}$$

$$R \text{ is\_well-ordering-relation } \& \\ a \in \operatorname{field} R \ \& \ b \in \operatorname{field} R \ \& \ (\textbf{for } c \textbf{ st } c \in R\,{-}\mathrm{Seg}\,(a) \textbf{ holds } \langle c, b\rangle \in R \ \& \ c \neq b) \\ \textbf{implies } \langle a, b\rangle \in R, \tag{42}$$

$$R \text{ is\_well-ordering-relation } \& \operatorname{dom} F = \operatorname{field} R \ \& \operatorname{rng} F \subseteq \operatorname{field} R \\ \& \ (\textbf{for } a{,}b \textbf{ st } \langle a, b\rangle \in R \ \& \ a \neq b \textbf{ holds } \langle F\,.a{,}F\,.b\rangle \in R \ \& \ F\,.a \neq F\,.b) \\ \textbf{implies for } a \textbf{ st } a \in \operatorname{field} R \textbf{ holds } \langle a, F\,.a\rangle \in R. \tag{43}$$

Let us consider R, S, F. The predicate

$$F \text{ is\_isomorphism\_of } R, S$$

is defined by

$$\operatorname{dom} F = \operatorname{field} R \ \& \operatorname{rng} F = \operatorname{field} S \ \& \\ F \text{ is\_one-to-one } \& \textbf{ for } a{,}b \textbf{ holds } \langle a, b\rangle \in R \textbf{ iff } a \in \operatorname{field} R \ \& \ b \in \operatorname{field} R \ \& \ \langle F\,.a{,}F\,.b\rangle \in S.$$

Next we state two propositions:

$$F \text{ is\_isomorphism\_of } R, S \textbf{ iff } \operatorname{dom} F = \operatorname{field} R \ \& \operatorname{rng} F = \operatorname{field} S \ \& \\ F \text{ is\_one-to-one} \\ \& \textbf{ for } a{,}b \textbf{ holds } \langle a, b\rangle \in R \textbf{ iff } a \in \operatorname{field} R \ \& \ b \in \operatorname{field} R \ \& \ \langle F\,.a{,}F\,.b\rangle \in S, \tag{44}$$

$$F \text{ is\_isomorphism\_of } R, S \\ \textbf{implies for } a{,}b \textbf{ st } \langle a, b\rangle \in R \ \& \ a \neq b \textbf{ holds } \langle F\,.a{,}F\,.b\rangle \in S \ \& \ F\,.a \neq F\,.b. \tag{45}$$

Let us consider R, S. The predicate

R, S are_isomorphic   is defined by $\textbf{ex } F \textbf{ st } F \text{ is\_isomorphism\_of } R, S.$

We now state a number of propositions:

$$R, S \text{ are\_isomorphic } \textbf{iff ex } F \textbf{ st } F \text{ is\_isomorphism\_of } R, S, \tag{46}$$

$$\operatorname{id}(\operatorname{field} R) \text{ is\_isomorphism\_of } R, R, \tag{47}$$

$$R, R \text{ are\_isomorphic}, \tag{48}$$

$$F \text{ is\_isomorphism\_of } R, S \textbf{ implies } F^{-1} \text{ is\_isomorphism\_of } S, R, \tag{49}$$

$$R, S \text{ are\_isomorphic } \textbf{implies } S, R \text{ are\_isomorphic}, \tag{50}$$

$$F \text{ is\_isomorphism\_of } R, S \ \& \ G \text{ is\_isomorphism\_of } S, T \\ \textbf{implies } G \cdot F \text{ is\_isomorphism\_of } R, T, \tag{51}$$

$$R, S \text{ are\_isomorphic } \& \ S, T \text{ are\_isomorphic } \textbf{implies } R, T \text{ are\_isomorphic}, \tag{52}$$

(53) $$F \text{ is\_isomorphism\_of } R, S \textbf{ implies } (R \text{ is\_reflexive } \textbf{implies } S \text{ is\_reflexive}) \ \& \\ (R \text{ is\_transitive } \textbf{implies } S \text{ is\_transitive}) \ \& \\ (R \text{ is\_connected } \textbf{implies } S \text{ is\_connected}) \ \& \\ (R \text{ is\_antisymmetric } \textbf{implies } S \text{ is\_antisymmetric}) \\ \& \ (R \text{ is\_well\_founded } \textbf{implies } S \text{ is\_well\_founded}),$$

(54) $$R \text{ is\_well-ordering-relation} \ \& \ F \text{ is\_isomorphism\_of } R, S \\ \textbf{implies } S \text{ is\_well-ordering-relation},$$

(55) $$R \text{ is\_well-ordering-relation } \textbf{implies for } F,G \\ \textbf{st } F \text{ is\_isomorphism\_of } R, S \ \& \ G \text{ is\_isomorphism\_of } R, S \textbf{ holds } F = G.$$

Let us consider R, S. Assume that the following holds

$$R \text{ is\_well-ordering-relation} \ \& \ R, S \text{ are\_isomorphic}.$$

The functor

$$\text{canonical\_isomorphism\_of}\,(R, S),$$

yields the type Function and is defined by

$$\textbf{it } \text{is\_isomorphism\_of } R, S.$$

The following propositions are true:

(56) $$R \text{ is\_well-ordering-relation} \ \& \ R, S \text{ are\_isomorphic} \\ \textbf{implies } (F = \text{canonical\_isomorphism\_of}\,(R, S) \textbf{ iff } F \text{ is\_isomorphism\_of } R, S),$$

(57) $$R \text{ is\_well-ordering-relation} \\ \textbf{implies for } a \textbf{ st } a \in \text{field}\, R \textbf{ holds not } R, R\,|^2\,(R - \text{Seg}\,(a)) \text{ are\_isomorphic},$$

(58) $$R \text{ is\_well-ordering-relation} \ \& \ a \in \text{field}\, R \ \& \ b \in \text{field}\, R \ \& \ a \neq b \\ \textbf{implies not } R\,|^2\,(R - \text{Seg}\,(a)), R\,|^2\,(R - \text{Seg}\,(b)) \text{ are\_isomorphic},$$

(59) $$R \text{ is\_well-ordering-relation} \ \& \ Z \subseteq \text{field}\, R \ \& \ F \text{ is\_isomorphism\_of } R, S \textbf{ implies} \\ F \,|\, Z \text{ is\_isomorphism\_of } R\,|^2\,Z, S\,|^2\,(F\,^{\circ}\,Z) \\ \& \ R\,|^2\,Z, S\,|^2\,(F\,^{\circ}\,Z) \text{ are\_isomorphic},$$

(60) $$R \text{ is\_well-ordering-relation} \ \& \ F \text{ is\_isomorphism\_of } R, S \textbf{ implies} \\ \textbf{for } a \textbf{ st } a \in \text{field}\, R \textbf{ ex } b \textbf{ st } b \in \text{field}\, S \ \& \ F\,^{\circ}\,(R - \text{Seg}\,(a)) = S - \text{Seg}\,(b),$$

(61) $$R \text{ is\_well-ordering-relation} \ \& \ F \text{ is\_isomorphism\_of } R, S \textbf{ implies for } a \textbf{ st} \\ a \in \text{field}\, R \\ \textbf{ex } b \textbf{ st } b \in \text{field}\, S \ \& \ R\,|^2\,(R - \text{Seg}\,(a)), S\,|^2\,(S - \text{Seg}\,(b)) \text{ are\_isomorphic},$$

(62) R is_well-ordering-relation & S is_well-ordering-relation & $a \in \operatorname{field} R$ &
$b \in \operatorname{field} S$ & $c \in \operatorname{field} S$ & $R, S\,|^2\,(S\,-\mathrm{Seg}\,(b))$ are_isomorphic
& $R\,|^2\,(R\,-\mathrm{Seg}\,(a)), S\,|^2\,(S\,-\mathrm{Seg}\,(c))$ are_isomorphic
implies $S\,-\mathrm{Seg}\,(c) \subseteq S\,-\mathrm{Seg}\,(b)$ & $\langle c, b\rangle \in S$,

(63) R is_well-ordering-relation & S is_well-ordering-relation **implies**
R, S are_isomorphic **or**
(**ex** a **st** $a \in \operatorname{field} R$ & $R\,|^2\,(R\,-\mathrm{Seg}\,(a)), S$ are_isomorphic)
or ex a **st** $a \in \operatorname{field} S$ & $R, S\,|^2\,(S\,-\mathrm{Seg}\,(a))$ are_isomorphic ,

(64) $Y \subseteq \operatorname{field} R$ & R is_well-ordering-relation **implies** $R, R\,|^2\,Y$ are_isomorphic
or ex a **st** $a \in \operatorname{field} R$ & $R\,|^2\,(R\,-\mathrm{Seg}\,(a)), R\,|^2\,Y$ are_isomorphic .

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A Model of ZF Set Theory Language

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Summary. The goal of this article is to construct a language of the ZF set theory and to develop a notational and conceptual base which facilitates a convenient usage of the language.

The articles [5], [6], [3], [4], [1], and [2] provide the terminology and notation for this paper. For simplicity we adopt the following convention: k, n will have the type Nat; D will have the type DOMAIN; a will have the type Any; p, q will have the type FinSequence **of** NAT. The constant VAR has the type SUBDOMAIN **of** NAT, and is defined by

$$\mathbf{it} = \{ k : 5 \leq k \}.$$

The following proposition is true

(1) $$\mathrm{VAR} = \{ k : 5 \leq k \}.$$

Variable stands for Element **of** VAR .

One can prove the following proposition

(2) a **is** Variable **iff** a **is** Element **of** VAR .

Let us consider n. The functor

$$\xi\, n,$$

with values of the type Variable, is defined by

$$\mathbf{it} = 5 + n.$$

One can prove the following proposition

(3) $$\xi\, n = 5 + n.$$

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In the sequel x, y, z, t denote objects of the type Variable. Let us consider x. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$<x> \quad \text{is} \quad \text{FinSequence } \textbf{of} \text{ NAT}.$$

We now define two new functors. Let us consider x, y. The functor

$$x \mathbin{\text{=}} y,$$

with values of the type FinSequence **of** NAT, is defined by

$$\textbf{it} = <0> \frown <x> \frown <y>.$$

The functor

$$x \epsilon y,$$

yields the type FinSequence **of** NAT and is defined by

$$\textbf{it} = <1> \frown <x> \frown <y>.$$

Next we state four propositions:

(4) $$x \mathbin{\text{=}} y = <0> \frown <x> \frown <y>,$$

(5) $$x \epsilon y = <1> \frown <x> \frown <y>,$$

(6) $$x \mathbin{\text{=}} y = z \mathbin{\text{=}} t \textbf{ implies } x = z \ \& \ y = t,$$

(7) $$x \epsilon y = z \epsilon t \textbf{ implies } x = z \ \& \ y = t.$$

We now define two new functors. Let us consider p. The functor

$$\neg p,$$

with values of the type FinSequence **of** NAT, is defined by

$$\textbf{it} = <2> \frown p.$$

Let us consider q. The functor

$$p \wedge q,$$

with values of the type FinSequence **of** NAT, is defined by

$$\textbf{it} = <3> \frown p \frown q.$$

Next we state three propositions:

(8) $$\neg p = <2> \frown p,$$

(9) $$p \wedge q = <3> \frown p \frown q,$$

(10) $$\neg p = \neg q \textbf{ implies } p = q.$$

Let us consider x, p. The functor

$$\forall\,(x,p),$$

yields the type FinSequence **of** NAT and is defined by

$$\textbf{it} = <4> ^\frown <x> ^\frown p.$$

The following propositions are true:

(11) $$\forall\,(x,p) = <4> ^\frown <x> ^\frown p,$$

(12) $$\forall\,(x,p) = \forall\,(y,q) \textbf{ implies } x = y \;\&\; p = q.$$

The constant WFF has the type DOMAIN, and is defined by

$$(\textbf{for } a \textbf{ st } a \in \textbf{it} \textbf{ holds } a \textbf{ is } \text{FinSequence } \textbf{of } \text{NAT}) \;\&$$
$$(\textbf{for } x,y \textbf{ holds } x \mathrel{\text{=}} y \in \textbf{it} \;\&\; x \mathrel{\epsilon} y \in \textbf{it}) \;\&\; (\textbf{for } p \textbf{ st } p \in \textbf{it} \textbf{ holds } \neg\, p \in \textbf{it}) \;\&$$
$$(\textbf{for } p,q \textbf{ st } p \in \textbf{it} \;\&\; q \in \textbf{it} \textbf{ holds } p \wedge q \in \textbf{it}) \;\&\; (\textbf{for } x,p \textbf{ st } p \in \textbf{it} \textbf{ holds } \forall\,(x,p) \in \textbf{it}) \;\&$$
$$\textbf{for } D \textbf{ st}$$
$$(\textbf{for } a \textbf{ st } a \in D \textbf{ holds } a \textbf{ is } \text{FinSequence } \textbf{of } \text{NAT}) \;\&$$
$$(\textbf{for } x,y \textbf{ holds } x \mathrel{\text{=}} y \in D \;\&\; x \mathrel{\epsilon} y \in D) \;\&\; (\textbf{for } p \textbf{ st } p \in D \textbf{ holds } \neg\, p \in D)$$
$$\&\; (\textbf{for } p,q \textbf{ st } p \in D \;\&\; q \in D \textbf{ holds } p \wedge q \in D) \;\&\; \textbf{for } x,p \textbf{ st } p \in D \textbf{ holds } \forall\,(x,p) \in D$$
$$\textbf{holds it} \subseteq D.$$

One can prove the following proposition

(13) $$(\textbf{for } a \textbf{ st } a \in \text{WFF} \textbf{ holds } a \textbf{ is } \text{FinSequence } \textbf{of } \text{NAT}) \;\&$$
$$(\textbf{for } x,y \textbf{ holds } x \mathrel{\text{=}} y \in \text{WFF} \;\&\; x \mathrel{\epsilon} y \in \text{WFF}) \;\&$$
$$(\textbf{for } p \textbf{ st } p \in \text{WFF} \textbf{ holds } \neg\, p \in \text{WFF}) \;\&$$
$$(\textbf{for } p,q \textbf{ st } p \in \text{WFF} \;\&\; q \in \text{WFF} \textbf{ holds } p \wedge q \in \text{WFF}) \;\&$$
$$(\textbf{for } x,p \textbf{ st } p \in \text{WFF} \textbf{ holds } \forall\,(x,p) \in \text{WFF}) \;\&\; \textbf{for } D \textbf{ st}$$
$$(\textbf{for } a \textbf{ st } a \in D \textbf{ holds } a \textbf{ is } \text{FinSequence } \textbf{of } \text{NAT}) \;\&$$
$$(\textbf{for } x,y \textbf{ holds } x \mathrel{\text{=}} y \in D \;\&\; x \mathrel{\epsilon} y \in D) \;\&\; (\textbf{for } p \textbf{ st } p \in D \textbf{ holds } \neg\, p \in D) \;\&$$
$$(\textbf{for } p,q \textbf{ st } p \in D \;\&\; q \in D \textbf{ holds } p \wedge q \in D)$$
$$\&\; \textbf{for } x,p \textbf{ st } p \in D \textbf{ holds } \forall\,(x,p) \in D$$
$$\textbf{holds } \text{WFF} \subseteq D.$$

The mode

$$\text{ZF-formula}\,,$$

which widens to the type FinSequence **of** NAT, is defined by

$$\textbf{it is } \text{Element } \textbf{of } \text{WFF}\,.$$

We now state two propositions:

(14) $$a \textbf{ is } \text{ZF-formula} \textbf{ iff } a \in \text{WFF},$$

(15) $$a \textbf{ is } \text{ZF-formula} \textbf{ iff } a \textbf{ is } \text{Element} \textbf{ of } \text{WFF}.$$

In the sequel F, $F1$, G, $G1$, H, $H1$ denote objects of the type ZF-formula. Let us consider x, y. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$x \mathrel{\text{=}} y \qquad \text{is} \qquad \text{ZF-formula},$$

$$x \mathrel{\epsilon} y \qquad \text{is} \qquad \text{ZF-formula}.$$

Let us consider H. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\neg H \qquad \text{is} \qquad \text{ZF-formula}.$$

Let us consider G. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$H \wedge G \qquad \text{is} \qquad \text{ZF-formula}.$$

Let us consider x, H. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\forall (x, H) \qquad \text{is} \qquad \text{ZF-formula}.$$

We now define five new predicates. Let us consider H. The predicate

$$H \text{ is\_a\_equality} \qquad \text{is defined by} \qquad \textbf{ex } x,y \textbf{ st } H = x \mathrel{\text{=}} y.$$

The predicate

$$H \text{ is\_a\_membership} \qquad \text{is defined by} \qquad \textbf{ex } x,y \textbf{ st } H = x \mathrel{\epsilon} y.$$

The predicate

$$H \text{ is\_negative} \qquad \text{is defined by} \qquad \textbf{ex } H1 \textbf{ st } H = \neg H1.$$

The predicate

$$H \text{ is\_conjunctive} \qquad \text{is defined by} \qquad \textbf{ex } F,G \textbf{ st } H = F \wedge G.$$

The predicate

$$H \text{ is\_universal} \qquad \text{is defined by} \qquad \textbf{ex } x,H1 \textbf{ st } H = \forall (x, H1).$$

The following proposition is true

(16) $$\begin{gathered}(H \text{ is\_a\_equality} \textbf{ iff ex } x,y \textbf{ st } H = x \mathrel{\text{=}} y)\ \&\\ (H \text{ is\_a\_membership} \textbf{ iff ex } x,y \textbf{ st } H = x \mathrel{\epsilon} y)\ \&\\ (H \text{ is\_negative} \textbf{ iff ex } H1 \textbf{ st } H = \neg H1)\ \&\\ (H \text{ is\_conjunctive} \textbf{ iff ex } F,G \textbf{ st } H = F \wedge G)\\ \&\ (H \text{ is\_universal} \textbf{ iff ex } x,H1 \textbf{ st } H = \forall (x, H1)).\end{gathered}$$

Let us consider H. The predicate

H is_atomic   is defined by H is_a_equality **or** H is_a_membership .

Next we state a proposition

(17) H is_atomic **iff** H is_a_equality **or** H is_a_membership .

We now define two new functors. Let us consider F, G. The functor

$$F \vee G,$$

yields the type ZF-formula and is defined by

$$\mathbf{it} = \neg\,(\neg\, F \wedge \neg\, G).$$

The functor

$$F \Rightarrow G,$$

yields the type ZF-formula and is defined by

$$\mathbf{it} = \neg\,(F \wedge \neg\, G).$$

The following two propositions are true:

(18) $$F \vee G = \neg\,(\neg\, F \wedge \neg\, G),$$

(19) $$F \Rightarrow G = \neg\,(F \wedge \neg\, G).$$

Let us consider F, G. The functor

$$F \Leftrightarrow G,$$

yields the type ZF-formula and is defined by

$$\mathbf{it} = (F \Rightarrow G) \wedge (G \Rightarrow F).$$

We now state a proposition

(20) $$F \Leftrightarrow G = (F \Rightarrow G) \wedge (G \Rightarrow F).$$

Let us consider x, H. The functor

$$\exists\,(x, H),$$

yields the type ZF-formula and is defined by

$$\mathbf{it} = \neg\,\forall\,(x, \neg\, H).$$

The following proposition is true

(21) $$\exists\,(x, H) = \neg\,\forall\,(x, \neg\, H).$$

We now define four new predicates. Let us consider H. The predicate

$$H \text{ is\_disjunctive} \qquad \text{is defined by} \qquad \mathbf{ex}\, F, G \;\mathbf{st}\; H = F \vee G.$$

The predicate

$$H \text{ is\_conditional} \qquad \text{is defined by} \qquad \mathbf{ex}\, F, G \;\mathbf{st}\; H = F \Rightarrow G.$$

The predicate

$$H \text{ is\_biconditional} \qquad \text{is defined by} \qquad \mathbf{ex}\, F, G \;\mathbf{st}\; H = F \Leftrightarrow G.$$

The predicate

$$H \text{ is\_existential} \qquad \text{is defined by} \qquad \mathbf{ex}\, x, H1 \;\mathbf{st}\; H = \exists\,(x, H1).$$

The following proposition is true

(22)
$$\begin{gathered}(H \text{ is\_disjunctive } \mathbf{iff}\ \mathbf{ex}\, F, G \;\mathbf{st}\; H = F \vee G)\ \&\\ (H \text{ is\_conditional } \mathbf{iff}\ \mathbf{ex}\, F, G \;\mathbf{st}\; H = F \Rightarrow G)\ \&\\ (H \text{ is\_biconditional } \mathbf{iff}\ \mathbf{ex}\, F, G \;\mathbf{st}\; H = F \Leftrightarrow G)\\ \&\ (H \text{ is\_existential } \mathbf{iff}\ \mathbf{ex}\, x, H1 \;\mathbf{st}\; H = \exists\,(x, H1)).\end{gathered}$$

We now define two new functors. Let us consider x, y, H. The functor

$$\forall\,(x, y, H),$$

yields the type ZF-formula and is defined by

$$\mathbf{it} = \forall\,(x, \forall\,(y, H)).$$

The functor

$$\exists\,(x, y, H),$$

yields the type ZF-formula and is defined by

$$\mathbf{it} = \exists\,(x, \exists\,(y, H)).$$

The following proposition is true

(23)
$$\forall\,(x, y, H) = \forall\,(x, \forall\,(y, H))\ \&\ \exists\,(x, y, H) = \exists\,(x, \exists\,(y, H)).$$

We now define two new functors. Let us consider x, y, z, H. The functor

$$\forall\,(x, y, z, H),$$

with values of the type ZF-formula, is defined by

$$\mathbf{it} = \forall\,(x, \forall\,(y, z, H)).$$

The functor

$$\exists\,(x, y, z, H),$$

with values of the type ZF-formula, is defined by

$$\mathbf{it} = \exists\,(x, \exists\,(y, z, H)).$$

We now state several propositions:

(24) $$\forall\,(x, y, z, H) = \forall\,(x, \forall\,(y, z, H))\ \&\ \exists\,(x, y, z, H) = \exists\,(x, \exists\,(y, z, H)),$$

(25) $$H \text{ is\_a\_equality } \mathbf{or}\ H \text{ is\_a\_membership } \mathbf{or}\ H \text{ is\_negative } \mathbf{or}\ H \text{ is\_conjunctive } \mathbf{or}\ H \text{ is\_universal}\,,$$

(26) $$H \text{ is\_atomic } \mathbf{or}\ H \text{ is\_negative } \mathbf{or}\ H \text{ is\_conjunctive } \mathbf{or}\ H \text{ is\_universal}\,,$$

(27) $$H \text{ is\_atomic } \mathbf{implies} \operatorname{len} H = 3,$$

(28) $$H \text{ is\_atomic } \mathbf{or}\ \mathbf{ex}\ H1\ \mathbf{st} \operatorname{len} H1 + 1 \le \operatorname{len} H,$$

(29) $$3 \le \operatorname{len} H,$$

(30) $$\operatorname{len} H = 3\ \mathbf{implies}\ H \text{ is\_atomic}\,.$$

One can prove the following propositions:

(31) $$\mathbf{for}\ x,y\ \mathbf{holds}\ (x = y).1 = 0\ \&\ (x \epsilon y).1 = 1,$$

(32) $$\mathbf{for}\ H\ \mathbf{holds}\ (\neg H).1 = 2,$$

(33) $$\mathbf{for}\ F,G\ \mathbf{holds}\ (F \wedge G).1 = 3,$$

(34) $$\mathbf{for}\ x,H\ \mathbf{holds}\ \forall\,(x, H).1 = 4,$$

(35) $$H \text{ is\_a\_equality } \mathbf{implies}\ H.1 = 0,$$

(36) $$H \text{ is\_a\_membership } \mathbf{implies}\ H.1 = 1,$$

(37) $$H \text{ is\_negative } \mathbf{implies}\ H.1 = 2,$$

(38) $$H \text{ is\_conjunctive } \mathbf{implies}\ H.1 = 3,$$

(39) $$H \text{ is\_universal } \mathbf{implies}\ H.1 = 4,$$

(40) $$H \text{ is\_a\_equality } \&\ H.1 = 0\ \mathbf{or}\ H \text{ is\_a\_membership } \&\ H.1 = 1\ \mathbf{or}\ H \text{ is\_negative } \&\ H.1 = 2\ \mathbf{or}\ H \text{ is\_conjunctive } \&\ H.1 = 3\ \mathbf{or}\ H \text{ is\_universal } \&\ H.1 = 4,$$

(41) $$H.1 = 0\ \mathbf{implies}\ H \text{ is\_a\_equality}\,,$$

(42) $$H.1 = 1\ \mathbf{implies}\ H \text{ is\_a\_membership}\,,$$

(43) $$H\,.1 = 2 \textbf{ implies } H \text{ is\_negative}\,,$$

(44) $$H\,.1 = 3 \textbf{ implies } H \text{ is\_conjunctive}\,,$$

(45) $$H\,.1 = 4 \textbf{ implies } H \text{ is\_universal}\,.$$

In the sequel sq denotes an object of the type FinSequence. We now state several propositions:

(46) $$H = F \frown sq \textbf{ implies } H = F,$$

(47) $$H \wedge G = H1 \wedge G1 \textbf{ implies } H = H1 \;\&\; G = G1,$$

(48) $$F \vee G = F1 \vee G1 \textbf{ implies } F = F1 \;\&\; G = G1,$$

(49) $$F \Rightarrow G = F1 \Rightarrow G1 \textbf{ implies } F = F1 \;\&\; G = G1,$$

(50) $$F \Leftrightarrow G = F1 \Leftrightarrow G1 \textbf{ implies } F = F1 \;\&\; G = G1,$$

(51) $$\exists\,(x, H) = \exists\,(y, G) \textbf{ implies } x = y \;\&\; H = G.$$

We now define two new functors. Let us consider H. Assume that the following holds

$$H \text{ is\_atomic}\,.$$

The functor

$$\mathrm{Var}_1\, H,$$

yields the type Variable and is defined by

$$\textbf{it} = H\,.2.$$

The functor

$$\mathrm{Var}_2\, H,$$

yields the type Variable and is defined by

$$\textbf{it} = H\,.3.$$

One can prove the following three propositions:

(52) $$H \text{ is\_atomic} \textbf{ implies } \mathrm{Var}_1\, H = H\,.2 \;\&\; \mathrm{Var}_2\, H = H\,.3,$$

(53) $$H \text{ is\_a\_equality} \textbf{ implies } H = (\mathrm{Var}_1\, H) = \mathrm{Var}_2\, H,$$

(54) $$H \text{ is\_a\_membership} \textbf{ implies } H = (\mathrm{Var}_1\, H) \in \mathrm{Var}_2\, H.$$

Let us consider H. Assume that the following holds

$$H \text{ is\_negative}\,.$$

The functor

$$\text{the\_argument\_of}\, H,$$

with values of the type ZF-formula, is defined by

$$\neg\, \mathbf{it} = H.$$

We now state a proposition

(55) $$H \text{ is\_negative } \mathbf{implies}\ H = \neg\, \text{the\_argument\_of}\, H.$$

We now define two new functors. Let us consider H. Assume that the following holds

$$H \text{ is\_conjunctive } \mathbf{or}\ H \text{ is\_disjunctive}\,.$$

The functor

$$\text{the\_left\_argument\_of}\, H,$$

with values of the type ZF-formula, is defined by

$$\begin{array}{lll} \mathbf{ex}\, H1\ \mathbf{st}\ \mathbf{it} \wedge H1 = H, & \mathbf{if} & H \text{ is\_conjunctive}\,, \\ \mathbf{ex}\, H1\ \mathbf{st}\ \mathbf{it} \vee H1 = H, & \mathbf{otherwise}. & \end{array}$$

The functor

$$\text{the\_right\_argument\_of}\, H,$$

with values of the type ZF-formula, is defined by

$$\begin{array}{lll} \mathbf{ex}\, H1\ \mathbf{st}\ H1 \wedge \mathbf{it} = H, & \mathbf{if} & H \text{ is\_conjunctive}\,, \\ \mathbf{ex}\, H1\ \mathbf{st}\ H1 \vee \mathbf{it} = H, & \mathbf{otherwise}. & \end{array}$$

One can prove the following propositions:

(56) $$H \text{ is\_conjunctive } \mathbf{implies}\ (F = \text{the\_left\_argument\_of}\, H\ \mathbf{iff}\ \mathbf{ex}\, G\ \mathbf{st}\ F \wedge G = H) \\ \&\ (F = \text{the\_right\_argument\_of}\, H\ \mathbf{iff}\ \mathbf{ex}\, G\ \mathbf{st}\ G \wedge F = H),$$

(57) $$H \text{ is\_disjunctive } \mathbf{implies}\ (F = \text{the\_left\_argument\_of}\, H\ \mathbf{iff}\ \mathbf{ex}\, G\ \mathbf{st}\ F \vee G = H) \\ \&\ (F = \text{the\_right\_argument\_of}\, H\ \mathbf{iff}\ \mathbf{ex}\, G\ \mathbf{st}\ G \vee F = H),$$

(58) $$H \text{ is\_conjunctive} \\ \mathbf{implies}\ H = (\text{the\_left\_argument\_of}\, H) \wedge \text{the\_right\_argument\_of}\, H,$$

(59) $$H \text{ is\_disjunctive} \\ \mathbf{implies}\ H = (\text{the\_left\_argument\_of}\, H) \vee \text{the\_right\_argument\_of}\, H.$$

We now define two new functors. Let us consider H. Assume that the following holds

$$H \text{ is\_universal } \mathbf{or}\ H \text{ is\_existential}\,.$$

The functor

$$\text{bound\_in}\,H,$$

with values of the type Variable, is defined by

$$\begin{array}{ll} \mathbf{ex}\,H1\ \mathbf{st}\ \forall\,(\mathbf{it}\,,H1) = H, & \mathbf{if} \qquad H\ \text{is\_universal}\,, \\ \mathbf{ex}\,H1\ \mathbf{st}\ \exists\,(\mathbf{it}\,,H1) = H, & \mathbf{otherwise}. \end{array}$$

The functor

$$\text{the\_scope\_of}\,H,$$

with values of the type ZF-formula, is defined by

$$\begin{array}{ll} \mathbf{ex}\,x\ \mathbf{st}\ \forall\,(x,\mathbf{it}) = H, & \mathbf{if} \qquad H\ \text{is\_universal}\,, \\ \mathbf{ex}\,x\ \mathbf{st}\ \exists\,(x,\mathbf{it}) = H, & \mathbf{otherwise}. \end{array}$$

Next we state four propositions:

(60) $$H\ \text{is\_universal}\ \mathbf{implies}\ (x = \text{bound\_in}\,H\ \mathbf{iff}\ \mathbf{ex}\,H1\ \mathbf{st}\ \forall\,(x,H1) = H) \\ \&\ (H1 = \text{the\_scope\_of}\,H\ \mathbf{iff}\ \mathbf{ex}\,x\ \mathbf{st}\ \forall\,(x,H1) = H),$$

(61) $$H\ \text{is\_existential}\ \mathbf{implies}\ (x = \text{bound\_in}\,H\ \mathbf{iff}\ \mathbf{ex}\,H1\ \mathbf{st}\ \exists\,(x,H1) = H) \\ \&\ (H1 = \text{the\_scope\_of}\,H\ \mathbf{iff}\ \mathbf{ex}\,x\ \mathbf{st}\ \exists\,(x,H1) = H),$$

(62) $$H\ \text{is\_universal}\ \mathbf{implies}\ H = \forall\,(\text{bound\_in}\,H,\text{the\_scope\_of}\,H),$$

(63) $$H\ \text{is\_existential}\ \mathbf{implies}\ H = \exists\,(\text{bound\_in}\,H,\text{the\_scope\_of}\,H).$$

We now define two new functors. Let us consider H. Assume that the following holds

$$H\ \text{is\_conditional}\,.$$

The functor

$$\text{the\_antecedent\_of}\,H,$$

with values of the type ZF-formula, is defined by

$$\mathbf{ex}\,H1\ \mathbf{st}\ H = \mathbf{it} \Rightarrow H1.$$

The functor

$$\text{the\_consequent\_of}\,H,$$

with values of the type ZF-formula, is defined by

$$\mathbf{ex}\,H1\ \mathbf{st}\ H = H1 \Rightarrow \mathbf{it}\,.$$

The following propositions are true:

(64) $$H\ \text{is\_conditional}\ \mathbf{implies}\ (F = \text{the\_antecedent\_of}\,H\ \mathbf{iff}\ \mathbf{ex}\,G\ \mathbf{st}\ H = F \Rightarrow G) \\ \&\ (F = \text{the\_consequent\_of}\,H\ \mathbf{iff}\ \mathbf{ex}\,G\ \mathbf{st}\ H = G \Rightarrow F),$$

(65) H is_conditional **implies** $H = (\text{the\_antecedent\_of}\, H) \Rightarrow \text{the\_consequent\_of}\, H$.

We now define two new functors. Let us consider H. Assume that the following holds

$$H \text{ is\_biconditional}\,.$$

The functor

$$\text{the\_left\_side\_of}\, H,$$

yields the type ZF-formula and is defined by

$$\mathbf{ex}\, H1 \;\mathbf{st}\; H = \mathbf{it} \Leftrightarrow H1.$$

The functor

$$\text{the\_right\_side\_of}\, H,$$

with values of the type ZF-formula, is defined by

$$\mathbf{ex}\, H1 \;\mathbf{st}\; H = H1 \Leftrightarrow \mathbf{it}\,.$$

We now state two propositions:

(66) H is_biconditional **implies** $(F = \text{the\_left\_side\_of}\, H \;\mathbf{iff}\; \mathbf{ex}\, G \;\mathbf{st}\; H = F \Leftrightarrow G)$
$\&\ (F = \text{the\_right\_side\_of}\, H \;\mathbf{iff}\; \mathbf{ex}\, G \;\mathbf{st}\; H = G \Leftrightarrow F)$,

(67) H is_biconditional **implies** $H = (\text{the\_left\_side\_of}\, H) \Leftrightarrow \text{the\_right\_side\_of}\, H$.

Let us consider H, F. The predicate

$$H \text{ is\_immediate\_constituent\_of } F$$

is defined by

$$F = \neg\, H \;\mathbf{or}\; (\mathbf{ex}\, H1 \;\mathbf{st}\; F = H \wedge H1 \;\mathbf{or}\; F = H1 \wedge H) \;\mathbf{or}\; \mathbf{ex}\, x \;\mathbf{st}\; F = \forall\,(x, H).$$

We now state a number of propositions:

(68) H is_immediate_constituent_of F **iff**
$F = \neg\, H \;\mathbf{or}\; (\mathbf{ex}\, H1 \;\mathbf{st}\; F = H \wedge H1 \;\mathbf{or}\; F = H1 \wedge H) \;\mathbf{or}\; \mathbf{ex}\, x \;\mathbf{st}\; F = \forall\,(x, H)$,

(69) **not** H is_immediate_constituent_of $x = y$,

(70) **not** H is_immediate_constituent_of $x \,\epsilon\, y$,

(71) F is_immediate_constituent_of $\neg\, H$ **iff** $F = H$,

(72) F is_immediate_constituent_of $G \wedge H$ **iff** $F = G$ **or** $F = H$,

(73) F is_immediate_constituent_of $\forall\,(x, H)$ **iff** $F = H$,

$$H \text{ is\_atomic } \textbf{implies not } F \text{ is\_immediate\_constituent\_of } H, \tag{74}$$

$$H \text{ is\_negative} \\ \textbf{implies } (F \text{ is\_immediate\_constituent\_of } H \textbf{ iff } F = \text{the\_argument\_of} H), \tag{75}$$

$$H \text{ is\_conjunctive } \textbf{implies } (F \text{ is\_immediate\_constituent\_of } H \\ \textbf{iff } F = \text{the\_left\_argument\_of} H \textbf{ or } F = \text{the\_right\_argument\_of} H), \tag{76}$$

$$H \text{ is\_universal} \\ \textbf{implies } (F \text{ is\_immediate\_constituent\_of } H \textbf{ iff } F = \text{the\_scope\_of} H). \tag{77}$$

In the sequel L will denote an object of the type FinSequence. Let us consider H, F. The predicate

$$H \text{ is\_subformula\_of } F$$

is defined by

$$\textbf{ex}\, n,L \textbf{ st } 1 \leq n \;\&\; \text{len}\, L = n \;\&\; L.1 = H \;\&\; L.n = F \;\&\; \textbf{for}\, k \textbf{ st } 1 \leq k \;\&\; k < n \\ \textbf{ex}\, H1,F1 \textbf{ st } L.k = H1 \;\&\; L.(k+1) = F1 \;\&\; H1 \text{ is\_immediate\_constituent\_of } F1.$$

Next we state two propositions:

(78) H is_subformula_of F **iff ex** n,L **st** $1 \leq n$ & len $L = n$ & $L.1 = H$ & $L.n = F$ &

$$\textbf{for}\, k \textbf{ st } 1 \leq k \;\&\; k < n \textbf{ ex}\, H1,F1 \\ \textbf{st } L.k = H1 \;\&\; L.(k+1) = F1 \;\&\; H1 \text{ is\_immediate\_constituent\_of } F1,$$

$$H \text{ is\_subformula\_of } H. \tag{79}$$

Let us consider H, F. The predicate

H is_proper_subformula_of F is defined by H is_subformula_of F & $H \neq F$.

We now state several propositions:

$$H \text{ is\_proper\_subformula\_of } F \textbf{ iff } H \text{ is\_subformula\_of } F \;\&\; H \neq F, \tag{80}$$

$$H \text{ is\_immediate\_constituent\_of } F \textbf{ implies } \text{len}\, H < \text{len}\, F, \tag{81}$$

$$H \text{ is\_immediate\_constituent\_of } F \textbf{ implies } H \text{ is\_proper\_subformula\_of } F, \tag{82}$$

$$H \text{ is\_proper\_subformula\_of } F \textbf{ implies } \text{len}\, H < \text{len}\, F, \tag{83}$$

$$H \text{ is\_proper\_subformula\_of } F \\ \textbf{implies ex}\, G \textbf{ st } G \text{ is\_immediate\_constituent\_of } F. \tag{84}$$

The following propositions are true:

$$F \text{ is\_proper\_subformula\_of } G \;\&\; G \text{ is\_proper\_subformula\_of } H \\ \textbf{implies } F \text{ is\_proper\_subformula\_of } H, \tag{85}$$

(86) F is_subformula_of G & G is_subformula_of H **implies** F is_subformula_of H,

(87) G is_subformula_of H & H is_subformula_of G **implies** $G = H$,

(88) **not** F is_proper_subformula_of $x \mathrel{=} y$,

(89) **not** F is_proper_subformula_of $x \mathrel{\epsilon} y$,

(90) F is_proper_subformula_of $\neg H$ **implies** F is_subformula_of H,

(91) F is_proper_subformula_of $G \wedge H$
implies F is_subformula_of G **or** F is_subformula_of H,

(92) F is_proper_subformula_of $\forall (x, H)$ **implies** F is_subformula_of H,

(93) H is_atomic **implies not** F is_proper_subformula_of H,

(94) H is_negative **implies** the_argument_of H is_proper_subformula_of H,

(95) H is_conjunctive **implies** the_left_argument_of H is_proper_subformula_of H
& the_right_argument_of H is_proper_subformula_of H,

(96) H is_universal **implies** the_scope_of H is_proper_subformula_of H,

(97) H is_subformula_of $x \mathrel{=} y$ **iff** $H = x \mathrel{=} y$,

(98) H is_subformula_of $x \mathrel{\epsilon} y$ **iff** $H = x \mathrel{\epsilon} y$.

Let us consider H. The functor

Subformulae H,

yields the type set and is defined by

$a \in$ **it iff ex** F **st** $F = a$ & F is_subformula_of H.

We now state a number of propositions:

(99) $a \in$ Subformulae H **iff ex** F **st** $F = a$ & F is_subformula_of H,

(100) $G \in$ Subformulae H **implies** G is_subformula_of H,

(101) F is_subformula_of H **implies** Subformulae $F \subseteq$ Subformulae H,

(102) Subformulae $x \mathrel{=} y = \{x \mathrel{=} y\}$,

(103) Subformulae $x \mathrel{\epsilon} y = \{x \mathrel{\epsilon} y\}$,

(104) Subformulae $\neg H =$ Subformulae $H \cup \{\neg H\}$,

$$\text{Subformulae}\,(H \wedge F) = \text{Subformulae}\,H \cup \text{Subformulae}\,F \cup \{H \wedge F\}, \tag{105}$$

$$\text{Subformulae}\,\forall\,(x, H) = \text{Subformulae}\,H \cup \{\forall\,(x, H)\}, \tag{106}$$

$$H\ \text{is\_atomic}\ \textbf{iff}\ \text{Subformulae}\,H = \{H\}, \tag{107}$$

$$\begin{gathered} H\ \text{is\_negative} \\ \textbf{implies}\ \text{Subformulae}\,H = \text{Subformulae the\_argument\_of}\,H \cup \{H\}, \end{gathered} \tag{108}$$

$$\begin{gathered} H\ \text{is\_conjunctive}\ \textbf{implies}\ \text{Subformulae}\,H = \text{Subformulae} \\ \text{the\_left\_argument\_of}\,H \cup \text{Subformulae the\_right\_argument\_of}\,H \cup \{H\}, \end{gathered} \tag{109}$$

$$H\ \text{is\_universal}\ \textbf{implies}\ \text{Subformulae}\,H = \text{Subformulae the\_scope\_of}\,H \cup \{H\}, \tag{110}$$

$$\begin{gathered} (H\ \text{is\_immediate\_constituent\_of}\ G \\ \textbf{or}\ H\ \text{is\_proper\_subformula\_of}\ G\ \textbf{or}\ H\ \text{is\_subformula\_of}\ G) \\ \&\ G \in \text{Subformulae}\,F \\ \textbf{implies}\ H \in \text{Subformulae}\,F. \end{gathered} \tag{111}$$

In the article we present several logical schemes. The scheme *ZF_Ind* deals with a unary predicate $\mathcal{P}$ states that the following holds

$$\textbf{for}\ H\ \textbf{holds}\ \mathcal{P}[H]$$

provided the parameter satisfies the following conditions:

- $\textbf{for}\ H\ \textbf{st}\ H\ \text{is\_atomic}\ \textbf{holds}\ \mathcal{P}[H],$
- $\textbf{for}\ H\ \textbf{st}\ H\ \text{is\_negative}\ \&\ \mathcal{P}[\text{the\_argument\_of}\,H]\ \textbf{holds}\ \mathcal{P}[H],$
- $\begin{gathered} \textbf{for}\ H\ \textbf{st} \\ H\ \text{is\_conjunctive}\ \&\ \mathcal{P}[\text{the\_left\_argument\_of}\,H]\ \&\ \mathcal{P}[\text{the\_right\_argument\_of}\,H] \\ \textbf{holds}\ \mathcal{P}[H], \end{gathered}$
- $\textbf{for}\ H\ \textbf{st}\ H\ \text{is\_universal}\ \&\ \mathcal{P}[\text{the\_scope\_of}\,H]\ \textbf{holds}\ \mathcal{P}[H].$

The scheme *ZF_CompInd* deals with a unary predicate $\mathcal{P}$ states that the following holds

$$\textbf{for}\ H\ \textbf{holds}\ \mathcal{P}[H]$$

provided the parameter satisfies the following condition:

- $\textbf{for}\ H\ \textbf{st}\ \textbf{for}\ F\ \textbf{st}\ F\ \text{is\_proper\_subformula\_of}\ H\ \textbf{holds}\ \mathcal{P}[F]\ \textbf{holds}\ \mathcal{P}[H].$

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Families of Sets

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Summary. The article contains definitions of the following concepts: family of sets, family of subsets of a set, the intersection of a family of sets. Functors $\cup$, $\cap$, and $\setminus$ are redefined for families of subsets of a set. Some properties of these notions are presented.

The terminology and notation used in this paper are introduced in the following papers: [1], [3], and [2]. For simplicity we adopt the following convention: X, Y, Z, $Z1$, D will denote objects of the type set; x, y will denote objects of the type Any. Let us consider X. The functor

$$\bigcap X,$$

with values of the type set, is defined by

$$\begin{array}{rll} \textbf{for}\, x \textbf{ holds } x \in \textbf{it iff for}\, Y \textbf{ holds } Y \in X \textbf{ implies } x \in Y, & \textbf{if} & X \neq \emptyset, \\ \textbf{it} = \emptyset, & \textbf{otherwise}. & \end{array}$$

The following propositions are true:

(1) $$X \neq \emptyset \textbf{ implies for}\, x \textbf{ holds } x \in \bigcap X \textbf{ iff for}\, Y \textbf{ st } Y \in X \textbf{ holds } x \in Y,$$

(2) $$\bigcap \emptyset = \emptyset,$$

(3) $$\bigcap X \subseteq \bigcup X,$$

(4) $$Z \in X \textbf{ implies } \bigcap X \subseteq Z,$$

(5) $$\emptyset \in X \textbf{ implies } \bigcap X = \emptyset,$$

(6) $$X \neq \emptyset \mathrel{\&} (\textbf{for}\, Z1 \textbf{ st } Z1 \in X \textbf{ holds } Z \subseteq Z1) \textbf{ implies } Z \subseteq \bigcap X,$$

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$$X \neq \emptyset \ \& \ X \subseteq Y \ \textbf{implies} \bigcap Y \subseteq \bigcap X, \tag{7}$$

$$X \in Y \ \& \ X \subseteq Z \ \textbf{implies} \bigcap Y \subseteq Z, \tag{8}$$

$$X \in Y \ \& \ X \cap Z = \emptyset \ \textbf{implies} \bigcap Y \cap Z = \emptyset, \tag{9}$$

$$X \neq \emptyset \ \& \ Y \neq \emptyset \ \textbf{implies} \bigcap (X \cup Y) = \bigcap X \cap \bigcap Y, \tag{10}$$

$$\bigcap \{x\} = x, \tag{11}$$

$$\bigcap \{X, Y\} = X \cap Y. \tag{12}$$

Set-Family stands for set .

In the sequel SFX, SFY, SFZ will have the type Set-Family. One can prove the following two propositions:

$$x \ \textbf{is} \ \text{Set-Family}, \tag{13}$$

$$SFX = SFY \ \textbf{iff} \ \textbf{for} \ X \ \textbf{holds} \ X \in SFX \ \textbf{iff} \ X \in SFY. \tag{14}$$

We now define two new predicates. Let us consider SFX, SFY. The predicate

$$SFX \ \text{is\_finer\_than} \ SFY$$

is defined by

$$\textbf{for} \ X \ \textbf{st} \ X \in SFX \ \textbf{ex} \ Y \ \textbf{st} \ Y \in SFY \ \& \ X \subseteq Y.$$

The predicate

$$SFX \ \text{is\_coarser\_than} \ SFY$$

is defined by

$$\textbf{for} \ Y \ \textbf{st} \ Y \in SFY \ \textbf{ex} \ X \ \textbf{st} \ X \in SFX \ \& \ X \subseteq Y.$$

Next we state several propositions:

$$SFX \ \text{is\_finer\_than} \ SFY \ \textbf{iff} \ \textbf{for} \ X \ \textbf{st} \ X \in SFX \ \textbf{ex} \ Y \ \textbf{st} \ Y \in SFY \ \& \ X \subseteq Y, \tag{15}$$

$$\begin{gathered} SFX \ \text{is\_coarser\_than} \ SFY \\ \textbf{iff} \ \textbf{for} \ Y \ \textbf{st} \ Y \in SFY \ \textbf{ex} \ X \ \textbf{st} \ X \in SFX \ \& \ X \subseteq Y, \end{gathered} \tag{16}$$

$$SFX \subseteq SFY \ \textbf{implies} \ SFX \ \text{is\_finer\_than} \ SFY, \tag{17}$$

$$SFX \ \text{is\_finer\_than} \ SFY \ \textbf{implies} \bigcup SFX \subseteq \bigcup SFY, \tag{18}$$

$$SFY \neq \emptyset \ \& \ SFX \ \text{is\_coarser\_than} \ SFY \ \textbf{implies} \bigcap SFX \subseteq \bigcap SFY. \tag{19}$$

Let us note that it makes sense to consider the following constant. Then $\emptyset$ is Set-Family . Let us consider x. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\{x\} \quad \text{is} \quad \text{Set-Family}\,.$$

Let us consider y. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\{x,y\} \quad \text{is} \quad \text{Set-Family}\,.$$

One can prove the following propositions:

(20) $$\emptyset \text{ is\_finer\_than } SFX,$$

(21) $$SFX \text{ is\_finer\_than } \emptyset \textbf{ implies } SFX = \emptyset,$$

(22) $$SFX \text{ is\_finer\_than } SFX,$$

(23) $$SFX \text{ is\_finer\_than } SFY \;\&\; SFY \text{ is\_finer\_than } SFZ \textbf{ implies } SFX \text{ is\_finer\_than } SFZ,$$

(24) $$SFX \text{ is\_finer\_than } \{Y\} \textbf{ implies for } X \textbf{ st } X \in SFX \textbf{ holds } X \subseteq Y,$$

(25) $$SFX \text{ is\_finer\_than } \{X,Y\} \textbf{ implies for } Z \textbf{ st } Z \in SFX \textbf{ holds } Z \subseteq X \textbf{ or } Z \subseteq Y.$$

We now define three new functors. Let us consider SFX, SFY. The functor

$$\text{UNION}\,(SFX,SFY),$$

yields the type Set-Family and is defined by

$$Z \in \textbf{it iff ex } X,Y \textbf{ st } X \in SFX \;\&\; Y \in SFY \;\&\; Z = X \cup Y.$$

The functor

$$\text{INTERSECTION}\,(SFX,SFY),$$

with values of the type Set-Family, is defined by

$$Z \in \textbf{it iff ex } X,Y \textbf{ st } X \in SFX \;\&\; Y \in SFY \;\&\; Z = X \cap Y.$$

The functor

$$\text{DIFFERENCE}\,(SFX,SFY),$$

with values of the type Set-Family, is defined by

$$Z \in \textbf{it iff ex } X,Y \textbf{ st } X \in SFX \;\&\; Y \in SFY \;\&\; Z = X \setminus Y.$$

One can prove the following propositions:

(26) $Z \in \text{UNION}\,(SFX,SFY) \textbf{ iff ex } X,Y \textbf{ st } X \in SFX \;\&\; Y \in SFY \;\&\; Z = X \cup Y$,

$$Z \in \mathrm{INTERSECTION}\,(SFX,SFY) \quad \textbf{iff ex}\ X,Y\ \textbf{st}\ X \in SFX\ \&\ Y \in SFY\ \&\ Z = X \cap Y, \tag{27}$$

$$Z \in \mathrm{DIFFERENCE}\,(SFX,SFY) \quad \textbf{iff ex}\ X,Y\ \textbf{st}\ X \in SFX\ \&\ Y \in SFY\ \&\ Z = X \setminus Y, \tag{28}$$

$$SFX\ \text{is\_finer\_than}\ \mathrm{UNION}\,(SFX,SFX), \tag{29}$$

$$\mathrm{INTERSECTION}\,(SFX,SFX)\ \text{is\_finer\_than}\ SFX, \tag{30}$$

$$\mathrm{DIFFERENCE}\,(SFX,SFX)\ \text{is\_finer\_than}\ SFX, \tag{31}$$

$$\mathrm{UNION}\,(SFX,SFY) = \mathrm{UNION}\,(SFY,SFX), \tag{32}$$

$$\mathrm{INTERSECTION}\,(SFX,SFY) = \mathrm{INTERSECTION}\,(SFY,SFX), \tag{33}$$

$$SFX \cap SFY \neq \emptyset \quad \textbf{implies} \bigcap SFX \cap \bigcap SFY = \bigcap \mathrm{INTERSECTION}\,(SFX,SFY), \tag{34}$$

$$SFY \neq \emptyset\ \textbf{implies}\ X \cup \bigcap SFY = \bigcap \mathrm{UNION}\,(\{X\},SFY), \tag{35}$$

$$X \cap \bigcup SFY = \bigcup \mathrm{INTERSECTION}\,(\{X\},SFY), \tag{36}$$

$$SFY \neq \emptyset\ \textbf{implies}\ X \setminus \bigcup SFY = \bigcap \mathrm{DIFFERENCE}\,(\{X\},SFY), \tag{37}$$

$$SFY \neq \emptyset\ \textbf{implies}\ X \setminus \bigcap SFY = \bigcup \mathrm{DIFFERENCE}\,(\{X\},SFY), \tag{38}$$

$$\bigcup \mathrm{INTERSECTION}\,(SFX,SFY) \subseteq \bigcup SFX \cap \bigcup SFY, \tag{39}$$

$$SFX \neq \emptyset\ \&\ SFY \neq \emptyset\ \textbf{implies} \bigcap SFX \cup \bigcap SFY \subseteq \bigcap \mathrm{UNION}\,(SFX,SFY), \tag{40}$$

$$SFX \neq \emptyset\ \&\ SFY \neq \emptyset \quad \textbf{implies} \bigcap \mathrm{DIFFERENCE}\,(SFX,SFY) \subseteq \bigcap SFX \setminus \bigcap SFY. \tag{41}$$

Let D have the type set.

Subset-Family **of** D stands for Subset **of** bool D.

We now state a proposition

$$\textbf{for}\ F\ \textbf{being}\ \text{Subset}\ \textbf{of}\ \text{bool}\,D\ \textbf{holds}\ F\ \textbf{is}\ \text{Subset-Family}\ \textbf{of}\ D. \tag{42}$$

In the sequel F, G have the type Subset-Family **of** D; P has the type Subset **of** D. Let us consider D, F, G. Let us note that it makes sense to consider the following functors on restricted areas. Then

$F \cup G$ is Subset-Family **of** D,

$$F \cap G \quad \text{is} \quad \text{Subset-Family } \mathbf{of}\ D,$$

$$F \setminus G \quad \text{is} \quad \text{Subset-Family } \mathbf{of}\ D.$$

Next we state a proposition

$$X \in F \ \mathbf{implies}\ X\ \mathbf{is}\ \text{Subset } \mathbf{of}\ D. \tag{43}$$

Let us consider D, F. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\bigcup F \quad \text{is} \quad \text{Subset } \mathbf{of}\ D.$$

Let us consider D, F. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\bigcap F \quad \text{is} \quad \text{Subset } \mathbf{of}\ D.$$

The following proposition is true

$$F = G\ \mathbf{iff}\ \mathbf{for}\ P\ \mathbf{holds}\ P \in F\ \mathbf{iff}\ P \in G. \tag{44}$$

The scheme *SubFamEx* deals with a constant $\mathcal{A}$ that has the type set and a unary predicate $\mathcal{P}$ and states that the following holds

$$\mathbf{ex}\ F\ \mathbf{being}\ \text{Subset-Family } \mathbf{of}\ \mathcal{A}\ \mathbf{st}\ \mathbf{for}\ B\ \mathbf{being}\ \text{Subset } \mathbf{of}\ \mathcal{A}\ \mathbf{holds}\ B \in F\ \mathbf{iff}\ \mathcal{P}[B]$$

for all values of the parameters.

Let us consider D, F. The functor

$$F^{\,\mathrm{c}},$$

yields the type Subset-Family **of** D and is defined by

$$\mathbf{for}\ P\ \mathbf{being}\ \text{Subset } \mathbf{of}\ D\ \mathbf{holds}\ P \in \mathbf{it}\ \mathbf{iff}\ P^{\,\mathrm{c}} \in F.$$

Next we state four propositions:

$$\mathbf{for}\ P\ \mathbf{holds}\ P \in F^{\,\mathrm{c}}\ \mathbf{iff}\ P^{\,\mathrm{c}} \in F, \tag{45}$$

$$F \neq \emptyset\ \mathbf{implies}\ F^{\,\mathrm{c}} \neq \emptyset, \tag{46}$$

$$F \neq \emptyset\ \mathbf{implies}\ \Omega\, D \setminus \bigcup F = \bigcap (F^{\,\mathrm{c}}), \tag{47}$$

$$F \neq \emptyset\ \mathbf{implies}\ \bigcup F^{\,\mathrm{c}} = \Omega\, D \setminus \bigcap F. \tag{48}$$

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Functions from a Set to a Set

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Summary. The article is a continuation of [1]. We define the following concepts: a function from a set X into a set Y, denoted by "Function of X,Y", the set of all functions from a set X into a set Y, denoted by Funcs(X,Y), and the permutation of a set (mode Permutation of X, where X is a set). Theorems and schemes included in the article are reformulations of the theorems of [1] in the new terminology. Also some basic facts about functions of two variables are proved.

The notation and terminology used in this paper are introduced in the following articles: [2], [3], and [1]. For simplicity we adopt the following convention: P, Q, X, Y, $Y1$, $Y2$, Z will denote objects of the type set; x, $x1$, $x2$, y, $y1$, $y2$, z, $z1$, $z2$ will denote objects of the type Any. Let us consider X, Y. Assume that the following holds

$$Y = \emptyset \textbf{ implies } X = \emptyset.$$

The mode

$$\text{Function } \textbf{of } X, Y,$$

which widens to the type Function, is defined by

$$X = \operatorname{dom} \textbf{it} \ \& \ \operatorname{rng} \textbf{it} \subseteq Y.$$

Next we state several propositions:

(1) $$(Y = \emptyset \textbf{ implies } X = \emptyset) \textbf{ implies for } f \textbf{ being } \text{Function} \\ \textbf{holds } f \textbf{ is } \text{Function } \textbf{of } X, Y \textbf{ iff } X = \operatorname{dom} f \ \& \ \operatorname{rng} f \subseteq Y,$$

(2) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \\ \textbf{st } Y = \emptyset \textbf{ implies } X = \emptyset \textbf{ holds } X = \operatorname{dom} f \ \& \ \operatorname{rng} f \subseteq Y,$$

(3) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{holds } f \textbf{ is } \text{Function } \textbf{of } \operatorname{dom} f, \operatorname{rng} f,$$

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(4) $$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ st } \mathrm{rng}\, f \subseteq Y \textbf{ holds } f \textbf{ is } \mathrm{Function} \textbf{ of } \mathrm{dom}\, f, Y,$$

(5) $$\textbf{for } f \textbf{ being } \mathrm{Function}$$
$$\textbf{st } \mathrm{dom}\, f = X \,\&\, \textbf{for } x \textbf{ st } x \in X \textbf{ holds } f\,.x \in Y \textbf{ holds } f \textbf{ is } \mathrm{Function} \textbf{ of } X, Y,$$

(6) $$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \textbf{ st } Y \neq \emptyset \,\&\, x \in X \textbf{ holds } f\,.x \in \mathrm{rng}\, f,$$

(7) $$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \textbf{ st } Y \neq \emptyset \,\&\, x \in X \textbf{ holds } f\,.x \in Y,$$

(8) $$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y$$
$$\textbf{st } (Y = \emptyset \textbf{ implies } X = \emptyset) \,\&\, \mathrm{rng}\, f \subseteq Z \textbf{ holds } f \textbf{ is } \mathrm{Function} \textbf{ of } X, Z,$$

(9) $$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y$$
$$\textbf{st } (Y = \emptyset \textbf{ implies } X = \emptyset) \,\&\, Y \subseteq Z \textbf{ holds } f \textbf{ is } \mathrm{Function} \textbf{ of } X, Z.$$

In the article we present several logical schemes. The scheme *FuncEx1* deals with a constant $\mathcal{A}$ that has the type set, a constant $\mathcal{B}$ that has the type set and a binary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex } f \textbf{ being } \mathrm{Function} \textbf{ of } \mathcal{A}, \mathcal{B} \textbf{ st for } x \textbf{ st } x \in \mathcal{A} \textbf{ holds } \mathcal{P}[x, f\,.x]$$

provided the parameters satisfy the following conditions:

- $$\textbf{for } x \textbf{ st } x \in \mathcal{A} \textbf{ ex } y \textbf{ st } y \in \mathcal{B} \,\&\, \mathcal{P}[x, y],$$
- $$\textbf{for } x, y1, y2 \textbf{ st } x \in \mathcal{A} \,\&\, \mathcal{P}[x, y1] \,\&\, \mathcal{P}[x, y2] \textbf{ holds } y1 = y2.$$

The scheme *Lambda1* concerns a constant $\mathcal{A}$ that has the type set, a constant $\mathcal{B}$ that has the type set and a unary functor $\mathcal{F}$ and states that the following holds

$$\textbf{ex } f \textbf{ being } \mathrm{Function} \textbf{ of } \mathcal{A}, \mathcal{B} \textbf{ st for } x \textbf{ st } x \in \mathcal{A} \textbf{ holds } f\,.x = \mathcal{F}(x)$$

provided the parameters satisfy the following condition:

- $$\textbf{for } x \textbf{ st } x \in \mathcal{A} \textbf{ holds } \mathcal{F}(x) \in \mathcal{B}.$$

Let us consider X, Y. The functor

$$\mathrm{Funcs}\,(X, Y),$$

yields the type set and is defined by

$$x \in \textbf{it iff ex } f \textbf{ being } \mathrm{Function} \textbf{ st } x = f \,\&\, \mathrm{dom}\, f = X \,\&\, \mathrm{rng}\, f \subseteq Y.$$

We now state a number of propositions:

(10) $$\textbf{for } F \textbf{ being } \mathrm{set} \textbf{ holds } F = \mathrm{Funcs}\,(X, Y) \textbf{ iff for } x$$
$$\textbf{holds } x \in F \textbf{ iff ex } f \textbf{ being } \mathrm{Function} \textbf{ st } x = f \,\&\, \mathrm{dom}\, f = X \,\&\, \mathrm{rng}\, f \subseteq Y,$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \textbf{ st } Y = \emptyset \textbf{ implies } X = \emptyset \textbf{ holds } f \in \mathrm{Funcs}\,(X, Y), \tag{11}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, X \textbf{ holds } f \in \mathrm{Funcs}\,(X, X), \tag{12}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } \emptyset, X \textbf{ holds } f \in \mathrm{Funcs}\,(\emptyset, X), \tag{13}$$

$$X \neq \emptyset \textbf{ implies } \mathrm{Funcs}\,(X, \emptyset) = \emptyset, \tag{14}$$

$$\mathrm{Funcs}\,(X, Y) = \emptyset \textbf{ implies } X \neq \emptyset \ \&\ Y = \emptyset, \tag{15}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \textbf{ st } Y \neq \emptyset \ \&\ \textbf{for } y \textbf{ st } y \in Y \textbf{ ex } x \textbf{ st } x \in X \ \&\ y = f.x \textbf{ holds } \mathrm{rng}\, f = Y, \tag{16}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \textbf{ st } y \in Y \ \&\ \mathrm{rng}\, f = Y \textbf{ ex } x \textbf{ st } x \in X \ \&\ f.x = y, \tag{17}$$

$$\textbf{for } f1, f2 \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \textbf{ st } Y \neq \emptyset \ \&\ \textbf{for } x \textbf{ st } x \in X \textbf{ holds } f1.x = f2.x \textbf{ holds } f1 = f2, \tag{18}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \textbf{ for } g \textbf{ being } \mathrm{Function} \textbf{ of } Y, Z \textbf{ st } (Z = \emptyset \textbf{ implies } Y = \emptyset) \ \&\ (Y = \emptyset \textbf{ implies } X = \emptyset) \textbf{ holds } g \cdot f \textbf{ is } \mathrm{Function} \textbf{ of } X, Z, \tag{19}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \textbf{ for } g \textbf{ being } \mathrm{Function} \textbf{ of } Y, Z \textbf{ st } Y \neq \emptyset \ \&\ Z \neq \emptyset \ \&\ \mathrm{rng}\, f = Y \ \&\ \mathrm{rng}\, g = Z \textbf{ holds } \mathrm{rng}\,(g \cdot f) = Z, \tag{20}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \textbf{ for } g \textbf{ being } \mathrm{Function} \textbf{ of } Y, Z \textbf{ st } Y \neq \emptyset \ \&\ Z \neq \emptyset \ \&\ x \in X \textbf{ holds } (g \cdot f).x = g.(f.x), \tag{21}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \textbf{ st } Y \neq \emptyset \textbf{ holds } \mathrm{rng}\, f = Y \textbf{ iff for } Z \textbf{ st } Z \neq \emptyset \textbf{ for } g, h \textbf{ being } \mathrm{Function} \textbf{ of } Y, Z \textbf{ st } g \cdot f = h \cdot f \textbf{ holds } g = h, \tag{22}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \textbf{ st } Y = \emptyset \textbf{ implies } X = \emptyset \textbf{ holds } f \cdot (\mathrm{id}\, X) = f \ \&\ (\mathrm{id}\, Y) \cdot f = f, \tag{23}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \textbf{ for } g \textbf{ being } \mathrm{Function} \textbf{ of } Y, X \textbf{ st } Y \neq \emptyset \ \&\ f \cdot g = \mathrm{id}\, Y \textbf{ holds } \mathrm{rng}\, f = Y, \tag{24}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \textbf{ st } Y = \emptyset \textbf{ implies } X = \emptyset \textbf{ holds } f \ \mathrm{is\_one\text{-}to\text{-}one} \textbf{ iff for } x1, x2 \textbf{ st } x1 \in X \ \&\ x2 \in X \ \&\ f.x1 = f.x2 \textbf{ holds } x1 = x2, \tag{25}$$

$$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X, Y \textbf{ for } g \textbf{ being } \mathrm{Function} \textbf{ of } Y, Z \textbf{ st } (Z = \emptyset \textbf{ implies } Y = \emptyset) \ \&\ (Y = \emptyset \textbf{ implies } X = \emptyset) \ \&\ g \cdot f \ \mathrm{is\_one\text{-}to\text{-}one} \textbf{ holds } f \ \mathrm{is\_one\text{-}to\text{-}one}, \tag{26}$$

(27) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ st } X \neq \emptyset \ \& \ Y \neq \emptyset \textbf{ holds } f \text{ is\_one-to-one} \\ \textbf{iff for } Z \textbf{ for } g, h \textbf{ being } \text{Function } \textbf{of } Z, X \textbf{ st } f \cdot g = f \cdot h \textbf{ holds } g = h,$$

(28) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ for } g \textbf{ being } \text{Function } \textbf{of } Y, Z \\ \textbf{st } Z \neq \emptyset \ \& \ Y \neq \emptyset \ \& \ \text{rng}\,(g \cdot f) = Z \ \& \ g \text{ is\_one-to-one} \textbf{ holds } \text{rng}\, f = Y,$$

(29) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ for } g \textbf{ being } \text{Function } \textbf{of } Y, X \\ \textbf{st } X \neq \emptyset \ \& \ Y \neq \emptyset \ \& \ g \cdot f = \text{id}\, X \textbf{ holds } f \text{ is\_one-to-one} \ \& \ \text{rng}\, g = X,$$

(30) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ for } g \textbf{ being } \text{Function } \textbf{of } Y, Z \textbf{ st} \\ (Z = \emptyset \textbf{ implies } Y = \emptyset) \ \& \ g \cdot f \text{ is\_one-to-one} \ \& \ \text{rng}\, f = Y \\ \textbf{holds } f \text{ is\_one-to-one} \ \& \ g \text{ is\_one-to-one},$$

(31) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ st} \\ f \text{ is\_one-to-one} \ \& \ (X = \emptyset \textbf{ iff } Y = \emptyset) \ \& \ \text{rng}\, f = Y \\ \textbf{holds } f^{-1} \textbf{ is } \text{Function } \textbf{of } Y, X,$$

(32) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \\ \textbf{st } Y \neq \emptyset \ \& \ f \text{ is\_one-to-one} \ \& \ x \in X \textbf{ holds } (f^{-1}).(f.x) = x,$$

(33) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \\ \textbf{st } \text{rng}\, f = Y \ \& \ f \text{ is\_one-to-one} \ \& \ y \in Y \textbf{ holds } f.((f^{-1}).y) = y,$$

(34) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ for } g \textbf{ being } \text{Function } \textbf{of } Y, X \textbf{ st} \\ X \neq \emptyset \ \& \ Y \neq \emptyset \ \& \ \text{rng}\, f = Y \\ \& \ f \text{ is\_one-to-one} \ \& \textbf{ for } y, x \textbf{ holds } y \in Y \ \& \ g.y = x \textbf{ iff } x \in X \ \& \ f.x = y \\ \textbf{holds } g = f^{-1},$$

(35) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \\ \textbf{st } Y \neq \emptyset \ \& \ \text{rng}\, f = Y \ \& \ f \text{ is\_one-to-one} \textbf{ holds } f^{-1} \cdot f = \text{id}\, X \ \& \ f \cdot f^{-1} = \text{id}\, Y,$$

(36) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ for } g \textbf{ being } \text{Function } \textbf{of } Y, X \textbf{ st} \\ X \neq \emptyset \ \& \ Y \neq \emptyset \ \& \ \text{rng}\, f = Y \ \& \ g \cdot f = \text{id}\, X \ \& \ f \text{ is\_one-to-one} \textbf{ holds } g = f^{-1},$$

(37) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ st} \\ Y \neq \emptyset \ \& \textbf{ ex } g \textbf{ being } \text{Function } \textbf{of } Y, X \textbf{ st } g \cdot f = \text{id}\, X \textbf{ holds } f \text{ is\_one-to-one},$$

(38) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \\ \textbf{st } (Y = \emptyset \textbf{ implies } X = \emptyset) \ \& \ Z \subseteq X \textbf{ holds } f \mid Z \textbf{ is } \text{Function } \textbf{of } Z, Y,$$

(39) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \\ \textbf{st } Y \neq \emptyset \ \& \ x \in X \ \& \ x \in Z \textbf{ holds } (f \mid Z).x = f.x,$$

(40) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X,Y \textbf{ st } (Y=\emptyset \textbf{ implies } X=\emptyset)\ \&\ X\subseteq Z \textbf{ holds } f \mid Z = f,$$

(41) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X,Y \textbf{ st } Y\neq\emptyset\ \&\ x\in X\ \&\ f.x\in Z \textbf{ holds } (Z \mid f).x = f.x,$$

(42) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X,Y \textbf{ st } (Y=\emptyset \textbf{ implies } X=\emptyset)\ \&\ Y\subseteq Z \textbf{ holds } Z \mid f = f,$$

(43) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X,Y \textbf{ st } Y\neq\emptyset \textbf{ for } y \textbf{ holds } y\in f \circ P \textbf{ iff } \textbf{ex } x \textbf{ st } x\in X\ \&\ x\in P\ \&\ y=f.x,$$

(44) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X,Y \textbf{ st } Y=\emptyset \textbf{ implies } X=\emptyset \textbf{ holds } f \circ P\subseteq Y,$$

(45) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X,Y \textbf{ st } Y=\emptyset \textbf{ implies } X=\emptyset \textbf{ holds } f \circ X=\operatorname{rng} f,$$

(46) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X,Y \textbf{ st } Y\neq\emptyset \textbf{ for } x \textbf{ holds } x\in f^{-1}\, Q \textbf{ iff } x\in X\ \&\ f.x\in Q,$$

(47) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X,Y \textbf{ st } Y=\emptyset \textbf{ implies } X=\emptyset \textbf{ holds } f^{-1}\, Q\subseteq X,$$

(48) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X,Y \textbf{ st } Y=\emptyset \textbf{ implies } X=\emptyset \textbf{ holds } f^{-1}\, Y=X,$$

(49) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X,Y \textbf{ st } Y\neq\emptyset \textbf{ holds } (\textbf{for } y \textbf{ st } y\in Y \textbf{ holds } f^{-1}\,\{y\}\neq\emptyset) \textbf{ iff } \operatorname{rng} f=Y,$$

(50) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X,Y \textbf{ st } (Y=\emptyset \textbf{ implies } X=\emptyset)\ \&\ P\subseteq X \textbf{ holds } P\subseteq f^{-1}\,(f \circ P),$$

(51) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X,Y \textbf{ st } Y=\emptyset \textbf{ implies } X=\emptyset \textbf{ holds } f^{-1}\,(f \circ X)=X,$$

(52) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X,Y \textbf{ st } (Y=\emptyset \textbf{ implies } X=\emptyset)\ \&\ \operatorname{rng} f=Y \textbf{ holds } f \circ (f^{-1}\, Y)=Y,$$

(53) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X,Y \textbf{ for } g \textbf{ being } \text{Function } \textbf{of } Y,Z \textbf{ st } (Z=\emptyset \textbf{ implies } Y=\emptyset)\ \&\ (Y=\emptyset \textbf{ implies } X=\emptyset) \textbf{ holds } f^{-1}\, Q\subseteq (g\cdot f)^{-1}\,(g \circ Q),$$

(54) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } \emptyset,Y \textbf{ holds } \operatorname{dom} f=\emptyset\ \&\ \operatorname{rng} f=\emptyset,$$

(55) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{st } \operatorname{dom} f=\emptyset \textbf{ holds } f \textbf{ is } \text{Function } \textbf{of } \emptyset,Y,$$

(56) $$\textbf{for } f1 \textbf{ being } \text{Function } \textbf{of } \emptyset,Y1 \textbf{ for } f2 \textbf{ being } \text{Function } \textbf{of } \emptyset,Y2 \textbf{ holds } f1=f2,$$

(57) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } \emptyset, Y \textbf{ for } g \textbf{ being } \text{Function } \textbf{of } Y, Z$$
$$\textbf{st } Z = \emptyset \textbf{ implies } Y = \emptyset \textbf{ holds } g \cdot f \textbf{ is } \text{Function } \textbf{of } \emptyset, Z,$$

(58) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } \emptyset, Y \textbf{ holds } f \text{ is\_one-to-one},$$

(59) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } \emptyset, Y \textbf{ holds } f \,^{\circ}\, P = \emptyset,$$

(60) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } \emptyset, Y \textbf{ holds } f \,^{-1}\, Q = \emptyset,$$

(61) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } \{x\}, Y \textbf{ st } Y \neq \emptyset \textbf{ holds } f.x \in Y,$$

(62) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } \{x\}, Y \textbf{ st } Y \neq \emptyset \textbf{ holds } \operatorname{rng} f = \{f.x\},$$

(63) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } \{x\}, Y \textbf{ st } Y \neq \emptyset \textbf{ holds } f \text{ is\_one-to-one},$$

(64) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } \{x\}, Y \textbf{ st } Y \neq \emptyset \textbf{ holds } f \,^{\circ}\, P \subseteq \{f.x\},$$

(65) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, \{y\} \textbf{ st } x \in X \textbf{ holds } f.x = y,$$

(66) $$\textbf{for } f1, f2 \textbf{ being } \text{Function } \textbf{of } X, \{y\} \textbf{ holds } f1 = f2.$$

The arguments of the notions defined below are the following: X which is an object of the type reserved above; f, g which are objects of the type Function **of** X, X. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$g \cdot f \qquad \text{is} \qquad \text{Function } \textbf{of } X, X.$$

Let us consider X. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\operatorname{id} X \qquad \text{is} \qquad \text{Function } \textbf{of } X, X.$$

The following propositions are true:

(67) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ holds } \operatorname{dom} f = X \ \&\ \operatorname{rng} f \subseteq X,$$

(68) $$\textbf{for } f \textbf{ being } \text{Function}$$
$$\textbf{st } \operatorname{dom} f = X \ \&\ \operatorname{rng} f \subseteq X \textbf{ holds } f \textbf{ is } \text{Function } \textbf{of } X, X,$$

(69) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ st } x \in X \textbf{ holds } f.x \in X,$$

(70) $$\textbf{for } f, g \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ st } x \in X \textbf{ holds } (g \cdot f).x = g.(f.x),$$

(71) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, X$$
$$\textbf{for } g \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ st } Y \neq \emptyset \ \&\ x \in X \textbf{ holds } (g \cdot f).x = g.(f.x),$$

(72) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y$$
$$\textbf{for } g \textbf{ being } \text{Function } \textbf{of } Y, Y \textbf{ st } Y \neq \emptyset \ \&\ x \in X \textbf{ holds } (g \cdot f).x = g.(f.x),$$

$$\textbf{for } f,g \textbf{ being } \text{Function } \textbf{of } X, X \quad \textbf{st } \operatorname{rng} f = X \ \& \ \operatorname{rng} g = X \textbf{ holds } \operatorname{rng}(g \cdot f) = X, \tag{73}$$

$$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ holds } f \cdot (\operatorname{id} X) = f \ \& \ (\operatorname{id} X) \cdot f = f, \tag{74}$$

$$\textbf{for } f,g \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ st } g \cdot f = f \ \& \ \operatorname{rng} f = X \textbf{ holds } g = \operatorname{id} X, \tag{75}$$

$$\textbf{for } f,g \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ st } f \cdot g = f \ \& \ f \text{ is\_one-to-one} \textbf{ holds } g = \operatorname{id} X, \tag{76}$$

$$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ holds } f \text{ is\_one-to-one} \quad \textbf{iff for } x1,x2 \textbf{ st } x1 \in X \ \& \ x2 \in X \ \& \ f\,.x1 = f\,.x2 \textbf{ holds } x1 = x2, \tag{77}$$

$$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ holds } f \,^\circ\, P \subseteq X. \tag{78}$$

The arguments of the notions defined below are the following: X which is an object of the type reserved above; f which is an object of the type Function **of** X, X; P which is an object of the type reserved above. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$f \,^\circ\, P \qquad \text{is} \qquad \text{Subset } \textbf{of } X.$$

One can prove the following propositions:

$$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ holds } f \,^\circ\, X = \operatorname{rng} f, \tag{79}$$

$$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ holds } f^{\,-1}\, Q \subseteq X. \tag{80}$$

The arguments of the notions defined below are the following: X which is an object of the type reserved above; f which is an object of the type Function **of** X, X; Q which is an object of the type reserved above. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$f^{\,-1}\, Q \qquad \text{is} \qquad \text{Subset } \textbf{of } X.$$

Next we state two propositions:

$$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ st } \operatorname{rng} f = X \textbf{ holds } f \,^\circ\, (f^{\,-1}\, X) = X, \tag{81}$$

$$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, X \textbf{ holds } f^{\,-1}\, (f \,^\circ\, X) = X. \tag{82}$$

Let us consider X. The mode

$$\text{Permutation } \textbf{of } X,$$

which widens to the type Function **of** X, X, is defined by

$$\textbf{it} \text{ is\_one-to-one} \ \& \ \operatorname{rng} \textbf{it} = X.$$

Next we state three propositions:

(83) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, X$$
$$\textbf{holds } f \textbf{ is } \text{Permutation } \textbf{of } X \textbf{ iff } f \text{ is\_one-to-one} \ \& \ \operatorname{rng} f = X,$$

(84) $$\textbf{for } f \textbf{ being } \text{Permutation } \textbf{of } X \textbf{ holds } f \text{ is\_one-to-one} \ \& \ \operatorname{rng} f = X,$$

(85) $$\textbf{for } f \textbf{ being } \text{Permutation } \textbf{of } X$$
$$\textbf{for } x1{,}x2 \textbf{ st } x1 \in X \ \& \ x2 \in X \ \& \ f\,.x1 = f\,.x2 \textbf{ holds } x1 = x2.$$

The arguments of the notions defined below are the following: X which is an object of the type reserved above; f, g which are objects of the type Permutation **of** X. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$g \cdot f \qquad \text{is} \qquad \text{Permutation } \textbf{of } X.$$

Let us consider X. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\operatorname{id} X \qquad \text{is} \qquad \text{Permutation } \textbf{of } X.$$

The arguments of the notions defined below are the following: X which is an object of the type reserved above; f which is an object of the type Permutation **of** X. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$f^{-1} \qquad \text{is} \qquad \text{Permutation } \textbf{of } X.$$

The following propositions are true:

(86) $$\textbf{for } f{,}g \textbf{ being } \text{Permutation } \textbf{of } X \textbf{ st } g \cdot f = g \textbf{ holds } f = \operatorname{id} X,$$

(87) $$\textbf{for } f{,}g \textbf{ being } \text{Permutation } \textbf{of } X \textbf{ st } g \cdot f = \operatorname{id} X \textbf{ holds } g = f^{-1},$$

(88) $$\textbf{for } f \textbf{ being } \text{Permutation } \textbf{of } X \textbf{ holds } (f^{-1}) \cdot f = \operatorname{id} X \ \& \ f \cdot (f^{-1}) = \operatorname{id} X,$$

(89) $$\textbf{for } f \textbf{ being } \text{Permutation } \textbf{of } X \textbf{ holds } (f^{-1})^{-1} = f,$$

(90) $$\textbf{for } f{,}g \textbf{ being } \text{Permutation } \textbf{of } X \textbf{ holds } (g \cdot f)^{-1} = f^{-1} \cdot g^{-1},$$

(91) $$\textbf{for } f \textbf{ being } \text{Permutation } \textbf{of } X \textbf{ st } P \cap Q = \emptyset \textbf{ holds } f^{\circ}\, P \cap f^{\circ}\, Q = \emptyset,$$

(92) $$\textbf{for } f \textbf{ being } \text{Permutation } \textbf{of } X$$
$$\textbf{st } P \subseteq X \textbf{ holds } f^{\circ}\,(f^{-1}\, P) = P \ \& \ f^{-1}\,(f^{\circ}\, P) = P,$$

(93) $$\textbf{for } f \textbf{ being } \text{Permutation } \textbf{of } X \textbf{ holds } f^{\circ}\, P = (f^{-1})^{-1}\, P \ \& \ f^{-1}\, P = (f^{-1})^{\circ}\, P.$$

In the sequel C, D, E denote objects of the type DOMAIN. The arguments of the notions defined below are the following: X, D, E which are objects of the type reserved above; f which is an object of the type Function **of** X, D; g which is an object of the type Function **of** D, E. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$g \cdot f \qquad \text{is} \qquad \text{Function } \mathbf{of}\ X, E.$$

Let us consider X, D. Let us note that one can characterize the mode

$$\text{Function } \mathbf{of}\ X, D$$

by the following (equivalent) condition:

$$X = \mathrm{dom}\,\mathbf{it}\ \&\ \mathrm{rng}\,\mathbf{it} \subseteq D.$$

We now state a number of propositions:

(94) $$\mathbf{for}\ f\ \mathbf{being}\ \text{Function}\ \mathbf{of}\ X, D\ \mathbf{holds}\ \mathrm{dom}\, f = X\ \&\ \mathrm{rng}\, f \subseteq D,$$

(95) $$\begin{gathered}\mathbf{for}\ f\ \mathbf{being}\ \text{Function}\\ \mathbf{st}\ \mathrm{dom}\, f = X\ \&\ \mathrm{rng}\, f \subseteq D\ \mathbf{holds}\ f\ \mathbf{is}\ \text{Function}\ \mathbf{of}\ X, D,\end{gathered}$$

(96) $$\mathbf{for}\ f\ \mathbf{being}\ \text{Function}\ \mathbf{of}\ X, D\ \mathbf{st}\ x \in X\ \mathbf{holds}\ f\,.x \in D,$$

(97) $$\mathbf{for}\ f\ \mathbf{being}\ \text{Function}\ \mathbf{of}\ \{x\}, D\ \mathbf{holds}\ f\,.x \in D,$$

(98) $$\begin{gathered}\mathbf{for}\ f1, f2\ \mathbf{being}\ \text{Function}\ \mathbf{of}\ X, D\\ \mathbf{st}\ \mathbf{for}\ x\ \mathbf{st}\ x \in X\ \mathbf{holds}\ f1.x = f2.x\ \mathbf{holds}\ f1 = f2,\end{gathered}$$

(99) $$\begin{gathered}\mathbf{for}\ f\ \mathbf{being}\ \text{Function}\ \mathbf{of}\ X, D\\ \mathbf{for}\ g\ \mathbf{being}\ \text{Function}\ \mathbf{of}\ D, E\ \mathbf{st}\ x \in X\ \mathbf{holds}\ (g \cdot f).x = g.(f\,.x),\end{gathered}$$

(100) $$\mathbf{for}\ f\ \mathbf{being}\ \text{Function}\ \mathbf{of}\ X, D\ \mathbf{holds}\ f \cdot (\mathrm{id}\, X) = f\ \&\ (\mathrm{id}\, D) \cdot f = f,$$

(101) $$\begin{gathered}\mathbf{for}\ f\ \mathbf{being}\ \text{Function}\ \mathbf{of}\ X, D\ \mathbf{holds}\ f\ \text{is\_one-to-one}\\ \mathbf{iff}\ \mathbf{for}\ x1, x2\ \mathbf{st}\ x1 \in X\ \&\ x2 \in X\ \&\ f\,.x1 = f\,.x2\ \mathbf{holds}\ x1 = x2,\end{gathered}$$

(102) $$\begin{gathered}\mathbf{for}\ f\ \mathbf{being}\ \text{Function}\ \mathbf{of}\ X, D\\ \mathbf{for}\ y\ \mathbf{holds}\ y \in f \circ P\ \mathbf{iff}\ \mathbf{ex}\ x\ \mathbf{st}\ x \in X\ \&\ x \in P\ \&\ y = f\,.x,\end{gathered}$$

(103) $$\mathbf{for}\ f\ \mathbf{being}\ \text{Function}\ \mathbf{of}\ X, D\ \mathbf{holds}\ f \circ P \subseteq D.$$

The arguments of the notions defined below are the following: X, D which are objects of the type reserved above; f which is an object of the type Function **of** X, D; P which is an object of the type reserved above. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$f \circ P \qquad \text{is} \qquad \text{Subset } \mathbf{of}\ D.$$

One can prove the following propositions:

(104) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D \textbf{ holds } f^{\circ}\, X = \operatorname{rng} f,$$

(105) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D \textbf{ st } f^{\circ}\, X = D \textbf{ holds } \operatorname{rng}(f) = D,$$

(106) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D \textbf{ for } x \textbf{ holds } x \in f^{-1}\, Q \textbf{ iff } x \in X \;\&\; f\,.x \in Q,$$

(107) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D \textbf{ holds } f^{-1}\, Q \subseteq X.$$

The arguments of the notions defined below are the following: X, D which are objects of the type reserved above; f which is an object of the type Function **of** X, D; Q which is an object of the type reserved above. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$f^{-1}\, Q \qquad \text{is} \qquad \text{Subset } \textbf{of } X.$$

One can prove the following propositions:

(108) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D \textbf{ holds } f^{-1}\, D = X,$$

(109) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D$$
$$\textbf{holds } (\textbf{for } y \textbf{ st } y \in D \textbf{ holds } f^{-1}\, \{y\} \neq \emptyset) \textbf{ iff } \operatorname{rng} f = D,$$

(110) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D \textbf{ holds } f^{-1}\, (f^{\circ}\, X) = X,$$

(111) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D \textbf{ st } \operatorname{rng} f = D \textbf{ holds } f^{\circ}\, (f^{-1}\, D) = D,$$

(112) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D$$
$$\textbf{for } g \textbf{ being } \text{Function } \textbf{of } D, E \textbf{ holds } f^{-1}\, Q \subseteq (g \cdot f)^{-1}\, (g^{\circ}\, Q).$$

In the sequel c denotes an object of the type Element **of** C; d denotes an object of the type Element **of** D. The arguments of the notions defined below are the following: C, D which are objects of the type reserved above; f which is an object of the type Function **of** C, D; c which is an object of the type reserved above. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$f\,.c \qquad \text{is} \qquad \text{Element } \textbf{of } D.$$

Now we present two schemes. The scheme *FuncExD* concerns a constant $\mathcal{A}$ that has the type DOMAIN, a constant $\mathcal{B}$ that has the type DOMAIN and a binary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex } f \textbf{ being } \text{Function } \textbf{of } \mathcal{A}, \mathcal{B} \textbf{ st for } x \textbf{ being } \text{Element } \textbf{of } \mathcal{A} \textbf{ holds } \mathcal{P}[x, f\,.x]$$

provided the parameters satisfy the following conditions:

- $\textbf{for } x \textbf{ being } \text{Element } \textbf{of } \mathcal{A} \textbf{ ex } y \textbf{ being } \text{Element } \textbf{of } \mathcal{B} \textbf{ st } \mathcal{P}[x, y],$

- $$\textbf{for } x \textbf{ being } \text{Element } \textbf{of } \mathcal{A}, y1, y2 \textbf{ being } \text{Element } \textbf{of } \mathcal{B} \textbf{ st } \mathcal{P}[x, y1] \mathbin{\&} \mathcal{P}[x, y2] \textbf{ holds } y1 = y2.$$

The scheme *LambdaD* concerns a constant $\mathcal{A}$ that has the type DOMAIN, a constant $\mathcal{B}$ that has the type DOMAIN and a unary functor $\mathcal{F}$ yielding values of the type Element **of** $\mathcal{B}$ and states that the following holds

$$\textbf{ex } f \textbf{ being } \text{Function } \textbf{of } \mathcal{A},\mathcal{B} \textbf{ st for } x \textbf{ being } \text{Element } \textbf{of } \mathcal{A} \textbf{ holds } f.x = \mathcal{F}(x)$$

for all values of the parameters.

One can prove the following propositions:

$$\textbf{for } f1,f2 \textbf{ being } \text{Function } \textbf{of } C, D \textbf{ st for } c \textbf{ holds } f1.c = f2.c \textbf{ holds } f1 = f2, \tag{113}$$

$$(\mathrm{id}\, C).c = c, \tag{114}$$

$$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } C, D \textbf{ for } g \textbf{ being } \text{Function } \textbf{of } D, E \textbf{ holds } (g \cdot f).c = g.(f.c), \tag{115}$$

$$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } C, D \textbf{ for } d \textbf{ holds } d \in f^{\circ}\, P \textbf{ iff ex } c \textbf{ st } c \in P \mathbin{\&} d = f.c, \tag{116}$$

$$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } C, D \textbf{ for } c \textbf{ holds } c \in f^{-1}\, Q \textbf{ iff } f.c \in Q, \tag{117}$$

$$\textbf{for } f1,f2 \textbf{ being } \text{Function } \textbf{of } [:X, Y:],Z \textbf{ st } Z \neq \emptyset \mathbin{\&} \textbf{for } x,y \textbf{ st } x \in X \mathbin{\&} y \in Y \textbf{ holds } f1.\langle x, y\rangle = f2.\langle x, y\rangle \textbf{ holds } f1 = f2, \tag{118}$$

$$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } [:X, Y:],Z \textbf{ st } x \in X \mathbin{\&} y \in Y \mathbin{\&} Z \neq \emptyset \textbf{ holds } f.\langle x, y\rangle \in Z. \tag{119}$$

Now we present two schemes. The scheme *FuncEx2* concerns a constant $\mathcal{A}$ that has the type set, a constant $\mathcal{B}$ that has the type set, a constant $\mathcal{C}$ that has the type set and a ternary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex } f \textbf{ being } \text{Function } \textbf{of } [:\mathcal{A},\mathcal{B}:],\mathcal{C} \textbf{ st for } x,y \textbf{ st } x \in \mathcal{A} \mathbin{\&} y \in \mathcal{B} \textbf{ holds } \mathcal{P}[x, y, f.\langle x, y\rangle]$$

provided the parameters satisfy the following conditions:

- $$\textbf{for } x,y \textbf{ st } x \in \mathcal{A} \mathbin{\&} y \in \mathcal{B} \textbf{ ex } z \textbf{ st } z \in \mathcal{C} \mathbin{\&} \mathcal{P}[x, y, z],$$
- $$\textbf{for } x,y,z1,z2 \textbf{ st } x \in \mathcal{A} \mathbin{\&} y \in \mathcal{B} \mathbin{\&} \mathcal{P}[x, y, z1] \mathbin{\&} \mathcal{P}[x, y, z2] \textbf{ holds } z1 = z2.$$

The scheme *Lambda2* concerns a constant $\mathcal{A}$ that has the type set, a constant $\mathcal{B}$ that has the type set, a constant $\mathcal{C}$ that has the type set and a binary functor $\mathcal{F}$ and states that the following holds

$$\textbf{ex } f \textbf{ being } \text{Function } \textbf{of } [:\mathcal{A},\mathcal{B}:],\mathcal{C} \textbf{ st for } x,y \textbf{ st } x \in \mathcal{A} \mathbin{\&} y \in \mathcal{B} \textbf{ holds } f.\langle x, y\rangle = \mathcal{F}(x, y)$$

provided the parameters satisfy the following condition:

- $$\textbf{for } x,y \textbf{ st } x \in \mathcal{A} \ \& \ y \in \mathcal{B} \textbf{ holds } \mathcal{F}(x, y) \in \mathcal{C}.$$

We now state a proposition

(120) $$\textbf{for } f1,f2 \textbf{ being } \text{Function } \textbf{of } [:C, D:],E$$
$$\textbf{st for } c,d \textbf{ holds } f1.\langle c, d\rangle = f2.\langle c, d\rangle \textbf{ holds } f1 = f2.$$

Now we present two schemes. The scheme *FuncEx2D* deals with a constant $\mathcal{A}$ that has the type DOMAIN, a constant $\mathcal{B}$ that has the type DOMAIN, a constant $\mathcal{C}$ that has the type DOMAIN and a ternary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex } f \textbf{ being } \text{Function } \textbf{of } [:\mathcal{A},\mathcal{B}:],\mathcal{C}$$
$$\textbf{st for } x \textbf{ being } \text{Element } \textbf{of } \mathcal{A} \textbf{ for } y \textbf{ being } \text{Element } \textbf{of } \mathcal{B} \textbf{ holds } \mathcal{P}[x, y, f.\langle x, y\rangle]$$

provided the parameters satisfy the following conditions:

- $$\textbf{for } x \textbf{ being } \text{Element } \textbf{of } \mathcal{A}$$
  $$\textbf{for } y \textbf{ being } \text{Element } \textbf{of } \mathcal{B} \textbf{ ex } z \textbf{ being } \text{Element } \textbf{of } \mathcal{C} \textbf{ st } \mathcal{P}[x, y, z],$$

- $$\textbf{for } x \textbf{ being } \text{Element } \textbf{of } \mathcal{A} \textbf{ for } y \textbf{ being } \text{Element } \textbf{of } \mathcal{B}$$
  $$\textbf{for } z1,z2 \textbf{ being } \text{Element } \textbf{of } \mathcal{C} \textbf{ st } \mathcal{P}[x, y, z1] \ \& \ \mathcal{P}[x, y, z2] \textbf{ holds } z1 = z2.$$

The scheme *Lambda2D* concerns a constant $\mathcal{A}$ that has the type DOMAIN, a constant $\mathcal{B}$ that has the type DOMAIN, a constant $\mathcal{C}$ that has the type DOMAIN and a binary functor $\mathcal{F}$ yielding values of the type Element **of** $\mathcal{C}$ and states that the following holds

$$\textbf{ex } f \textbf{ being } \text{Function } \textbf{of } [:\mathcal{A},\mathcal{B}:],\mathcal{C}$$
$$\textbf{st for } x \textbf{ being } \text{Element } \textbf{of } \mathcal{A} \textbf{ for } y \textbf{ being } \text{Element } \textbf{of } \mathcal{B} \textbf{ holds } f.\langle x, y\rangle = \mathcal{F}(x, y)$$

for all values of the parameters.

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Finite Sets

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Summary. The article contains the definition of a finite set based on the notion of finite sequence. Some theorems about properties of finite sets and finite families of sets are proved.

The terminology and notation used here are introduced in the following papers: [5], [6], [4], [2], [1], and [3]. Let A have the type set. The predicate

$$A\,\text{is\_finite} \qquad \text{is defined by} \qquad \mathbf{ex}\,p\ \mathbf{being}\ \text{FinSequence}\ \mathbf{st}\ \text{rng}\,p = A.$$

For simplicity we adopt the following convention: A, B, C, D, X, Y have the type set; x, y, z, $x1$, $x2$, $x3$, $x4$, $x5$, $x6$, $x7$, $x8$ have the type Any; f has the type Function; n has the type Nat. The following propositions are true:

$$A\,\text{is\_finite}\ \mathbf{iff}\ \mathbf{ex}\,p\ \mathbf{being}\ \text{FinSequence}\ \mathbf{st}\ \text{rng}\,p = A, \tag{1}$$

$$\mathbf{for}\,p\ \mathbf{being}\ \text{FinSequence}\ \mathbf{holds}\ \text{rng}\,p\,\text{is\_finite}\,, \tag{2}$$

$$\text{Seg}\,n\,\text{is\_finite}\,, \tag{3}$$

$$\emptyset\,\text{is\_finite}\,, \tag{4}$$

$$\{x\}\,\text{is\_finite}\,, \tag{5}$$

$$\{x, y\}\,\text{is\_finite}\,, \tag{6}$$

$$\{x, y, z\}\,\text{is\_finite}\,, \tag{7}$$

$$\{x1,x2,x3,x4\}\,\text{is\_finite}\,, \tag{8}$$

$$\{x1,x2,x3,x4,x5\}\,\text{is\_finite}\,, \tag{9}$$

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(10) $$\{x1,x2,x3,x4,x5,x6\}\ \text{is\_finite}\,,$$

(11) $$\{x1,x2,x3,x4,x5,x6,x7\}\ \text{is\_finite}\,,$$

(12) $$\{x1,x2,x3,x4,x5,x6,x7,x8\}\ \text{is\_finite}\,,$$

(13) $$A \subseteq B\ \&\ B\ \text{is\_finite}\ \textbf{implies}\ A\ \text{is\_finite}\,,$$

(14) $$A\ \text{is\_finite}\ \&\ B\ \text{is\_finite}\ \textbf{implies}\ A \cup B\ \text{is\_finite}\,,$$

(15) $$A\ \text{is\_finite}\ \textbf{implies}\ A \cap B\ \text{is\_finite}\ \&\ B \cap A\ \text{is\_finite}\,,$$

(16) $$A\ \text{is\_finite}\ \textbf{implies}\ A \setminus B\ \text{is\_finite}\,,$$

(17) $$A\ \text{is\_finite}\ \textbf{implies}\ f\ ^{\circ}\ A\ \text{is\_finite}\,,$$

(18) A is_finite **implies for** X **being** Subset-Family **of** A **st** $X \neq \emptyset$ **ex** x **being** set **st** $x \in X$ & **for** B **being** set **st** $B \in X$ **holds** $x \subseteq B$ **implies** $B = x$.

The scheme *Finite* deals with a constant $\mathcal{A}$ that has the type set and a unary predicate $\mathcal{P}$ and states that the following holds

$$\mathcal{P}[\mathcal{A}]$$

provided the parameters satisfy the following conditions:

- $$\mathcal{A}\ \text{is\_finite}\,,$$
- $$\mathcal{P}[\emptyset],$$
- **for** x,B **being** set **st** $x \in \mathcal{A}$ & $B \subseteq \mathcal{A}$ & $\mathcal{P}[B]$ **holds** $\mathcal{P}[B \cup \{x\}]$.

We now state several propositions:

(19) $$A\ \text{is\_finite}\ \&\ B\ \text{is\_finite}\ \textbf{implies}\ [:A, B:]\ \text{is\_finite}\,,$$

(20) $$A\ \text{is\_finite}\ \&\ B\ \text{is\_finite}\ \&\ C\ \text{is\_finite}\ \textbf{implies}\ [:A, B, C:]\ \text{is\_finite}\,,$$

(21) $$A\ \text{is\_finite}\ \&\ B\ \text{is\_finite}\ \&\ C\ \text{is\_finite}\ \&\ D\ \text{is\_finite}\ \textbf{implies}\ [:A, B, C, D:]\ \text{is\_finite}\,,$$

(22) $$B \neq \emptyset\ \&\ [:A, B:]\ \text{is\_finite}\ \textbf{implies}\ A\ \text{is\_finite}\,,$$

(23) $$A \neq \emptyset\ \&\ [:A, B:]\ \text{is\_finite}\ \textbf{implies}\ B\ \text{is\_finite}\,,$$

(24) $$A\ \text{is\_finite}\ \textbf{iff}\ \text{bool}\ A\ \text{is\_finite}\,,$$

(25) $$A\ \text{is\_finite}\ \&\ (\textbf{for}\ X\ \textbf{st}\ X \in A\ \textbf{holds}\ X\ \text{is\_finite})\ \textbf{iff}\ \bigcup A\ \text{is\_finite}\,,$$

(26) $$\text{dom}\, f\ \text{is\_finite}\ \textbf{implies}\ \text{rng}\, f\ \text{is\_finite}\,,$$

(27) $$Y \subseteq \text{rng}\, f\ \&\ f\ ^{-1}\ Y\ \text{is\_finite}\ \textbf{implies}\ Y\ \text{is\_finite}\,.$$

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Graphs of Functions.

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Summary. The graph of a function is defined in [1]. In this paper the graph of a function is redefined as a Relation. Operations on functions are interpreted as the corresponding operations on relations. Some theorems about graphs of functions are proved.

The terminology and notation used in this paper have been introduced in the following papers: [2], [3], [1], and [4]. For simplicity we adopt the following convention: X, $X1$, $X2$, Y, $Y1$, $Y2$ denote objects of the type set; x, $x1$, $x2$, y, $y1$, $y2$, z denote objects of the type Any; f, $f1$, $f2$, g, $g1$, $g2$, h, $h1$ denote objects of the type Function. Let us consider f. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\mathrm{graph}\, f \quad \text{is} \quad \mathrm{Relation}\,.$$

Next we state a number of propositions:

$$\textbf{for } R \textbf{ being } \mathrm{Relation} \textbf{ st } \textbf{for } x,y1,y2 \textbf{ st } \langle x, y1\rangle \in R \;\&\; \langle x, y2\rangle \in R \textbf{ holds } y1 = y2 \textbf{ ex } f \textbf{ st } \mathrm{graph}\, f = R, \tag{1}$$

$$y \in \mathrm{rng}\, f \textbf{ iff } \textbf{ex } x \textbf{ st } \langle x, y\rangle \in \mathrm{graph}\, f, \tag{2}$$

$$\mathrm{dom}\,\mathrm{graph}\, f = \mathrm{dom}\, f \;\&\; \mathrm{rng}\,\mathrm{graph}\, f = \mathrm{rng}\, f, \tag{3}$$

$$\mathrm{graph}\, f \subseteq [\!:\mathrm{dom}\, f, \mathrm{rng}\, f:\!], \tag{4}$$

$$(\textbf{for } x,y \textbf{ holds } \langle x, y\rangle \in \mathrm{graph}\, f1 \textbf{ iff } \langle x, y\rangle \in \mathrm{graph}\, f2) \textbf{ implies } f1 = f2, \tag{5}$$

$$\textbf{for } G \textbf{ being } \mathrm{set} \textbf{ st } G \subseteq \mathrm{graph}\, f \textbf{ ex } g \textbf{ st } \mathrm{graph}\, g = G, \tag{6}$$

$$\mathrm{graph}\, f \subseteq \mathrm{graph}\, g \textbf{ implies } \mathrm{dom}\, f \subseteq \mathrm{dom}\, g \;\&\; \mathrm{rng}\, f \subseteq \mathrm{rng}\, g, \tag{7}$$

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$$\text{graph}\, f \subseteq \text{graph}\, g \textbf{ iff } \text{dom}\, f \subseteq \text{dom}\, g \;\&\; \textbf{for } x \textbf{ st } x \in \text{dom}\, f \textbf{ holds } f.x = g.x, \tag{8}$$

$$\text{dom}\, f = \text{dom}\, g \;\&\; \text{graph}\, f \subseteq \text{graph}\, g \textbf{ implies } f = g, \tag{9}$$

$$\langle x, z\rangle \in \text{graph}\,(g \cdot f) \textbf{ iff ex } y \textbf{ st } \langle x, y\rangle \in \text{graph}\, f \;\&\; \langle y, z\rangle \in \text{graph}\, g, \tag{10}$$

$$(\text{graph}\, f) \cdot (\text{graph}\, g) = \text{graph}\,(g \cdot f), \tag{11}$$

$$\langle x, z\rangle \in \text{graph}\,(g \cdot f) \textbf{ implies } \langle x, f.x\rangle \in \text{graph}\, f \;\&\; \langle f.x, z\rangle \in \text{graph}\, g, \tag{12}$$

$$\begin{gathered}\text{graph}\, h \subseteq \text{graph}\, f \\ \textbf{implies } \text{graph}\,(g \cdot h) \subseteq \text{graph}\,(g \cdot f) \;\&\; \text{graph}\,(h \cdot g) \subseteq \text{graph}\,(f \cdot g),\end{gathered} \tag{13}$$

$$\begin{gathered}\text{graph}\, g2 \subseteq \text{graph}\, g1 \;\&\; \text{graph}\, f2 \subseteq \text{graph}\, f1 \\ \textbf{implies } \text{graph}\,(g2 \cdot f2) \subseteq \text{graph}\,(g1 \cdot f1),\end{gathered} \tag{14}$$

$$\textbf{ex } f \textbf{ st } \text{graph}\, f = \{\langle x, y\rangle\}, \tag{15}$$

$$\text{graph}\, f = \{\langle x, y\rangle\} \textbf{ implies } f.x = y, \tag{16}$$

$$\text{graph}\, f = \{\langle x, y\rangle\} \textbf{ implies } \text{dom}\, f = \{x\} \;\&\; \text{rng}\, f = \{y\}, \tag{17}$$

$$\text{dom}\, f = \{x\} \textbf{ implies } \text{graph}\, f = \{\langle x, f.x\rangle\}, \tag{18}$$

$$(\textbf{ex } f \textbf{ st } \text{graph}\, f = \{\langle x1, y1\rangle, \langle x2, y2\rangle\}) \textbf{ iff } (x1 = x2 \textbf{ implies } y1 = y2), \tag{19}$$

$$\textbf{ex } f \textbf{ st } \text{graph}\, f = \emptyset, \tag{20}$$

$$\text{graph}\, f = \emptyset \textbf{ implies } \text{dom}\, f = \emptyset \;\&\; \text{rng}\, f = \emptyset, \tag{21}$$

$$\text{rng}\, f = \emptyset \textbf{ or } \text{dom}\, f = \emptyset \textbf{ implies } \text{graph}\, f = \emptyset, \tag{22}$$

$$\text{rng}\, f \cap \text{dom}\, g = \emptyset \textbf{ implies } \text{graph}\,(g \cdot f) = \emptyset, \tag{23}$$

$$\text{graph}\, g = \emptyset \textbf{ implies } \text{graph}\,(g \cdot f) = \emptyset \;\&\; \text{graph}\,(f \cdot g) = \emptyset, \tag{24}$$

$$\begin{gathered} f \text{ is\_one-to-one} \\ \textbf{iff for } x1, x2, y \textbf{ st } \langle x1, y\rangle \in \text{graph}\, f \;\&\; \langle x2, y\rangle \in \text{graph}\, f \textbf{ holds } x1 = x2,\end{gathered} \tag{25}$$

$$\text{graph}\, g \subseteq \text{graph}\, f \;\&\; f \text{ is\_one-to-one} \textbf{ implies } g \text{ is\_one-to-one}, \tag{26}$$

$$(\textbf{ex } g \textbf{ st } \text{graph}\, g = \text{graph}\, f \cap X) \;\&\; \textbf{ex } g \textbf{ st } \text{graph}\, g = X \cap \text{graph}\, f, \tag{27}$$

$$\begin{gathered}\text{graph}\, h = \text{graph}\, f \cap \text{graph}\, g \\ \textbf{implies } \text{dom}\, h \subseteq \text{dom}\, f \cap \text{dom}\, g \;\&\; \text{rng}\, h \subseteq \text{rng}\, f \cap \text{rng}\, g,\end{gathered} \tag{28}$$

$$\text{graph}\, h = \text{graph}\, f \cap \text{graph}\, g \;\&\; x \in \text{dom}\, h \textbf{ implies } h.x = f.x \;\&\; h.x = g.x, \tag{29}$$

$$(f \text{ is\_one-to-one } \mathbf{or}\ g \text{ is\_one-to-one}) \ \&\ \operatorname{graph} h = \operatorname{graph} f \cap \operatorname{graph} g \quad \mathbf{implies}\ h \text{ is\_one-to-one}, \tag{30}$$

$$\operatorname{dom} f \cap \operatorname{dom} g = \emptyset \ \mathbf{implies}\ \mathbf{ex}\ h\ \mathbf{st}\ \operatorname{graph} h = \operatorname{graph} f \cup \operatorname{graph} g, \tag{31}$$

$$\operatorname{graph} f \subseteq \operatorname{graph} h \ \&\ \operatorname{graph} g \subseteq \operatorname{graph} h \quad \mathbf{implies}\ \mathbf{ex}\ h1\ \mathbf{st}\ \operatorname{graph} h1 = \operatorname{graph} f \cup \operatorname{graph} g, \tag{32}$$

$$\operatorname{graph} h = \operatorname{graph}(f) \cup \operatorname{graph}(g) \quad \mathbf{implies}\ \operatorname{dom} h = \operatorname{dom} f \cup \operatorname{dom} g \ \&\ \operatorname{rng} h = \operatorname{rng} f \cup \operatorname{rng} g, \tag{33}$$

$$x \in \operatorname{dom} f \ \&\ \operatorname{graph} h = \operatorname{graph} f \cup \operatorname{graph} g \ \mathbf{implies}\ h.x = f.x, \tag{34}$$

$$x \in \operatorname{dom} g \ \&\ \operatorname{graph} h = \operatorname{graph} f \cup \operatorname{graph} g \ \mathbf{implies}\ h.x = g.x, \tag{35}$$

$$x \in \operatorname{dom} h \ \&\ \operatorname{graph} h = \operatorname{graph} f \cup \operatorname{graph} g \ \mathbf{implies}\ h.x = f.x \ \mathbf{or}\ h.x = g.x, \tag{36}$$

$$f \text{ is\_one-to-one} \ \&\ g \text{ is\_one-to-one} \ \&\ \operatorname{graph} h = \operatorname{graph} f \cup \operatorname{graph} g \ \&\ \operatorname{rng} f \cap \operatorname{rng} g = \emptyset \quad \mathbf{implies}\ h \text{ is\_one-to-one}, \tag{37}$$

$$\mathbf{ex}\ g\ \mathbf{st}\ \operatorname{graph} g = \operatorname{graph}(f) \setminus X, \tag{38}$$

$$\langle x, y \rangle \in \operatorname{graph} \operatorname{id}(X) \ \mathbf{iff}\ x \in X \ \&\ x = y, \tag{39}$$

$$\operatorname{graph} \operatorname{id} X = \triangle X, \tag{40}$$

$$x \in X \ \mathbf{iff}\ \langle x, x \rangle \in \operatorname{graph} \operatorname{id}(X), \tag{41}$$

$$\langle x, y \rangle \in \operatorname{graph}(f \cdot \operatorname{id}(X)) \ \mathbf{iff}\ x \in X \ \&\ \langle x, y \rangle \in \operatorname{graph} f, \tag{42}$$

$$\langle x, y \rangle \in \operatorname{graph}(\operatorname{id}(Y) \cdot f) \ \mathbf{iff}\ \langle x, y \rangle \in \operatorname{graph} f \ \&\ y \in Y, \tag{43}$$

$$\operatorname{graph}(f \cdot \operatorname{id}(X)) \subseteq \operatorname{graph} f \ \&\ \operatorname{graph}(\operatorname{id}(X) \cdot f) \subseteq \operatorname{graph}(f), \tag{44}$$

$$\operatorname{graph} \operatorname{id} \emptyset = \emptyset, \tag{45}$$

$$\operatorname{graph} f = \emptyset \ \mathbf{implies}\ f \text{ is\_one-to-one}, \tag{46}$$

$$f \text{ is\_one-to-one } \mathbf{implies\ for}\ x,y\ \mathbf{holds}\ \langle y, x \rangle \in \operatorname{graph}(f^{-1}) \ \mathbf{iff}\ \langle x, y \rangle \in \operatorname{graph} f, \tag{47}$$

$$f \text{ is\_one-to-one } \mathbf{implies}\ \operatorname{graph}(f^{-1}) = (\operatorname{graph} f)^{\tilde{}}, \tag{48}$$

$$\operatorname{graph} f = \emptyset \ \mathbf{implies}\ \operatorname{graph}(f^{-1}) = \emptyset, \tag{49}$$

$$\langle x, y \rangle \in \operatorname{graph}(f \mid X) \ \mathbf{iff}\ x \in X \ \&\ \langle x, y \rangle \in \operatorname{graph} f, \tag{50}$$

$$\operatorname{graph}(f \mid X) = (\operatorname{graph} f) \mid X, \tag{51}$$

$$x \in \operatorname{dom} f \ \& \ x \in X \ \textbf{iff} \ \langle x, f.x \rangle \in \operatorname{graph}(f \mid X), \tag{52}$$

$$\operatorname{graph}(f \mid X) \subseteq \operatorname{graph} f, \tag{53}$$

$$\operatorname{graph}((f \mid X) \cdot h) \subseteq \operatorname{graph}(f \cdot h) \ \& \ \operatorname{graph}(g \cdot (f \mid X)) \subseteq \operatorname{graph}(g \cdot f), \tag{54}$$

$$\operatorname{graph}(f \mid X) = \operatorname{graph}(f) \cap [\!: X, \operatorname{rng} f :\!], \tag{55}$$

$$X \subseteq Y \ \textbf{implies} \ \operatorname{graph}(f \mid X) \subseteq \operatorname{graph}(f \mid Y), \tag{56}$$

$$\operatorname{graph} f1 \subseteq \operatorname{graph} f2 \ \textbf{implies} \ \operatorname{graph}(f1 \mid X) \subseteq \operatorname{graph}(f2 \mid X), \tag{57}$$

$$\operatorname{graph} f1 \subseteq \operatorname{graph} f2 \ \& \ X1 \subseteq X2 \ \textbf{implies} \ \operatorname{graph}(f1 \mid X1) \subseteq \operatorname{graph}(f2 \mid X2), \tag{58}$$

$$\operatorname{graph}(f \mid (X \cup Y)) = \operatorname{graph}(f \mid X) \cup \operatorname{graph}(f \mid Y), \tag{59}$$

$$\operatorname{graph}(f \mid (X \cap Y)) = \operatorname{graph}(f \mid X) \cap \operatorname{graph}(f \mid Y), \tag{60}$$

$$\operatorname{graph}(f \mid (X \setminus Y)) = \operatorname{graph}(f \mid X) \setminus \operatorname{graph}(f \mid Y), \tag{61}$$

$$\operatorname{graph}(f \mid \emptyset) = \emptyset, \tag{62}$$

$$\operatorname{graph} f = \emptyset \ \textbf{implies} \ \operatorname{graph}(f \mid X) = \emptyset, \tag{63}$$

$$\operatorname{graph} g \subseteq \operatorname{graph} f \ \textbf{implies} \ f \mid \operatorname{dom} g = g, \tag{64}$$

$$\langle x, y \rangle \in \operatorname{graph}(Y \mid f) \ \textbf{iff} \ y \in Y \ \& \ \langle x, y \rangle \in \operatorname{graph} f, \tag{65}$$

$$\operatorname{graph}(Y \mid f) = Y \mid (\operatorname{graph} f), \tag{66}$$

$$x \in \operatorname{dom} f \ \& \ f.x \in Y \ \textbf{iff} \ \langle x, f.x \rangle \in \operatorname{graph}(Y \mid f), \tag{67}$$

$$\operatorname{graph}(Y \mid f) \subseteq \operatorname{graph}(f), \tag{68}$$

$$\operatorname{graph}((Y \mid f) \cdot h) \subseteq \operatorname{graph}(f \cdot h) \ \& \ \operatorname{graph}(g \cdot (Y \mid f)) \subseteq \operatorname{graph}(g \cdot f), \tag{69}$$

$$\operatorname{graph}(Y \mid f) = \operatorname{graph}(f) \cap [\!: \operatorname{dom} f, Y :\!], \tag{70}$$

$$X \subseteq Y \ \textbf{implies} \ \operatorname{graph}(X \mid f) \subseteq \operatorname{graph}(Y \mid f), \tag{71}$$

$$\operatorname{graph} f1 \subseteq \operatorname{graph} f2 \ \textbf{implies} \ \operatorname{graph}(Y \mid f1) \subseteq \operatorname{graph}(Y \mid f2), \tag{72}$$

$$\operatorname{graph} f1 \subseteq \operatorname{graph} f2 \ \& \ Y1 \subseteq Y2 \ \textbf{implies} \ \operatorname{graph}(Y1 \mid f1) \subseteq \operatorname{graph}(Y2 \mid f2), \tag{73}$$

$$\operatorname{graph}((X \cup Y) \mid f) = \operatorname{graph}(X \mid f) \cup \operatorname{graph}(Y \mid f), \tag{74}$$

$$\operatorname{graph}((X \cap Y) \mid f) = \operatorname{graph}(X \mid f) \cap \operatorname{graph}(Y \mid f), \tag{75}$$

$$\text{graph}\,((X \setminus Y) \mid f) = \text{graph}\,(X \mid f) \setminus \text{graph}\,(Y \mid f), \tag{76}$$

$$\text{graph}\,(\emptyset \mid f) = \emptyset, \tag{77}$$

$$\text{graph}\,f = \emptyset \textbf{ implies } \text{graph}\,(Y \mid f) = \emptyset, \tag{78}$$

$$\text{graph}\,g \subseteq \text{graph}\,f \;\&\; f \text{ is\_one-to-one } \textbf{implies } \text{rng}\,g \mid f = g, \tag{79}$$

$$y \in f^{\circ}\, X \textbf{ iff ex } x \textbf{ st } \langle x, y\rangle \in \text{graph}\,f \;\&\; x \in X, \tag{80}$$

$$f^{\circ}\, X = (\text{graph}\,f)^{\circ}\, X, \tag{81}$$

$$\text{graph}\,f = \emptyset \textbf{ implies } f^{\circ}\, X = \emptyset, \tag{82}$$

$$\text{graph}\,f1 \subseteq \text{graph}\,f2 \textbf{ implies } f1^{\circ}\, X \subseteq f2^{\circ}\, X, \tag{83}$$

$$\text{graph}\,f1 \subseteq \text{graph}\,f2 \;\&\; X1 \subseteq X2 \textbf{ implies } f1^{\circ}\, X1 \subseteq f2^{\circ}\, X2, \tag{84}$$

$$x \in f^{-1}\, Y \textbf{ iff ex } y \textbf{ st } \langle x, y\rangle \in \text{graph}\,f \;\&\; y \in Y, \tag{85}$$

$$f^{-1}\, Y = (\text{graph}\,f)^{-1}\, Y, \tag{86}$$

$$x \in f^{-1}\, Y \textbf{ iff } \langle x, f.x\rangle \in \text{graph}\,f \;\&\; f.x \in Y, \tag{87}$$

$$\text{graph}\,f = \emptyset \textbf{ implies } f^{-1}\, Y = \emptyset, \tag{88}$$

$$\text{graph}\,f1 \subseteq \text{graph}\,f2 \textbf{ implies } f1^{-1}\, Y \subseteq f2^{-1}\, Y, \tag{89}$$

$$\text{graph}\,f1 \subseteq \text{graph}\,f2 \;\&\; Y1 \subseteq Y2 \textbf{ implies } f1^{-1}\, Y1 \subseteq f2^{-1}\, Y2. \tag{90}$$

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Binary Operations

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Summary. In this paper we define binary and unary operations on domains. We also define the following predicates concerning the operations: ... is commutative, ... is associative, ... is the unity of ..., and ... is distributive wrt .... A number of schemes useful in justifying the existence of the operations are proved.

The articles [3], [1], and [2] provide the notation and terminology for this paper. The arguments of the notions defined below are the following: f which is an object of the type Function; a, b which are objects of the type Any. The functor

$$f\,.(a,b),$$

with values of the type Any, is defined by

$$\mathbf{it} = f\,.\langle a,b\rangle.$$

One can prove the following proposition

(1) **for** f **being** Function **for** a,b **being** Any **holds** $f\,.(a,b) = f\,.\langle a,b\rangle$.

In the sequel A, B, C will denote objects of the type DOMAIN. The arguments of the notions defined below are the following: A, B, C which are objects of the type reserved above; f which is an object of the type Function **of** $[:A,B:]$, C; a which is an object of the type Element **of** A; b which is an object of the type Element **of** B. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$f\,.(a,b) \quad \text{is} \quad \text{Element } \mathbf{of}\ C.$$

The following proposition is true

(2) **for** $f1$,$f2$ **being** Function **of** $[:A,B:]$,C **st**
for a **being** Element **of** A

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$$\textbf{for}\, b\ \textbf{being}\ \text{Element}\ \textbf{of}\ B\ \textbf{holds}\ f1.(a,b) = f2.(a,b)$$
$$\textbf{holds}\ f1 = f2.$$

We now define two new modes. Let us consider A.

Unary_Operation **of** A stands for Function **of** A, A.

Binary_Operation **of** A stands for Function **of** $[:A, A:], A$.

We now state a proposition

(3) **for** f **being** Function **of** A, A **holds** f **is** Unary_Operation **of** A.

In the sequel u denotes an object of the type Unary_Operation **of** A. Next we state a proposition

(4) **for** f **being** Function **of** $[:A, A:], A$ **holds** f **is** Binary_Operation **of** A.

In the article we present several logical schemes. The scheme *UnOpEx* concerns a constant $\mathcal{A}$ that has the type DOMAIN and a binary predicate $\mathcal{P}$ and states that the following holds

ex u **being** Unary_Operation **of** $\mathcal{A}$ **st for** x **being** Element **of** $\mathcal{A}$ **holds** $\mathcal{P}[x, u.x]$

provided the parameters satisfy the following conditions:

- **for** x **being** Element **of** $\mathcal{A}$ **ex** y **being** Element **of** $\mathcal{A}$ **st** $\mathcal{P}[x, y]$,
- **for** x,$y1$,$y2$ **being** Element **of** $\mathcal{A}$ **st** $\mathcal{P}[x, y1]$ & $\mathcal{P}[x, y2]$ **holds** $y1 = y2$.

The scheme *UnOpLambda* concerns a constant $\mathcal{A}$ that has the type DOMAIN and a unary functor $\mathcal{F}$ yielding values of the type Element **of** $\mathcal{A}$ and states that the following holds

ex u **being** Unary_Operation **of** $\mathcal{A}$ **st for** x **being** Element **of** $\mathcal{A}$ **holds** $u.x = \mathcal{F}(x)$

for all values of the parameters.

For simplicity we adopt the following convention: o, o' will have the type Binary_Operation **of** A; a, b, c, e, $e1$, $e2$ will have the type Element **of** A. Let us consider A, o, a, b. Let us note that it makes sense to consider the following functor on a restricted area. Then

$o.(a, b)$ is Element **of** A.

Now we present two schemes. The scheme *BinOpEx* concerns a constant $\mathcal{A}$ that has the type DOMAIN and a ternary predicate $\mathcal{P}$ and states that the following holds

ex o **being** Binary_Operation **of** $\mathcal{A}$
st for a,b **being** Element **of** $\mathcal{A}$ **holds** $\mathcal{P}[a, b, o.(a, b)]$

provided the parameters satisfy the following conditions:

- **for** x,y **being** Element **of** $\mathcal{A}$ **ex** z **being** Element **of** $\mathcal{A}$ **st** $\mathcal{P}[x, y, z]$,
- **for** x,y **being** Element **of** $\mathcal{A}$
  for $z1$,$z2$ **being** Element **of** $\mathcal{A}$ **st** $\mathcal{P}[x, y, z1]$ & $\mathcal{P}[x, y, z2]$ **holds** $z1 = z2$.

The scheme *BinOpLambda* concerns a constant $\mathcal{A}$ that has the type DOMAIN and a binary functor $\mathcal{F}$ yielding values of the type Element **of** $\mathcal{A}$ and states that the following holds

ex o **being** Binary_Operation **of** $\mathcal{A}$
st for a,b **being** Element **of** $\mathcal{A}$ **holds** $o.(a, b) = \mathcal{F}(a, b)$

for all values of the parameters.

We now define three new predicates. Let us consider A, o. The predicate

o is_commutative   is defined by   **for** a,b **holds** $o.(a, b) = o.(b, a)$.

The predicate

o is_associative   is defined by   **for** a,b,c **holds** $o.(a, o.(b, c)) = o.(o.(a, b),c)$.

The predicate

o is_an_idempotentOp   is defined by   **for** a **holds** $o.(a, a) = a$.

Next we state three propositions:

(5) o is_commutative **iff for** a,b **holds** $o.(a, b) = o.(b, a)$,

(6) o is_associative **iff for** a,b,c **holds** $o.(a, o.(b, c)) = o.(o.(a, b),c)$,

(7) o is_an_idempotentOp **iff for** a **holds** $o.(a, a) = a$.

We now define two new predicates. Let us consider A, e, o. The predicate

e is_a_left_unity_wrt o is defined by   **for** a **holds** $o.(e, a) = a$.

The predicate

e is_a_right_unity_wrt o is defined by   **for** a **holds** $o.(a, e) = a$.

Let us consider A, e, o. The predicate

e is_a_unity_wrt o is defined by e is_a_left_unity_wrt o & e is_a_right_unity_wrt o.

We now state a number of propositions:

(8) e is_a_left_unity_wrt o **iff for** a **holds** $o.(e, a) = a$,

(9) e is_a_right_unity_wrt o **iff for** a **holds** $o.(a,e) = a$,

(10) e is_a_unity_wrt o **iff** e is_a_left_unity_wrt o & e is_a_right_unity_wrt o,

(11) e is_a_unity_wrt o **iff for** a **holds** $o.(e,a) = a$ & $o.(a,e) = a$,

(12) o is_commutative **implies** (e is_a_unity_wrt o **iff for** a **holds** $o.(e,a) = a$),

(13) o is_commutative **implies** (e is_a_unity_wrt o **iff for** a **holds** $o.(a,e) = a$),

(14) o is_commutative **implies** (e is_a_unity_wrt o **iff** e is_a_left_unity_wrt o),

(15) o is_commutative **implies** (e is_a_unity_wrt o **iff** e is_a_right_unity_wrt o),

(16) o is_commutative **implies** (e is_a_left_unity_wrt o **iff** e is_a_right_unity_wrt o),

(17) $e1$ is_a_left_unity_wrt o & $e2$ is_a_right_unity_wrt o **implies** $e1 = e2$,

(18) $e1$ is_a_unity_wrt o & $e2$ is_a_unity_wrt o **implies** $e1 = e2$.

Let us consider A, o. Assume that the following holds

ex e **st** e is_a_unity_wrt o.

The functor

the_unity_wrt o,

with values of the type Element **of** A, is defined by

it is_a_unity_wrt o.

One can prove the following proposition

(19) (**ex** e **st** e is_a_unity_wrt o)
implies for e **holds** e = the_unity_wrt o **iff** e is_a_unity_wrt o.

We now define two new predicates. Let us consider A, o', o. The predicate

o' is_left_distributive_wrt o

is defined by

for a,b,c **holds** $o'.(a, o.(b,c)) = o.(o'.(a,b), o'.(a,c))$.

The predicate

o' is_right_distributive_wrt o

is defined by

for a,b,c **holds** $o'.(o.(a,b), c) = o.(o'.(a,c), o'.(b,c))$.

Let us consider A, o', o. The predicate

$$o' \text{ is\_distributive\_wrt } o$$

is defined by

$$o' \text{ is\_left\_distributive\_wrt } o \ \& \ o' \text{ is\_right\_distributive\_wrt } o.$$

We now state several propositions:

(20) $$o' \text{ is\_left\_distributive\_wrt } o \\ \textbf{iff for } a,b,c \textbf{ holds } o'.(a, o.(b, c)) = o.(o'.(a, b),o'.(a, c)),$$

(21) $$o' \text{ is\_right\_distributive\_wrt } o \\ \textbf{iff for } a,b,c \textbf{ holds } o'.(o.(a, b),c) = o.(o'.(a, c),o'.(b, c)),$$

(22) $$o' \text{ is\_distributive\_wrt } o \\ \textbf{iff } o' \text{ is\_left\_distributive\_wrt } o \ \& \ o' \text{ is\_right\_distributive\_wrt } o,$$

(23) $$o' \text{ is\_distributive\_wrt } o \textbf{ iff for } a,b,c \textbf{ holds} \\ o'.(a, o.(b, c)) = o.(o'.(a, b),o'.(a, c)) \ \& \ o'.(o.(a, b),c) = o.(o'.(a, c),o'.(b, c)),$$

(24) $$o' \text{ is\_commutative } \textbf{implies } (o' \text{ is\_distributive\_wrt } o \\ \textbf{iff for } a,b,c \textbf{ holds } o'.(a, o.(b, c)) = o.(o'.(a, b),o'.(a, c))),$$

(25) $$o' \text{ is\_commutative } \textbf{implies } (o' \text{ is\_distributive\_wrt } o \\ \textbf{iff for } a,b,c \textbf{ holds } o'.(o.(a, b),c) = o.(o'.(a, c),o'.(b, c))),$$

(26) $$o' \text{ is\_commutative} \\ \textbf{implies } (o' \text{ is\_distributive\_wrt } o \textbf{ iff } o' \text{ is\_left\_distributive\_wrt } o),$$

(27) $$o' \text{ is\_commutative} \\ \textbf{implies } (o' \text{ is\_distributive\_wrt } o \textbf{ iff } o' \text{ is\_right\_distributive\_wrt } o),$$

(28) $$o' \text{ is\_commutative} \\ \textbf{implies } (o' \text{ is\_right\_distributive\_wrt } o \textbf{ iff } o' \text{ is\_left\_distributive\_wrt } o).$$

Let us consider A, u, o. The predicate

u is_distributive_wrt o is defined by **for** a,b **holds** $u.(o.(a, b)) = o.((u.a),(u.b))$.

The following proposition is true

(29) u is_distributive_wrt o **iff for** a,b **holds** $u.(o.(a, b)) = o.((u.a),(u.b))$.

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Relations Defined on Sets

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Summary. The article includes theorems concerning properties of relations defined as a subset of the Cartesian product of two sets (mode Relation of X,Y where X,Y are sets). Some notions, introduced in [3] such as domain, codomain, field of a relation, composition of relations, image and inverse image of a set under a relation are redefined.

The articles [1], [2], and [3] provide the terminology and notation for this paper. For simplicity we adopt the following convention: A, B, X, $X1$, Y, $Y1$, Z will denote objects of the type set; a, x, y will denote objects of the type Any. Let us consider X, Y. The mode

$$\text{Relation } \mathbf{of}\ X, Y,$$

which widens to the type Relation, is defined by

$$\mathbf{it} \subseteq [:X, Y:].$$

The following proposition is true

(1) $\mathbf{for}\ R\ \mathbf{being}\ \text{Relation}\ \mathbf{holds}\ R \subseteq [:X, Y:]\ \mathbf{iff}\ R\ \mathbf{is}\ \text{Relation}\ \mathbf{of}\ X, Y.$

In the sequel P, R will denote objects of the type Relation **of** X, Y. The following propositions are true:

(2) $A \subseteq R\ \mathbf{implies}\ A \subseteq [:X, Y:],$

(3) $A \subseteq [:X, Y:]\ \mathbf{implies}\ A\ \mathbf{is}\ \text{Relation}\ \mathbf{of}\ X, Y,$

(4) $A \subseteq R\ \mathbf{implies}\ A\ \mathbf{is}\ \text{Relation}\ \mathbf{of}\ X, Y,$

(5) $[:X, Y:]\ \mathbf{is}\ \text{Relation}\ \mathbf{of}\ X, Y,$

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$$a \in R \textbf{ implies ex } x{,}y \textbf{ st } a = \langle x, y\rangle \ \&\ x \in X \ \&\ y \in Y, \tag{6}$$

$$\langle x, y\rangle \in R \textbf{ implies } x \in X \ \&\ y \in Y, \tag{7}$$

$$x \in X \ \&\ y \in Y \textbf{ implies } \{\langle x, y\rangle\} \textbf{ is } \text{Relation } \textbf{of } X, Y, \tag{8}$$

$$\textbf{for } R \textbf{ being } \text{Relation } \textbf{st } \operatorname{dom} R \subseteq X \textbf{ holds } R \textbf{ is } \text{Relation } \textbf{of } X, \operatorname{rng} R, \tag{9}$$

$$\textbf{for } R \textbf{ being } \text{Relation } \textbf{st } \operatorname{rng} R \subseteq Y \textbf{ holds } R \textbf{ is } \text{Relation } \textbf{of } \operatorname{dom} R{,}Y, \tag{10}$$

$$\begin{gathered}\textbf{for } R \textbf{ being } \text{Relation}\\ \textbf{st } \operatorname{dom} R \subseteq X \ \&\ \operatorname{rng} R \subseteq Y \textbf{ holds } R \textbf{ is } \text{Relation } \textbf{of } X, Y,\end{gathered} \tag{11}$$

$$\operatorname{dom} R \subseteq X \ \&\ \operatorname{rng} R \subseteq Y, \tag{12}$$

$$\operatorname{dom} R \subseteq X1 \textbf{ implies } R \textbf{ is } \text{Relation } \textbf{of } X1{,}Y, \tag{13}$$

$$\operatorname{rng} R \subseteq Y1 \textbf{ implies } R \textbf{ is } \text{Relation } \textbf{of } X, Y1, \tag{14}$$

$$X \subseteq X1 \textbf{ implies } R \textbf{ is } \text{Relation } \textbf{of } X1{,}Y, \tag{15}$$

$$Y \subseteq Y1 \textbf{ implies } R \textbf{ is } \text{Relation } \textbf{of } X, Y1, \tag{16}$$

$$X \subseteq X1 \ \&\ Y \subseteq Y1 \textbf{ implies } R \textbf{ is } \text{Relation } \textbf{of } X1{,}Y1. \tag{17}$$

Let us consider X, Y, P, R. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$\begin{array}{ccc} P \cup R & \text{is} & \text{Relation } \textbf{of } X, Y, \\ P \cap R & \text{is} & \text{Relation } \textbf{of } X, Y, \\ P \setminus R & \text{is} & \text{Relation } \textbf{of } X, Y. \end{array}$$

We now state a proposition

$$R \cap [\!:X, Y:\!] = R. \tag{18}$$

Let us consider X, Y, R. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$\begin{array}{ccc} \operatorname{dom} R & \text{is} & \text{Subset } \textbf{of } X, \\ \operatorname{rng} R & \text{is} & \text{Subset } \textbf{of } Y. \end{array}$$

Next we state several propositions:

$$\operatorname{field} R \subseteq X \cup Y, \tag{19}$$

$$\textbf{for } R \textbf{ being } \text{Relation } \textbf{holds } R \textbf{ is } \text{Relation } \textbf{of } \operatorname{dom} R{,}\operatorname{rng} R, \tag{20}$$

(21) $$\mathrm{dom}\, R \subseteq X1 \;\&\; \mathrm{rng}\, R \subseteq Y1 \;\textbf{implies}\; R \;\textbf{is}\; \text{Relation} \;\textbf{of}\; X1,Y1,$$

(22) $$(\textbf{for}\, x \;\textbf{st}\; x \in X \;\textbf{ex}\, y \;\textbf{st}\; \langle x, y\rangle \in R) \;\textbf{iff}\; \mathrm{dom}\, R = X,$$

(23) $$(\textbf{for}\, y \;\textbf{st}\; y \in Y \;\textbf{ex}\, x \;\textbf{st}\; \langle x, y\rangle \in R) \;\textbf{iff}\; \mathrm{rng}\, R = Y.$$

Let us consider X, Y, R. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$R\,\tilde{} \qquad \text{is} \qquad \text{Relation} \;\textbf{of}\; Y, X.$$

The arguments of the notions defined below are the following: X, Y, Z which are objects of the type reserved above; P which is an object of the type Relation **of** X, Y; R which is an object of the type Relation **of** Y, Z. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$P \cdot R \qquad \text{is} \qquad \text{Relation} \;\textbf{of}\; X, Z.$$

One can prove the following propositions:

(24) $$\mathrm{dom}\,(R\,\tilde{}) = \mathrm{rng}\, R \;\&\; \mathrm{rng}\,(R\,\tilde{}) = \mathrm{dom}\, R,$$

(25) $$\varnothing \;\textbf{is}\; \text{Relation} \;\textbf{of}\; X, Y,$$

(26) $$R \;\textbf{is}\; \text{Relation} \;\textbf{of}\; \emptyset, Y \;\textbf{implies}\; R = \varnothing\,,$$

(27) $$R \;\textbf{is}\; \text{Relation} \;\textbf{of}\; X, \emptyset \;\textbf{implies}\; R = \varnothing\,,$$

(28) $$\triangle\, X \subseteq [\!:X, X:\!],$$

(29) $$\triangle\, X \;\textbf{is}\; \text{Relation} \;\textbf{of}\; X, X,$$

(30) $$\triangle\, A \subseteq R \;\textbf{implies}\; A \subseteq \mathrm{dom}\, R \;\&\; A \subseteq \mathrm{rng}\, R,$$

(31) $$\triangle\, X \subseteq R \;\textbf{implies}\; X = \mathrm{dom}\, R \;\&\; X \subseteq \mathrm{rng}\, R,$$

(32) $$\triangle\, Y \subseteq R \;\textbf{implies}\; Y \subseteq \mathrm{dom}\, R \;\&\; Y = \mathrm{rng}\, R.$$

Let us consider X, Y, R, A. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$R \,|\, A \qquad \text{is} \qquad \text{Relation} \;\textbf{of}\; X, Y.$$

Let us consider X, Y, B, R. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$B \,|\, R \qquad \text{is} \qquad \text{Relation} \;\textbf{of}\; X, Y.$$

The following four propositions are true:

(33) $$R \,|\, X1 \;\textbf{is}\; \text{Relation} \;\textbf{of}\; X1, Y,$$

(34) $$X \subseteq X1 \textbf{ implies } R \mid X1 = R,$$

(35) $$Y1 \mid R \textbf{ is } \text{Relation} \textbf{ of } X, Y1,$$

(36) $$Y \subseteq Y1 \textbf{ implies } Y1 \mid R = R.$$

Let us consider X, Y, R, A. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$R^{\circ}\, A \quad \text{is} \quad \text{Subset} \textbf{ of } Y,$$

$$R^{-1}\, A \quad \text{is} \quad \text{Subset} \textbf{ of } X.$$

Next we state three propositions:

(37) $$R^{\circ}\, A \subseteq Y \;\&\; R^{-1}\, A \subseteq X,$$

(38) $$R^{\circ}\, X = \operatorname{rng} R \;\&\; R^{-1}\, Y = \operatorname{dom} R,$$

(39) $$R^{\circ}\, (R^{-1}\, Y) = \operatorname{rng} R \;\&\; R^{-1}\, (R^{\circ}\, X) = \operatorname{dom} R.$$

The scheme *Rel_On_Set_Ex* deals with a constant $\mathcal{A}$ that has the type set, a constant $\mathcal{B}$ that has the type set and a binary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex } R \textbf{ being } \text{Relation} \textbf{ of } \mathcal{A},\mathcal{B} \textbf{ st for } x,y \textbf{ holds } \langle x, y\rangle \in R \textbf{ iff } x \in \mathcal{A} \;\&\; y \in \mathcal{B} \;\&\; \mathcal{P}[x, y]$$

for all values of the parameters.

Let us consider X.

$$\text{Relation} \textbf{ of } X \quad \text{stands for} \quad \text{Relation} \textbf{ of } X, X.$$

We now state three propositions:

(40) $$\textbf{for } R \textbf{ being } \text{Relation} \textbf{ of } X, X \textbf{ holds } R \subseteq [\!:X, X:\!] \textbf{ iff } R \textbf{ is } \text{Relation} \textbf{ of } X,$$

(41) $$[\!:X, X:\!] \textbf{ is } \text{Relation} \textbf{ of } X,$$

(42) $$\textbf{for } R \textbf{ being } \text{Relation} \textbf{ of } X, X \textbf{ holds } R \textbf{ is } \text{Relation} \textbf{ of } X.$$

In the sequel R denotes an object of the type Relation **of** X. One can prove the following propositions:

(43) $$\triangle\, X \textbf{ is } \text{Relation} \textbf{ of } X,$$

(44) $$\triangle\, X \subseteq R \textbf{ implies } X = \operatorname{dom} R \;\&\; X = \operatorname{rng} R,$$

(45) $$R \cdot (\triangle\, X) = R \;\&\; (\triangle\, X) \cdot R = R.$$

For simplicity we adopt the following convention: D, $D1$, $D2$, E, F denote objects of the type DOMAIN; R denotes an object of the type Relation **of** D, E; x denotes an object of the type Element **of** D; y denotes an object of the type Element **of** E. We now state a proposition

$$\triangle\, D \neq \emptyset\,. \tag{46}$$

Let us consider D, E, R. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$\operatorname{dom} R \quad \text{is} \quad \text{Element } \mathbf{of} \operatorname{bool} D,$$

$$\operatorname{rng} R \quad \text{is} \quad \text{Element } \mathbf{of} \operatorname{bool} E.$$

Next we state several propositions:

$$\begin{gathered}\mathbf{for}\, x\, \mathbf{being}\ \text{Element } \mathbf{of}\ D \\ \mathbf{holds}\ x \in \operatorname{dom} R\ \mathbf{iff}\ \mathbf{ex}\, y\ \mathbf{being}\ \text{Element } \mathbf{of}\ E\ \mathbf{st}\ \langle x, y\rangle \in R,\end{gathered} \tag{47}$$

$$\begin{gathered}\mathbf{for}\, y\, \mathbf{being}\ \text{Element } \mathbf{of}\ E \\ \mathbf{holds}\ y \in \operatorname{rng} R\ \mathbf{iff}\ \mathbf{ex}\, x\ \mathbf{being}\ \text{Element } \mathbf{of}\ D\ \mathbf{st}\ \langle x, y\rangle \in R,\end{gathered} \tag{48}$$

$$\begin{gathered}\mathbf{for}\, x\, \mathbf{being}\ \text{Element } \mathbf{of}\ D \\ \mathbf{holds}\ x \in \operatorname{dom} R\ \mathbf{implies}\ \mathbf{ex}\, y\ \mathbf{being}\ \text{Element } \mathbf{of}\ E\ \mathbf{st}\ y \in \operatorname{rng} R,\end{gathered} \tag{49}$$

$$\begin{gathered}\mathbf{for}\, y\, \mathbf{being}\ \text{Element } \mathbf{of}\ E \\ \mathbf{holds}\ y \in \operatorname{rng} R\ \mathbf{implies}\ \mathbf{ex}\, x\ \mathbf{being}\ \text{Element } \mathbf{of}\ D\ \mathbf{st}\ x \in \operatorname{dom} R,\end{gathered} \tag{50}$$

$$\begin{gathered}\mathbf{for}\ P\ \mathbf{being}\ \text{Relation } \mathbf{of}\ D, E, R\ \mathbf{being}\ \text{Relation } \mathbf{of}\ E, F \\ \mathbf{for}\ x\ \mathbf{being}\ \text{Element } \mathbf{of}\ D, z\ \mathbf{being}\ \text{Element } \mathbf{of}\ F \\ \mathbf{holds}\ \langle x, z\rangle \in P \cdot R\ \mathbf{iff}\ \mathbf{ex}\, y\ \mathbf{being}\ \text{Element } \mathbf{of}\ E\ \mathbf{st}\ \langle x, y\rangle \in P\ \&\ \langle y, z\rangle \in R.\end{gathered} \tag{51}$$

Let us consider D, E, R, $D1$. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$R\,^{\circ}\, D1 \quad \text{is} \quad \text{Element } \mathbf{of} \operatorname{bool} E,$$

$$R\,^{-1}\, D1 \quad \text{is} \quad \text{Element } \mathbf{of} \operatorname{bool} D.$$

We now state two propositions:

$$y \in R\,^{\circ}\, D1\ \mathbf{iff}\ \mathbf{ex}\, x\ \mathbf{being}\ \text{Element } \mathbf{of}\ D\ \mathbf{st}\ \langle x, y\rangle \in R\ \&\ x \in D1, \tag{52}$$

$$x \in R\,^{-1}\, D2\ \mathbf{iff}\ \mathbf{ex}\, y\ \mathbf{being}\ \text{Element } \mathbf{of}\ E\ \mathbf{st}\ \langle x, y\rangle \in R\ \&\ y \in D2. \tag{53}$$

The scheme *Rel_On_Dom_Ex* concerns a constant $\mathcal{A}$ that has the type DOMAIN, a constant $\mathcal{B}$ that has the type DOMAIN and a binary predicate $\mathcal{P}$ and states that the following holds

$$\begin{gathered}\mathbf{ex}\ R\ \mathbf{being}\ \text{Relation } \mathbf{of}\ \mathcal{A}, \mathcal{B}\ \mathbf{st}\ \mathbf{for}\ x\ \mathbf{being}\ \text{Element } \mathbf{of}\ \mathcal{A}, y\ \mathbf{being}\ \text{Element } \mathbf{of}\ \mathcal{B} \\ \mathbf{holds}\ \langle x, y\rangle \in R\ \mathbf{iff}\ x \in \mathcal{A}\ \&\ y \in \mathcal{B}\ \&\ \mathcal{P}[x, y]\end{gathered}$$

for all values of the parameters.

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Boolean Domains

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Summary. BOOLE DOMAIN is a SET DOMAIN that is closed under union and difference. This condition is equivalent to being closed under symmetric difference and one of the following operations: union, intersection or difference. We introduce the set of all finite subsets of a set A, denoted by Fin A. The mode Finite Subset of a set A is introduced with the mother type: Element of Fin A. In consequence, "Finite Subset of ..." is an elementary type, therefore one may use such types as "set of Finite Subset of A", "[(Finite Subset of A), Finite Subset of A]", and so on. The article begins with some auxiliary theorems that belong really to [5] or [1] but are missing there. Moreover, bool A is redefined as a SET DOMAIN, for an arbitrary set A.

The articles [4], [5], [3], and [2] provide the notation and terminology for this paper. In the sequel X, Y will denote objects of the type set. The following propositions are true:

(1) $$X \text{ misses } Y \textbf{ implies } X \setminus Y = X \ \& \ Y \setminus X = Y,$$

(2) $$X \text{ misses } Y \textbf{ implies } (X \cup Y) \setminus Y = X \ \& \ (X \cup Y) \setminus X = Y,$$

(3) $$X \cup Y = X \dot{-} (Y \setminus X),$$

(4) $$X \cup Y = X \dot{-} Y \dot{-} X \cap Y,$$

(5) $$X \setminus Y = X \dot{-} (X \cap Y),$$

(6) $$X \cap Y = X \dot{-} Y \dot{-} (X \cup Y),$$

(7) $$(\textbf{for } x \textbf{ being } \text{set } \textbf{st } x \in X \textbf{ holds } x \in Y) \textbf{ implies } X \subseteq Y.$$

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Let us consider X. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\mathrm{bool}\,X \qquad \text{is} \qquad \text{SET\_DOMAIN}\,.$$

The following proposition is true

(8) $$\textbf{for}\,Y\ \textbf{being}\ \text{Element}\ \textbf{of}\ \mathrm{bool}\,X\ \textbf{holds}\ Y \subseteq X.$$

The mode

$$\text{BOOLE\_DOMAIN}\,,$$

which widens to the type SET_DOMAIN, is defined by

$$\textbf{for}\ X{,}Y\ \textbf{being}\ \text{Element}\ \textbf{of it holds}\ X \cup Y \in \textbf{it}\ \&\ X \setminus Y \in \textbf{it}\,.$$

The following proposition is true

(9) $$\begin{gathered}\textbf{for}\ A\ \textbf{being}\ \text{SET\_DOMAIN}\ \textbf{holds}\ A\ \textbf{is}\ \text{BOOLE\_DOMAIN}\\ \textbf{iff for}\ X{,}Y\ \textbf{being}\ \text{Element}\ \textbf{of}\ A\ \textbf{holds}\ X \cup Y \in A\ \&\ X \setminus Y \in A.\end{gathered}$$

In the sequel A will denote an object of the type BOOLE_DOMAIN. One can prove the following propositions:

(10) $$X \in A\ \&\ Y \in A\ \textbf{implies}\ X \cup Y \in A\ \&\ X \setminus Y \in A,$$

(11) $$X\ \textbf{is}\ \text{Element}\ \textbf{of}\ A\ \&\ Y\ \textbf{is}\ \text{Element}\ \textbf{of}\ A\ \textbf{implies}\ X \cup Y\ \textbf{is}\ \text{Element}\ \textbf{of}\ A,$$

(12) $$X\ \textbf{is}\ \text{Element}\ \textbf{of}\ A\ \&\ Y\ \textbf{is}\ \text{Element}\ \textbf{of}\ A\ \textbf{implies}\ X \setminus Y\ \textbf{is}\ \text{Element}\ \textbf{of}\ A.$$

The arguments of the notions defined below are the following: A which is an object of the type reserved above; X, Y which are objects of the type Element **of** A. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$X \cup Y \qquad \text{is} \qquad \text{Element}\ \textbf{of}\ A,$$

$$X \setminus Y \qquad \text{is} \qquad \text{Element}\ \textbf{of}\ A.$$

The following propositions are true:

(13) $$X\ \textbf{is}\ \text{Element}\ \textbf{of}\ A\ \&\ Y\ \textbf{is}\ \text{Element}\ \textbf{of}\ A\ \textbf{implies}\ X \cap Y\ \textbf{is}\ \text{Element}\ \textbf{of}\ A,$$

(14) $$X\ \textbf{is}\ \text{Element}\ \textbf{of}\ A\ \&\ Y\ \textbf{is}\ \text{Element}\ \textbf{of}\ A\ \textbf{implies}\ X \mathbin{\dot-} Y\ \textbf{is}\ \text{Element}\ \textbf{of}\ A,$$

(15) $$\begin{gathered}\textbf{for}\ A\ \textbf{being}\ \text{SET\_DOMAIN}\ \textbf{st}\\ \textbf{for}\ X{,}Y\ \textbf{being}\ \text{Element}\ \textbf{of}\ A\ \textbf{holds}\ X \mathbin{\dot-} Y \in A\ \&\ X \setminus Y \in A\\ \textbf{holds}\ A\ \textbf{is}\ \text{BOOLE\_DOMAIN}\,,\end{gathered}$$

$$\text{(16)}\quad \textbf{for } A \textbf{ being } \text{SET\_DOMAIN } \textbf{st} \textbf{ for } X,Y \textbf{ being } \text{Element } \textbf{of } A \textbf{ holds } X \mathbin{\dot{-}} Y \in A \mathrel{\&} X \cap Y \in A \textbf{ holds } A \textbf{ is } \text{BOOLE\_DOMAIN}\,,$$

$$\text{(17)}\quad \textbf{for } A \textbf{ being } \text{SET\_DOMAIN } \textbf{st} \textbf{ for } X,Y \textbf{ being } \text{Element } \textbf{of } A \textbf{ holds } X \mathbin{\dot{-}} Y \in A \mathrel{\&} X \cup Y \in A \textbf{ holds } A \textbf{ is } \text{BOOLE\_DOMAIN}\,.$$

The arguments of the notions defined below are the following: A which is an object of the type reserved above; X, Y which are objects of the type Element **of** A. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$X \cap Y \quad \text{is} \quad \text{Element } \textbf{of } A,$$

$$X \mathbin{\dot{-}} Y \quad \text{is} \quad \text{Element } \textbf{of } A.$$

We now state four propositions:

$$\text{(18)}\quad \emptyset \in A,$$

$$\text{(19)}\quad \emptyset \textbf{ is } \text{Element } \textbf{of } A,$$

$$\text{(20)}\quad \text{bool}\, A \textbf{ is } \text{BOOLE\_DOMAIN}\,,$$

$$\text{(21)}\quad \textbf{for } A,B \textbf{ being } \text{BOOLE\_DOMAIN } \textbf{holds } A \cap B \textbf{ is } \text{BOOLE\_DOMAIN}\,.$$

In the sequel A, B will denote objects of the type set. Let us consider A. The functor

$$\text{Fin}\, A,$$

with values of the type BOOLE_DOMAIN, is defined by

$$\textbf{for } X \textbf{ being } \text{set } \textbf{holds } X \in \textbf{it iff } X \subseteq A \mathrel{\&} X \text{ is\_finite}\,.$$

The following propositions are true:

$$\text{(22)}\quad B \in \text{Fin}\, A \textbf{ iff } B \subseteq A \mathrel{\&} B \text{ is\_finite}\,,$$

$$\text{(23)}\quad A \subseteq B \textbf{ implies } \text{Fin}\, A \subseteq \text{Fin}\, B,$$

$$\text{(24)}\quad \text{Fin}\,(A \cap B) = \text{Fin}\, A \cap \text{Fin}\, B,$$

$$\text{(25)}\quad \text{Fin}\, A \cup \text{Fin}\, B \subseteq \text{Fin}\,(A \cup B),$$

$$\text{(26)}\quad \text{Fin}\, A \subseteq \text{bool}\, A,$$

$$\text{(27)}\quad A \text{ is\_finite } \textbf{implies } \text{Fin}\, A = \text{bool}\, A,$$

$$\text{(28)}\quad \text{Fin}\, \emptyset = \{\emptyset\}.$$

Let us consider A.

$$\text{Finite\_Subset } \mathbf{of}\ A \quad \text{stands for} \quad \text{Element } \mathbf{of}\ \text{Fin}\, A.$$

Next we state a proposition

(29) $\mathbf{for}\ X\ \mathbf{being}\ \text{Element } \mathbf{of}\ \text{Fin}\, A\ \mathbf{holds}\ X\ \mathbf{is}\ \text{Finite\_Subset } \mathbf{of}\ A.$

The arguments of the notions defined below are the following: A which is an object of the type reserved above; X, Y which are objects of the type Finite_Subset **of** A. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$X \cup Y \quad \text{is} \quad \text{Finite\_Subset } \mathbf{of}\ A,$$

$$X \cap Y \quad \text{is} \quad \text{Finite\_Subset } \mathbf{of}\ A,$$

$$X \setminus Y \quad \text{is} \quad \text{Finite\_Subset } \mathbf{of}\ A,$$

$$X \mathbin{\dot{-}} Y \quad \text{is} \quad \text{Finite\_Subset } \mathbf{of}\ A.$$

One can prove the following propositions:

(30) $\mathbf{for}\ X\ \mathbf{being}\ \text{Finite\_Subset } \mathbf{of}\ A\ \mathbf{holds}\ X\ \text{is\_finite}\,,$

(31) $\mathbf{for}\ X\ \mathbf{being}\ \text{Finite\_Subset } \mathbf{of}\ A\ \mathbf{holds}\ X \subseteq A,$

(32) $\mathbf{for}\ X\ \mathbf{being}\ \text{Finite\_Subset } \mathbf{of}\ A\ \mathbf{holds}\ X\ \mathbf{is}\ \text{Subset } \mathbf{of}\ A,$

(33) $\emptyset\ \mathbf{is}\ \text{Finite\_Subset } \mathbf{of}\ A,$

(34) $A\ \text{is\_finite}\ \mathbf{implies}\ \mathbf{for}\ X\ \mathbf{being}\ \text{Subset } \mathbf{of}\ A\ \mathbf{holds}\ X\ \mathbf{is}\ \text{Finite\_Subset } \mathbf{of}\ A.$

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Models and Satisfiability

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Summary. The article includes schemes of defining by structural induction, and definitions and theorems related to: the set of variables which have free occurrences in a ZF-formula, the set of all valuations of variables in a model, the set of all valuations which satisfy a ZF-formula in a model, the satisfiability of a ZF-formula in a model by a valuation, the validity of a ZF-formula in a model, the axioms of ZF-language, the model of the ZF set theory.

The articles [6], [7], [3], [1], [4], [5], and [2] provide the notation and terminology for this paper. For simplicity we adopt the following convention: H, H' will have the type ZF-formula; x, y, z will have the type Variable; a, b, c will have the type Any; A, X will have the type set. In the article we present several logical schemes. The scheme *ZFsch_ex* deals with a binary functor $\mathcal{F}$, a binary functor $\mathcal{G}$, a unary functor $\mathcal{H}$, a binary functor $\mathcal{I}$, a binary functor $\mathcal{J}$ and a constant $\mathcal{A}$ that has the type ZF-formula, and states that the following holds

$$\mathbf{ex}\, a{,}A\ \mathbf{st}\ (\mathbf{for}\, x{,}y\ \mathbf{holds}\ \langle x \mathrel{=} y{,}\mathcal{F}(x,y)\rangle \in A\ \&\ \langle x \mathrel{\epsilon} y{,}\mathcal{G}(x,y)\rangle \in A)\ \&\ \langle \mathcal{A}{,}a\rangle \in A\ \&$$
$$\mathbf{for}\, H{,}a\ \mathbf{st}\ \langle H, a\rangle \in A\ \mathbf{holds}\ (H\ \text{is\_a\_equality}\ \mathbf{implies}\ a = \mathcal{F}(\mathrm{Var}_1\, H{,}\mathrm{Var}_2\, H))\ \&$$
$$(H\ \text{is\_a\_membership}\ \mathbf{implies}\ a = \mathcal{G}(\mathrm{Var}_1\, H{,}\mathrm{Var}_2\, H))\ \&$$
$$(H\ \text{is\_negative}\ \mathbf{implies}\ \mathbf{ex}\, b\ \mathbf{st}\ a = \mathcal{H}(b)\ \&\ \langle \text{the\_argument\_of}\, H{,}b\rangle \in A)\ \&$$
$$(H\ \text{is\_conjunctive}\ \mathbf{implies}\ \mathbf{ex}\, b{,}c$$
$$\mathbf{st}\ a = \mathcal{I}(b,c)\ \&\ \langle \text{the\_left\_argument\_of}\, H{,}b\rangle \in A\ \&\ \langle \text{the\_right\_argument\_of}\, H{,}c\rangle \in A)$$
$$\&\ (H\ \text{is\_universal}$$
$$\mathbf{implies}\ \mathbf{ex}\, b{,}x\ \mathbf{st}\ x = \text{bound\_in}\, H\ \&\ a = \mathcal{J}(x,b)\ \&\ \langle \text{the\_scope\_of}\, H{,}b\rangle \in A)$$

for all values of the parameters.

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The scheme *ZFsch_uniq* deals with a binary functor $\mathcal{F}$, a binary functor $\mathcal{G}$, a unary functor $\mathcal{H}$, a binary functor $\mathcal{I}$, a binary functor $\mathcal{J}$, a constant $\mathcal{A}$ that has the type ZF-formula, a constant $\mathcal{B}$ and a constant $\mathcal{C}$ and states that the following holds

$$\mathcal{B} = \mathcal{C}$$

provided the parameters satisfy the following conditions:

- **ex** A **st** (**for** x,y **holds** $\langle x \texttt{=} y, \mathcal{F}(x,y)\rangle \in A$ & $\langle x \,\epsilon\, y, \mathcal{G}(x,y)\rangle \in A$) & $\langle \mathcal{A}, \mathcal{B}\rangle \in A$ & **for** H,a **st** $\langle H, a\rangle \in A$ **holds** (H is_a_equality **implies** $a = \mathcal{F}(\mathrm{Var}_1\, H, \mathrm{Var}_2\, H)$) & ($H$ is_a_membership **implies** $a = \mathcal{G}(\mathrm{Var}_1\, H, \mathrm{Var}_2\, H)$) & ($H$ is_negative **implies** **ex** b **st** $a = \mathcal{H}(b)$ & $\langle$the_argument_of H,$b\rangle \in A$) & (H is_conjunctive **implies** **ex** b,c **st** $a = \mathcal{I}(b,c)$ & $\langle$the_left_argument_of H,$b\rangle \in A$ & $\langle$the_right_argument_of H,$c\rangle \in A$) & (H is_universal **implies** **ex** b,x **st** x = bound_in H & $a = \mathcal{J}(x,b)$ & $\langle$the_scope_of H,$b\rangle \in A$),
- **ex** A **st** (**for** x,y **holds** $\langle x \texttt{=} y, \mathcal{F}(x,y)\rangle \in A$ & $\langle x \,\epsilon\, y, \mathcal{G}(x,y)\rangle \in A$) & $\langle \mathcal{A}, \mathcal{C}\rangle \in A$ & **for** H,a **st** $\langle H, a\rangle \in A$ **holds** (H is_a_equality **implies** $a = \mathcal{F}(\mathrm{Var}_1\, H, \mathrm{Var}_2\, H)$) & ($H$ is_a_membership **implies** $a = \mathcal{G}(\mathrm{Var}_1\, H, \mathrm{Var}_2\, H)$) & ($H$ is_negative **implies** **ex** b **st** $a = \mathcal{H}(b)$ & $\langle$the_argument_of H,$b\rangle \in A$) & (H is_conjunctive **implies** **ex** b,c **st** $a = \mathcal{I}(b,c)$ & $\langle$the_left_argument_of H,$b\rangle \in A$ & $\langle$the_right_argument_of H,$c\rangle \in A$) & (H is_universal **implies** **ex** b,x **st** x = bound_in H & $a = \mathcal{J}(x,b)$ & $\langle$the_scope_of H,$b\rangle \in A$).

The scheme *ZFsch_result* deals with a binary functor $\mathcal{F}$, a binary functor $\mathcal{G}$, a unary functor $\mathcal{H}$, a binary functor $\mathcal{I}$, a binary functor $\mathcal{J}$, a constant $\mathcal{A}$ that has the type ZF-formula and a unary functor $\mathcal{K}$ and states that the following holds

($\mathcal{A}$ is_a_equality **implies** $\mathcal{K}(\mathcal{A}) = \mathcal{F}(\mathrm{Var}_1\, \mathcal{A}, \mathrm{Var}_2\, \mathcal{A})$) & ($\mathcal{A}$ is_a_membership **implies** $\mathcal{K}(\mathcal{A}) = \mathcal{G}(\mathrm{Var}_1\, \mathcal{A}, \mathrm{Var}_2\, \mathcal{A})$) & ($\mathcal{A}$ is_negative **implies** $\mathcal{K}(\mathcal{A}) = \mathcal{H}(\mathcal{K}($the_argument_of $\mathcal{A}))$) & ($\mathcal{A}$ is_conjunctive **implies** **for** a,b **st** $a = \mathcal{K}($the_left_argument_of $\mathcal{A})$ & $b = \mathcal{K}($the_right_argument_of $\mathcal{A})$ **holds** $\mathcal{K}(\mathcal{A}) = \mathcal{I}(a,b)$) & ($\mathcal{A}$ is_universal **implies** $\mathcal{K}(\mathcal{A}) = \mathcal{J}($bound_in $\mathcal{A}$,$\mathcal{K}($the_scope_of $\mathcal{A})))$

provided the parameters satisfy the following condition:

- $$\textbf{for } H',a \textbf{ holds } a = \mathcal{K}(H') \textbf{ iff ex } A \textbf{ st}$$
  $$(\textbf{for } x,y \textbf{ holds } \langle x = y, \mathcal{F}(x,y)\rangle \in A \mathrel{\&} \langle x \epsilon y, \mathcal{G}(x,y)\rangle \in A) \mathrel{\&} \langle H',a\rangle \in A \mathrel{\&}$$
  $$\textbf{for } H,a \textbf{ st } \langle H,a\rangle \in A \textbf{ holds } (H \text{ is\_a\_equality } \textbf{implies } a = \mathcal{F}(\mathrm{Var}_1\, H, \mathrm{Var}_2\, H))$$
  $$\mathrel{\&} (H \text{ is\_a\_membership } \textbf{implies } a = \mathcal{G}(\mathrm{Var}_1\, H, \mathrm{Var}_2\, H)) \mathrel{\&}$$
  $$(H \text{ is\_negative } \textbf{implies ex } b \textbf{ st } a = \mathcal{H}(b) \mathrel{\&} \langle \text{the\_argument\_of } H, b\rangle \in A) \mathrel{\&}$$
  $$(H \text{ is\_conjunctive } \textbf{implies ex } b,c \textbf{ st } a = \mathcal{I}(b,c)$$
  $$\mathrel{\&} \langle \text{the\_left\_argument\_of } H, b\rangle \in A \mathrel{\&} \langle \text{the\_right\_argument\_of } H, c\rangle \in A)$$
  $$\mathrel{\&} (H \text{ is\_universal}$$
  $$\textbf{implies ex } b,x \textbf{ st } x = \text{bound\_in } H \mathrel{\&} a = \mathcal{J}(x,b) \mathrel{\&} \langle \text{the\_scope\_of } H, b\rangle \in A).$$

The scheme *ZFsch_property* concerns a binary functor $\mathcal{F}$, a binary functor $\mathcal{G}$, a unary functor $\mathcal{H}$, a binary functor $\mathcal{I}$, a binary functor $\mathcal{J}$, a unary functor $\mathcal{K}$, a constant $\mathcal{A}$ that has the type ZF-formula and a unary predicate $\mathcal{P}$ and states that the following holds

$$\mathcal{P}[\mathcal{K}(\mathcal{A})]$$

provided the parameters satisfy the following conditions:

- $$\textbf{for } H',a \textbf{ holds } a = \mathcal{K}(H') \textbf{ iff ex } A \textbf{ st}$$
  $$(\textbf{for } x,y \textbf{ holds } \langle x = y, \mathcal{F}(x,y)\rangle \in A \mathrel{\&} \langle x \epsilon y, \mathcal{G}(x,y)\rangle \in A) \mathrel{\&} \langle H',a\rangle \in A \mathrel{\&}$$
  $$\textbf{for } H,a \textbf{ st } \langle H,a\rangle \in A \textbf{ holds } (H \text{ is\_a\_equality } \textbf{implies } a = \mathcal{F}(\mathrm{Var}_1\, H, \mathrm{Var}_2\, H))$$
  $$\mathrel{\&} (H \text{ is\_a\_membership } \textbf{implies } a = \mathcal{G}(\mathrm{Var}_1\, H, \mathrm{Var}_2\, H)) \mathrel{\&}$$
  $$(H \text{ is\_negative } \textbf{implies ex } b \textbf{ st } a = \mathcal{H}(b) \mathrel{\&} \langle \text{the\_argument\_of } H, b\rangle \in A) \mathrel{\&}$$
  $$(H \text{ is\_conjunctive } \textbf{implies ex } b,c \textbf{ st } a = \mathcal{I}(b,c)$$
  $$\mathrel{\&} \langle \text{the\_left\_argument\_of } H, b\rangle \in A \mathrel{\&} \langle \text{the\_right\_argument\_of } H, c\rangle \in A)$$
  $$\mathrel{\&} (H \text{ is\_universal}$$
  $$\textbf{implies ex } b,x \textbf{ st } x = \text{bound\_in } H \mathrel{\&} a = \mathcal{J}(x,b) \mathrel{\&} \langle \text{the\_scope\_of } H, b\rangle \in A),$$
- $$\textbf{for } x,y \textbf{ holds } \mathcal{P}[\mathcal{F}(x,y)] \mathrel{\&} \mathcal{P}[\mathcal{G}(x,y)],$$
- $$\textbf{for } a \textbf{ st } \mathcal{P}[a] \textbf{ holds } \mathcal{P}[\mathcal{H}(a)],$$
- $$\textbf{for } a,b \textbf{ st } \mathcal{P}[a] \mathrel{\&} \mathcal{P}[b] \textbf{ holds } \mathcal{P}[\mathcal{I}(a,b)],$$
- $$\textbf{for } a,x \textbf{ st } \mathcal{P}[a] \textbf{ holds } \mathcal{P}[\mathcal{J}(x,a)].$$

Let us consider H. The functor

$$\mathrm{Free}\, H,$$

yields the type Any and is defined by

$$\begin{gathered}\mathbf{ex}\,A\ \mathbf{st}\ (\mathbf{for}\,x,y\ \mathbf{holds}\ \langle x = y,\{x,y\}\rangle \in A\ \&\ \langle x \epsilon\, y,\{x,y\}\rangle \in A)\ \&\ \langle H,\mathbf{it}\rangle \in A\ \&\\ \mathbf{for}\,H',a\ \mathbf{st}\ \langle H',a\rangle \in A\ \mathbf{holds}\ (H'\ \text{is\_a\_equality}\ \mathbf{implies}\ a = \{\mathrm{Var}_1\,H',\mathrm{Var}_2\,H'\})\ \&\\ (H'\ \text{is\_a\_membership}\ \mathbf{implies}\ a = \{\mathrm{Var}_1\,H',\mathrm{Var}_2\,H'\})\ \&\\ (H'\ \text{is\_negative}\ \mathbf{implies}\ \mathbf{ex}\,b\ \mathbf{st}\ a = b\ \&\ \langle \text{the\_argument\_of}\,H',b\rangle \in A)\ \&\\ (H'\ \text{is\_conjunctive}\ \mathbf{implies}\ \mathbf{ex}\,b,c\\ \mathbf{st}\ a = \bigcup\{b,c\}\ \&\ \langle \text{the\_left\_argument\_of}\,H',b\rangle \in A\ \&\ \langle \text{the\_right\_argument\_of}\,H',c\rangle \in A)\\ \&\ (H'\ \text{is\_universal}\\ \mathbf{implies}\ \mathbf{ex}\,b,x\ \mathbf{st}\ x = \text{bound\_in}\,H'\ \&\ a = (\bigcup\{b\}) \setminus \{x\}\ \&\ \langle \text{the\_scope\_of}\,H',b\rangle \in A).\end{gathered}$$

Let us consider H. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\mathrm{Free}\,H \qquad \text{is} \qquad \mathbf{set\ of}\ \mathrm{Variable}\,.$$

One can prove the following proposition

$$\begin{gathered}\mathbf{for}\,H\ \mathbf{holds}\ (H\ \text{is\_a\_equality}\ \mathbf{implies}\ \mathrm{Free}\,H = \{\mathrm{Var}_1\,H,\mathrm{Var}_2\,H\})\ \&\\ (H\ \text{is\_a\_membership}\ \mathbf{implies}\ \mathrm{Free}\,H = \{\mathrm{Var}_1\,H,\mathrm{Var}_2\,H\})\ \&\\ (H\ \text{is\_negative}\ \mathbf{implies}\ \mathrm{Free}\,H = \mathrm{Free}\,\text{the\_argument\_of}\,H)\ \&\\ (H\ \text{is\_conjunctive}\ \mathbf{implies}\\ \mathrm{Free}\,H = \mathrm{Free}\,\text{the\_left\_argument\_of}\,H \cup \mathrm{Free}\,\text{the\_right\_argument\_of}\,H)\\ \&\ (H\ \text{is\_universal}\ \mathbf{implies}\ \mathrm{Free}\,H = (\mathrm{Free}\,\text{the\_scope\_of}\,H) \setminus \{\text{bound\_in}\,H\}).\end{gathered} \tag{1}$$

Let D have the type SET_DOMAIN. The functor

$$\mathrm{VAL}\,D,$$

with values of the type DOMAIN, is defined by

$$a \in \mathbf{it}\ \mathbf{iff}\ a\ \mathbf{is}\ \mathrm{Function}\ \mathbf{of}\ \mathrm{VAR}\,,D.$$

The arguments of the notions defined below are the following: $D1$ which is an object of the type SET_DOMAIN; f which is an object of the type Function **of** VAR, $D1$; x which is an object of the type reserved above. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$f\,.x \qquad \text{is} \qquad \mathrm{Element}\ \mathbf{of}\ D1.$$

For simplicity we adopt the following convention: E will denote an object of the type SET_DOMAIN; f, g will denote objects of the type Function **of** VAR, E; $v1$, $v2$, $v3$, $v4$, $v5$ will denote objects of the type Element **of** VAL E. Let us consider H, E. The functor

$$\mathrm{St}\,(H,E),$$

yields the type Any and is defined by

$$
\begin{gathered}
\textbf{ex}\, A\ \textbf{st} \\
(\textbf{for}\ x{,}y\ \textbf{holds}\ \langle x = y{,}\{\, v1 : \textbf{for}\ f\ \textbf{st}\ f = v1\ \textbf{holds}\ f\,.x = f\,.y\,\}\rangle \in A \\
\&\ \langle x \in y{,}\{\, v2 : \textbf{for}\ f\ \textbf{st}\ f = v2\ \textbf{holds}\ f\,.x \in f\,.y\,\}\rangle \in A) \\
\&\ \langle H, \textbf{it}\rangle \in A\ \&\ \textbf{for}\ H'{,}a\ \textbf{st}\ \langle H'{,}a\rangle \in A\ \textbf{holds} \\
(H'\ \text{is\_a\_equality} \\
\textbf{implies}\ a = \{\, v3 : \textbf{for}\ f\ \textbf{st}\ f = v3\ \textbf{holds}\ f\,.(\mathrm{Var}_1\, H') = f\,.(\mathrm{Var}_2\, H')\,\}) \\
\& \\
(H'\ \text{is\_a\_membership} \\
\textbf{implies}\ a = \{\, v4 : \textbf{for}\ f\ \textbf{st}\ f = v4\ \textbf{holds}\ f\,.(\mathrm{Var}_1\, H') \in f\,.(\mathrm{Var}_2\, H')\,\}) \\
\&\ (H'\ \text{is\_negative}\ \textbf{implies}\ \textbf{ex}\, b\ \textbf{st}\ a = (\mathrm{VAL}\, E) \setminus \bigcup\{b\}\ \&\ \langle \text{the\_argument\_of}\, H'{,}b\rangle \in A) \\
\& \\
(H'\ \text{is\_conjunctive}\ \textbf{implies}\ \textbf{ex}\, b{,}c\ \textbf{st}\ a = (\bigcup\{b\}) \cap \bigcup\{c\} \\
\&\ \langle \text{the\_left\_argument\_of}\, H'{,}b\rangle \in A\ \&\ \langle \text{the\_right\_argument\_of}\, H'{,}c\rangle \in A) \\
\&\ (H'\ \text{is\_universal}\ \textbf{implies}\ \textbf{ex}\, b{,}x\ \textbf{st}\ x = \text{bound\_in}\, H'\ \& \\
a = \{\, v5 : \\
\textbf{for}\ X{,}f\ \textbf{st}\ X = b\ \&\ f = v5 \\
\textbf{holds}\ f \in X\ \&\ \textbf{for}\ g\ \textbf{st}\ \textbf{for}\ y\ \textbf{st}\ g\,.y \neq f\,.y\ \textbf{holds}\ x = y\ \textbf{holds}\ g \in X\,\} \\
\&\ \langle \text{the\_scope\_of}\, H'{,}b\rangle \in A).
\end{gathered}
$$

Let us consider H, E. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\mathrm{St}\,(H, E) \qquad \text{is} \qquad \text{Subset}\ \textbf{of}\ \mathrm{VAL}\, E.$$

We now state a number of propositions:

$$\textbf{for}\ x{,}y{,}f\ \textbf{holds}\ f\,.x = f\,.y\ \textbf{iff}\ f \in \mathrm{St}\,(x = y{,}E), \tag{2}$$

$$\textbf{for}\ x{,}y{,}f\ \textbf{holds}\ f\,.x \in f\,.y\ \textbf{iff}\ f \in \mathrm{St}\,(x \in y{,}E), \tag{3}$$

$$\textbf{for}\ H{,}f\ \textbf{holds}\ \textbf{not}\ f \in \mathrm{St}\,(H, E)\ \textbf{iff}\ f \in \mathrm{St}\,(\neg\, H{,}E), \tag{4}$$

$$\textbf{for}\ H{,}H'{,}f\ \textbf{holds}\ f \in \mathrm{St}\,(H, E)\ \&\ f \in \mathrm{St}\,(H'{,}E)\ \textbf{iff}\ f \in \mathrm{St}\,(H \wedge H'{,}E), \tag{5}$$

$$
\begin{gathered}
\textbf{for}\ x{,}H{,}f\ \textbf{holds} \\
f \in \mathrm{St}\,(H, E)\ \&\ (\textbf{for}\ g\ \textbf{st}\ \textbf{for}\ y\ \textbf{st}\ g\,.y \neq f\,.y\ \textbf{holds}\ x = y\ \textbf{holds}\ g \in \mathrm{St}\,(H, E)) \\
\textbf{iff}\ f \in \mathrm{St}\,(\forall\,(x, H){,}E),
\end{gathered} \tag{6}
$$

$$
\begin{gathered}
H\ \text{is\_a\_equality} \\
\textbf{implies}\ \textbf{for}\ f\ \textbf{holds}\ f\,.(\mathrm{Var}_1\, H) = f\,.(\mathrm{Var}_2\, H)\ \textbf{iff}\ f \in \mathrm{St}\,(H, E),
\end{gathered} \tag{7}
$$

$$H \text{ is\_a\_membership} \quad \textbf{implies for } f \textbf{ holds } f\,.(\mathrm{Var}_1\,H) \in f\,.(\mathrm{Var}_2\,H) \textbf{ iff } f \in \mathrm{St}\,(H,E), \tag{8}$$

$$H \text{ is\_negative} \quad \textbf{implies for } f \textbf{ holds not } f \in \mathrm{St}\,(\text{the\_argument\_of}\,H{,}E) \textbf{ iff } f \in \mathrm{St}\,(H,E), \tag{9}$$

$$H \text{ is\_conjunctive } \textbf{implies for } f \textbf{ holds } f \in \mathrm{St}\,(\text{the\_left\_argument\_of}\,H{,}E) \;\&\; f \in \mathrm{St}\,(\text{the\_right\_argument\_of}\,H{,}E) \textbf{ iff } f \in \mathrm{St}\,(H,E), \tag{10}$$

$$H \text{ is\_universal } \textbf{implies for } f \textbf{ holds } f \in \mathrm{St}\,(\text{the\_scope\_of}\,H{,}E) \;\&\; (\textbf{for } g \textbf{ st for } y \textbf{ st } g.y \neq f\,.y \textbf{ holds } \text{bound\_in}\,H = y \textbf{ holds } g \in \mathrm{St}\,(\text{the\_scope\_of}\,H{,}E)) \textbf{ iff } f \in \mathrm{St}\,(H,E). \tag{11}$$

The arguments of the notions defined below are the following: D which is an object of the type SET_DOMAIN; f which is an object of the type Function **of** VAR, D; H which is an object of the type reserved above. The predicate

$$D, f \models H \qquad \text{is defined by} \qquad f \in \mathrm{St}\,(H,D).$$

Next we state a number of propositions:

$$\textbf{for } E{,}f{,}x{,}y \textbf{ holds } E, f \models x = y \textbf{ iff } f\,.x = f\,.y, \tag{12}$$

$$\textbf{for } E{,}f{,}x{,}y \textbf{ holds } E, f \models x \,\epsilon\, y \textbf{ iff } f\,.x \in f\,.y, \tag{13}$$

$$\textbf{for } E{,}f{,}H \textbf{ holds } E, f \models H \textbf{ iff not } E, f \models \neg H, \tag{14}$$

$$\textbf{for } E{,}f{,}H{,}H' \textbf{ holds } E, f \models H \wedge H' \textbf{ iff } E, f \models H \;\&\; E, f \models H', \tag{15}$$

$$\textbf{for } E{,}f{,}H{,}x \textbf{ holds } E, f \models \forall\,(x,H) \textbf{ iff for } g \textbf{ st for } y \textbf{ st } g.y \neq f\,.y \textbf{ holds } x = y \textbf{ holds } E, g \models H, \tag{16}$$

$$\textbf{for } E{,}f{,}H{,}H' \textbf{ holds } E, f \models H \vee H' \textbf{ iff } E, f \models H \textbf{ or } E, f \models H', \tag{17}$$

$$\textbf{for } E{,}f{,}H{,}H' \textbf{ holds } E, f \models H \Rightarrow H' \textbf{ iff } (E, f \models H \textbf{ implies } E, f \models H'), \tag{18}$$

$$\textbf{for } E{,}f{,}H{,}H' \textbf{ holds } E, f \models H \Leftrightarrow H' \textbf{ iff } (E, f \models H \textbf{ iff } E, f \models H'), \tag{19}$$

$$\textbf{for } E{,}f{,}H{,}x \textbf{ holds } E, f \models \exists\,(x,H) \textbf{ iff ex } g \textbf{ st } (\textbf{for } y \textbf{ st } g.y \neq f\,.y \textbf{ holds } x = y) \;\&\; E, g \models H, \tag{20}$$

$$\textbf{for } E{,}f{,}x \;\; \textbf{for } e \textbf{ being } \text{Element } \textbf{of } E \textbf{ ex } g \textbf{ st } g.x = e \;\&\; \textbf{for } z \textbf{ st } z \neq x \textbf{ holds } g.z = f\,.z, \tag{21}$$

(22) $$E, f \models \forall(x, y, H)$$
$$\textbf{iff for } g \textbf{ st for } z \textbf{ st } g.z \neq f.z \textbf{ holds } x = z \textbf{ or } y = z \textbf{ holds } E, g \models H,$$

(23) $$E, f \models \exists(x, y, H)$$
$$\textbf{iff ex } g \textbf{ st } (\textbf{for } z \textbf{ st } g.z \neq f.z \textbf{ holds } x = z \textbf{ or } y = z) \;\&\; E, g \models H.$$

Let us consider E, H. The predicate

$$E \models H \qquad \text{is defined by} \qquad \textbf{for } f \textbf{ holds } E, f \models H.$$

One can prove the following propositions:

(24) $$E \models H \textbf{ iff for } f \textbf{ holds } E, f \models H,$$

(25) $$E \models \forall(x, H) \textbf{ iff } E \models H.$$

We now define five new functors. The constant the_axiom_of_extensionality has the type ZF-formula, and is defined by

$$\textbf{it} = \forall(\xi\,0,\xi\,1,\forall(\xi\,2,\xi\,2 \in \xi\,0 \Leftrightarrow \xi\,2 \in \xi\,1) \Rightarrow \xi\,0 = \xi\,1).$$

The constant the_axiom_of_pairs has the type ZF-formula, and is defined by

$$\textbf{it} = \forall(\xi\,0,\xi\,1,\exists(\xi\,2,\forall(\xi\,3,\xi\,3 \in \xi\,2 \Leftrightarrow (\xi\,3 = \xi\,0 \lor \xi\,3 = \xi\,1)))).$$

The constant the_axiom_of_unions has the type ZF-formula, and is defined by

$$\textbf{it} = \forall(\xi\,0,\exists(\xi\,1,\forall(\xi\,2,\xi\,2 \in \xi\,1 \Leftrightarrow \exists(\xi\,3,\xi\,2 \in \xi\,3 \land \xi\,3 \in \xi\,0)))).$$

The constant the_axiom_of_infinity has the type ZF-formula, and is defined by

$$\textbf{it} = \exists(\xi\,0,$$
$$\xi\,1, \xi\,1 \in \xi\,0 \land \forall(\xi\,2,\xi\,2 \in \xi\,0 \Rightarrow \exists(\xi\,3,\xi\,3 \in \xi\,0 \land \neg\,\xi\,3 = \xi\,2 \land \forall(\xi\,4,\xi\,4 \in \xi\,2 \Rightarrow \xi\,4 \in \xi\,3)))).$$

The constant the_axiom_of_power_sets has the type ZF-formula, and is defined by

$$\textbf{it} = \forall(\xi\,0,\exists(\xi\,1,\forall(\xi\,2,\xi\,2 \in \xi\,1 \Leftrightarrow \forall(\xi\,3,\xi\,3 \in \xi\,2 \Rightarrow \xi\,3 \in \xi\,0)))).$$

Let H have the type ZF-formula. Assume that the following holds

$$\{\xi\,0,\xi\,1,\xi\,2\} \text{ misses Free}\, H.$$

The functor

$$\text{the\_axiom\_of\_substitution\_for}\, H,$$

with values of the type ZF-formula, is defined by

$$\textbf{it} =$$
$$\forall(\xi\,3,\exists(\xi\,0,\forall(\xi\,4,H \Leftrightarrow \xi\,4 = \xi\,0))) \Rightarrow \forall(\xi\,1,\exists(\xi\,2,\forall(\xi\,4,\xi\,4 \in \xi\,2 \Leftrightarrow \exists(\xi\,3,\xi\,3 \in \xi\,1 \land H)))).$$

We now state several propositions:

(26) $$\text{the\_axiom\_of\_extensionality} = \forall(\xi\,0,\xi\,1,\forall(\xi\,2,\xi\,2\,\epsilon\,\xi\,0 \Leftrightarrow \xi\,2\,\epsilon\,\xi\,1) \Rightarrow \xi\,0 \mathrel{=} \xi\,1),$$

(27) $$\text{the\_axiom\_of\_pairs} = \forall(\xi\,0,\xi\,1,\exists(\xi\,2,\forall(\xi\,3,\xi\,3\,\epsilon\,\xi\,2 \Leftrightarrow (\xi\,3 \mathrel{=} \xi\,0 \vee \xi\,3 \mathrel{=} \xi\,1)))),$$

(28) $$\text{the\_axiom\_of\_unions} = \forall(\xi\,0,\exists(\xi\,1,\forall(\xi\,2,\xi\,2\,\epsilon\,\xi\,1 \Leftrightarrow \exists(\xi\,3,\xi\,2\,\epsilon\,\xi\,3 \wedge \xi\,3\,\epsilon\,\xi\,0)))),$$

(29) $$\text{the\_axiom\_of\_infinity} = \exists(\xi\,0,\xi\,1,\xi\,1\,\epsilon\,\xi\,0 \wedge \forall(\xi\,2, \xi\,2\,\epsilon\,\xi\,0 \Rightarrow \exists(\xi\,3,\xi\,3\,\epsilon\,\xi\,0 \wedge \neg\,\xi\,3 \mathrel{=} \xi\,2 \wedge \forall(\xi\,4,\xi\,4\,\epsilon\,\xi\,2 \Rightarrow \xi\,4\,\epsilon\,\xi\,3)))),$$

(30) $$\text{the\_axiom\_of\_power\_sets} = \forall(\xi\,0,\exists(\xi\,1,\forall(\xi\,2,\xi\,2\,\epsilon\,\xi\,1 \Leftrightarrow \forall(\xi\,3,\xi\,3\,\epsilon\,\xi\,2 \Rightarrow \xi\,3\,\epsilon\,\xi\,0)))),$$

(31) $$\{\xi\,0,\xi\,1,\xi\,2\} \text{ misses Free}\,H \textbf{ implies } \text{the\_axiom\_of\_substitution\_for}\,H = \forall(\xi\,3,\exists(\xi\,0, \forall(\xi\,4,H \Leftrightarrow \xi\,4 \mathrel{=} \xi\,0))) \Rightarrow \forall(\xi\,1,\exists(\xi\,2,\forall(\xi\,4,\xi\,4\,\epsilon\,\xi\,2 \Leftrightarrow \exists(\xi\,3,\xi\,3\,\epsilon\,\xi\,1 \wedge H)))).$$

Let us consider E. The predicate

$$E \text{ is\_a\_model\_of\_ZF}$$

is defined by

$$E \text{ is\_}\in\text{-transitive} \,\&\, E \models \text{the\_axiom\_of\_pairs} \,\&\, E \models \text{the\_axiom\_of\_unions} \,\&\, E \models \text{the\_axiom\_of\_infinity} \,\&\, E \models \text{the\_axiom\_of\_power\_sets} \,\&\, \textbf{for}\ H\ \textbf{st}\ \{\xi\,0,\xi\,1,\xi\,2\} \text{ misses Free}\,H\ \textbf{holds}\ E \models \text{the\_axiom\_of\_substitution\_for}\,H.$$

The following proposition is true

(32) $$E \text{ is\_a\_model\_of\_ZF} \textbf{ iff } E \text{ is\_}\in\text{-transitive} \,\&\, E \models \text{the\_axiom\_of\_pairs} \,\&\, E \models \text{the\_axiom\_of\_unions} \,\&\, E \models \text{the\_axiom\_of\_infinity} \,\&\, E \models \text{the\_axiom\_of\_power\_sets} \,\&\, \textbf{for}\ H\ \textbf{st}\ \{\xi\,0,\xi\,1,\xi\,2\} \text{ misses Free}\,H\ \textbf{holds}\ E \models \text{the\_axiom\_of\_substitution\_for}\,H.$$

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The Contraction Lemma

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Summary. The article includes the proof of the contraction lemma which claims that every class in which the axiom of extensionality is valid is isomorphic with a transitive class. In this article the isomorphism (wrt membership relation) of two sets is defined. It is based on [6].

The articles [7], [8], [4], [1], [5], [3], and [2] provide the terminology and notation for this paper. For simplicity we adopt the following convention: X, Y, Z denote objects of the type set; x, y denote objects of the type Any; E denotes an object of the type SET_DOMAIN; A, B, C denote objects of the type Ordinal; L denotes an object of the type Transfinite-Sequence; f denotes an object of the type Function; d, $d1$, d' denote objects of the type Element **of** E. Let us consider E, A. The functor

$$\mathrm{M}_{\mu}\,(E,A),$$

with values of the type set, is defined by

$$\begin{gathered}\mathbf{ex}\,L\ \mathbf{st}\ \mathbf{it}=\{\,d:\mathbf{for}\,d1\ \mathbf{st}\ d1\in d\,\mathbf{ex}\,B\ \mathbf{st}\ B\in\mathrm{dom}\,L\ \&\ d1\in\bigcup\{L.B\}\,\}\ \&\ \mathrm{dom}\,L=A\\ \&\ \mathbf{for}\,B\ \mathbf{st}\ B\in A\\ \mathbf{holds}\ L.B=\{\,d1:\mathbf{for}\,d\ \mathbf{st}\ d\in d1\,\mathbf{ex}\,C\ \mathbf{st}\ C\in\mathrm{dom}\,(L\restriction B)\ \&\ d\in\bigcup\{L\restriction B.C\}\,\}.\end{gathered}$$

One can prove the following propositions:

(1) $$\mathrm{M}_{\mu}\,(E,A)=\{\,d:\mathbf{for}\,d1\ \mathbf{st}\ d1\in d\,\mathbf{ex}\,B\ \mathbf{st}\ B\in A\ \&\ d1\in\mathrm{M}_{\mu}\,(E,B)\,\},$$

(2) $$\mathbf{not}\,(\mathbf{ex}\,d1\ \mathbf{st}\ d1\in d)\ \mathbf{iff}\ d\in\mathrm{M}_{\mu}\,(E,\mathbf{0}),$$

(3) $$d\cap E\subseteq\mathrm{M}_{\mu}\,(E,A)\ \mathbf{iff}\ d\in\mathrm{M}_{\mu}\,(E,\mathrm{succ}\,A),$$

(4) $$A\subseteq B\ \mathbf{implies}\ \mathrm{M}_{\mu}\,(E,A)\subseteq\mathrm{M}_{\mu}\,(E,B),$$

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(5) $$\mathbf{ex}\, A\ \mathbf{st}\ d \in \mathrm{M}_\mu\,(E, A),$$

(6) $$d' \in d\ \&\ d \in \mathrm{M}_\mu\,(E, A)$$
$$\mathbf{implies}\ d' \in \mathrm{M}_\mu\,(E, A)\ \&\ \mathbf{ex}\, B\ \mathbf{st}\ B \in A\ \&\ d' \in \mathrm{M}_\mu\,(E, B),$$

(7) $$\mathrm{M}_\mu\,(E, A) \subseteq E,$$

(8) $$\mathbf{ex}\, A\ \mathbf{st}\ E = \mathrm{M}_\mu\,(E, A),$$

(9) $$\mathbf{ex}\, f\ \mathbf{st}\ \mathrm{dom}\, f = E\ \&\ \mathbf{for}\, d\ \mathbf{holds}\ f\,.d = f\ ^\circ\ d.$$

Let us consider f, X, Y. The predicate

$$f \text{ is\_}{\in}\text{-isomorphism\_of } X, Y$$

is defined by

$$\mathrm{dom}\, f = X\ \&\ \mathrm{rng}\, f = Y\ \&\ f \text{ is\_one-to-one}\ \&\ \mathbf{for}\, x{,}y$$
$$\mathbf{st}\ x \in X\ \&\ y \in X\ \mathbf{holds}\ (\mathbf{ex}\, Z\ \mathbf{st}\ Z = y\ \&\ x \in Z)\ \mathbf{iff}\ \mathbf{ex}\, Z\ \mathbf{st}\ f\,.y = Z\ \&\ f\,.x \in Z.$$

Next we state a proposition

(10) $$f \text{ is\_}{\in}\text{-isomorphism\_of } X, Y\ \mathbf{iff}\ \mathrm{dom}\, f = X\ \&\ \mathrm{rng}\, f = Y\ \&\ f \text{ is\_one-to-one}\ \&$$
$$\mathbf{for}\, x{,}y\ \mathbf{st}\ x \in X\ \&\ y \in X$$
$$\mathbf{holds}\ (\mathbf{ex}\, Z\ \mathbf{st}\ Z = y\ \&\ x \in Z)\ \mathbf{iff}\ \mathbf{ex}\, Z\ \mathbf{st}\ f\,.y = Z\ \&\ f\,.x \in Z.$$

Let us consider X, Y. The predicate

X, Y are_$\in$-isomorphic is defined by $\mathbf{ex}\, f\ \mathbf{st}\ f$ is_$\in$-isomorphism_of X, Y.

Next we state two propositions:

(11) $$X, Y \text{ are\_}{\in}\text{-isomorphic}\ \mathbf{iff}\ \mathbf{ex}\, f\ \mathbf{st}\ f \text{ is\_}{\in}\text{-isomorphism\_of } X, Y,$$

(12) $$\mathrm{dom}\, f = E\ \&\ (\mathbf{for}\, d\ \mathbf{holds}\ f\,.d = f\ ^\circ\ d)\ \mathbf{implies}\ \mathrm{rng}\, f \text{ is\_}{\in}\text{-transitive}\,.$$

In the sequel u, v, w will denote objects of the type Element **of** E. Next we state two propositions:

(13) $$E \models \text{the\_axiom\_of\_extensionality}$$
$$\mathbf{implies}\ \mathbf{for}\, u{,}v\ \mathbf{st}\ \mathbf{for}\, w\ \mathbf{holds}\ w \in u\ \mathbf{iff}\ w \in v\ \mathbf{holds}\ u = v,$$

(14) $$E \models \text{the\_axiom\_of\_extensionality}$$
$$\mathbf{implies}\ \mathbf{ex}\, X\ \mathbf{st}\ X \text{ is\_}{\in}\text{-transitive}\ \&\ E, X \text{ are\_}{\in}\text{-isomorphic}\,.$$

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Axioms of Incidence

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Summary. This article is based on *"Foundations of Geometry"* by Karol Borsuk and Wanda Szmielew ([1]). The fourth axiom of incidence is weakened. In [1] it has the form *for any plane there exist three non-collinear points in the plane* and in the article *for any plane there exists one point in the plane.* The original axiom is proved. The article includes: theorems concerning collinearity of points and coplanarity of points and lines, basic theorems concerning lines and planes, fundamental existence theorems, theorems concerning intersection of lines and planes.

The articles [3], [2], and [4] provide the terminology and notation for this paper. We consider structures IncStruct, which are systems

$$\langle\!\langle \mathrm{Points}, \mathrm{Lines}, \mathrm{Planes}, \mathrm{Inc}_1, \mathrm{Inc}_2, \mathrm{Inc}_3 \rangle\!\rangle$$

where Points, Lines, Planes have the type DOMAIN, Inc_1 has the type Relation **of the** Points, **the** Lines, Inc_2 has the type Relation **of the** Points, **the** Planes, and Inc_3 has the type Relation **of the** Lines, **the** Planes. We now define three new modes. Let S have the type IncStruct.

POINT **of** S stands for Element **of the** Points **of** S.

LINE **of** S stands for Element **of the** Lines **of** S.

PLANE **of** S stands for Element **of the** Planes **of** S.

In the sequel S will have the type IncStruct; A will have the type Element **of the** Points **of** S; L will have the type Element **of the** Lines **of** S; P will have the type Element **of the** Planes **of** S. The following propositions are true:

(1) A **is** POINT **of** S,

(2) L **is** LINE **of** S,

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$$P \textbf{ is } \mathrm{PLANE} \textbf{ of } S. \tag{3}$$

For simplicity we adopt the following convention: A, B, C, D will denote objects of the type POINT **of** S; L will denote an object of the type LINE **of** S; P will denote an object of the type PLANE **of** S; F, G will denote objects of the type Subset **of the** Points **of** S. The arguments of the notions defined below are the following: S which is an object of the type reserved above; A which is an object of the type POINT **of** S; L which is an object of the type LINE **of** S. The predicate

$$A \text{ on } L \qquad \text{is defined by} \qquad \langle A, L\rangle \in \textbf{the}\,\mathrm{Inc}_1 \textbf{ of } S.$$

The arguments of the notions defined below are the following: S which is an object of the type reserved above; A which is an object of the type POINT **of** S; P which is an object of the type PLANE **of** S. The predicate

$$A \text{ on } P \qquad \text{is defined by} \qquad \langle A, P\rangle \in \textbf{the}\,\mathrm{Inc}_2 \textbf{ of } S.$$

The arguments of the notions defined below are the following: S which is an object of the type reserved above; L which is an object of the type LINE **of** S; P which is an object of the type PLANE **of** S. The predicate

$$L \text{ on } P \qquad \text{is defined by} \qquad \langle L, P\rangle \in \textbf{the}\,\mathrm{Inc}_3 \textbf{ of } S.$$

The arguments of the notions defined below are the following: S which is an object of the type reserved above; F which is an object of the type **set of** POINT **of** S; L which is an object of the type LINE **of** S. The predicate

$$F \text{ on } L \qquad \text{is defined by} \qquad \textbf{for } A \textbf{ being } \mathrm{POINT} \textbf{ of } S \textbf{ st } A \in F \textbf{ holds } A \text{ on } L.$$

The arguments of the notions defined below are the following: S which is an object of the type reserved above; F which is an object of the type **set of** POINT **of** S; P which is an object of the type PLANE **of** S. The predicate

$$F \text{ on } P \qquad \text{is defined by} \qquad \textbf{for } A \textbf{ st } A \in F \textbf{ holds } A \text{ on } P.$$

The arguments of the notions defined below are the following: S which is an object of the type reserved above; F which is an object of the type **set of** POINT **of** S. The predicate

$$F \text{ is\_linear} \qquad \text{is defined by} \qquad \textbf{ex}\, L \textbf{ st } F \text{ on } L.$$

The arguments of the notions defined below are the following: S which is an object of the type reserved above; F which is an object of the type **set of** POINT **of** S. The predicate

$$F \text{ is\_planar} \qquad \text{is defined by} \qquad \textbf{ex}\, P \textbf{ st } F \text{ on } P.$$

Next we state a number of propositions:

$$A \text{ on } L \textbf{ iff } \langle A, L\rangle \in \textbf{the}\,\mathrm{Inc}_1 \textbf{ of } S, \tag{4}$$

$$A \text{ on } P \textbf{ iff } \langle A, P\rangle \in \textbf{the} \operatorname{Inc}_2 \textbf{ of } S, \tag{5}$$

$$L \text{ on } P \textbf{ iff } \langle L, P\rangle \in \textbf{the} \operatorname{Inc}_3 \textbf{ of } S, \tag{6}$$

$$F \text{ on } L \textbf{ iff for } A \textbf{ st } A \in F \textbf{ holds } A \text{ on } L, \tag{7}$$

$$F \text{ on } P \textbf{ iff for } A \textbf{ st } A \in F \textbf{ holds } A \text{ on } P, \tag{8}$$

$$F \text{ is\_linear } \textbf{iff ex } L \textbf{ st } F \text{ on } L, \tag{9}$$

$$F \text{ is\_planar } \textbf{iff ex } P \textbf{ st } F \text{ on } P, \tag{10}$$

$$\{A, B\} \text{ on } L \textbf{ iff } A \text{ on } L \mathbin{\&} B \text{ on } L, \tag{11}$$

$$\{A, B, C\} \text{ on } L \textbf{ iff } A \text{ on } L \mathbin{\&} B \text{ on } L \mathbin{\&} C \text{ on } L, \tag{12}$$

$$\{A, B\} \text{ on } P \textbf{ iff } A \text{ on } P \mathbin{\&} B \text{ on } P, \tag{13}$$

$$\{A, B, C\} \text{ on } P \textbf{ iff } A \text{ on } P \mathbin{\&} B \text{ on } P \mathbin{\&} C \text{ on } P, \tag{14}$$

$$\{A, B, C, D\} \text{ on } P \textbf{ iff } A \text{ on } P \mathbin{\&} B \text{ on } P \mathbin{\&} C \text{ on } P \mathbin{\&} D \text{ on } P, \tag{15}$$

$$G \subseteq F \mathbin{\&} F \text{ on } L \textbf{ implies } G \text{ on } L, \tag{16}$$

$$G \subseteq F \mathbin{\&} F \text{ on } P \textbf{ implies } G \text{ on } P, \tag{17}$$

$$F \text{ on } L \mathbin{\&} A \text{ on } L \textbf{ iff } F \cup \{A\} \text{ on } L, \tag{18}$$

$$F \text{ on } P \mathbin{\&} A \text{ on } P \textbf{ iff } F \cup \{A\} \text{ on } P, \tag{19}$$

$$F \cup G \text{ on } L \textbf{ iff } F \text{ on } L \mathbin{\&} G \text{ on } L, \tag{20}$$

$$F \cup G \text{ on } P \textbf{ iff } F \text{ on } P \mathbin{\&} G \text{ on } P, \tag{21}$$

$$G \subseteq F \mathbin{\&} F \text{ is\_linear } \textbf{implies } G \text{ is\_linear}\,, \tag{22}$$

$$G \subseteq F \mathbin{\&} F \text{ is\_planar } \textbf{implies } G \text{ is\_planar}\,. \tag{23}$$

The mode

$$\text{IncSpace}\,,$$

which widens to the type IncStruct, is defined by

$$(\textbf{for } L \textbf{ being } \text{LINE } \textbf{of it ex } A{,}B \textbf{ being } \text{POINT } \textbf{of it st } A \neq B \mathbin{\&} \{A, B\} \text{ on } L) \mathbin{\&}$$
$$(\textbf{for } A{,}B \textbf{ being } \text{POINT } \textbf{of it ex } L \textbf{ being } \text{LINE } \textbf{of it st } \{A, B\} \text{ on } L) \mathbin{\&}$$
$$(\textbf{for } A{,}B \textbf{ being } \text{POINT } \textbf{of it}, K{,}L \textbf{ being } \text{LINE } \textbf{of it}$$
$$\textbf{st } A \neq B \mathbin{\&} \{A, B\} \text{ on } K \mathbin{\&} \{A, B\} \text{ on } L \textbf{ holds } K = L)$$
$$\mathbin{\&} (\textbf{for } P \textbf{ being } \text{PLANE } \textbf{of it ex } A \textbf{ being } \text{POINT } \textbf{of it st } A \text{ on } P) \mathbin{\&}$$

$$\begin{gathered}
(\textbf{for } A,B,C \textbf{ being } \text{POINT} \textbf{ of it ex } P \textbf{ being } \text{PLANE} \textbf{ of it st } \{A,B,C\} \text{ on } P)\ \& \\
(\textbf{for } A,B,C \textbf{ being } \text{POINT} \textbf{ of it}, P,Q \textbf{ being } \text{PLANE} \textbf{ of it} \\
\textbf{st not } \{A,B,C\} \text{ is_linear } \& \{A,B,C\} \text{ on } P \ \& \{A,B,C\} \text{ on } Q \textbf{ holds } P = Q) \\
\& \\
(\textbf{for } L \textbf{ being } \text{LINE} \textbf{ of it}, P \textbf{ being } \text{PLANE} \textbf{ of it} \\
\textbf{st ex } A,B \textbf{ being } \text{POINT} \textbf{ of it st } A \neq B \ \& \{A,B\} \text{ on } L \ \& \{A,B\} \text{ on } P \textbf{ holds } L \text{ on } P) \\
\& \\
(\textbf{for } A \textbf{ being } \text{POINT} \textbf{ of it}, P,Q \textbf{ being } \text{PLANE} \textbf{ of it} \\
\textbf{st } A \text{ on } P \ \& \ A \text{ on } Q \textbf{ ex } B \textbf{ being } \text{POINT} \textbf{ of it st } A \neq B \ \& \ B \text{ on } P \ \& \ B \text{ on } Q) \\
\& \ (\textbf{ex } A,B,C,D \textbf{ being } \text{POINT} \textbf{ of it st not } \{A,B,C,D\} \text{ is_planar})\ \& \\
\textbf{for } A \textbf{ being } \text{POINT} \textbf{ of it}, L \textbf{ being } \text{LINE} \textbf{ of it}, P \textbf{ being } \text{PLANE} \textbf{ of it} \\
\textbf{st } A \text{ on } L \ \& \ L \text{ on } P \textbf{ holds } A \text{ on } P.
\end{gathered}$$

The following proposition is true

(24)
$$\begin{gathered}
(\textbf{for } L \textbf{ being } \text{LINE} \textbf{ of } S \textbf{ ex } A,B \textbf{ being } \text{POINT} \textbf{ of } S \textbf{ st } A \neq B \ \& \{A,B\} \text{ on } L) \\
\& \ (\textbf{for } A,B \textbf{ being } \text{POINT} \textbf{ of } S \textbf{ ex } L \textbf{ being } \text{LINE} \textbf{ of } S \textbf{ st } \{A,B\} \text{ on } L)\ \& \\
(\textbf{for } A,B \textbf{ being } \text{POINT} \textbf{ of } S, K,L \textbf{ being } \text{LINE} \textbf{ of } S \\
\textbf{st } A \neq B \ \& \{A,B\} \text{ on } K \ \& \{A,B\} \text{ on } L \textbf{ holds } K = L) \\
\& \ (\textbf{for } P \textbf{ being } \text{PLANE} \textbf{ of } S \textbf{ ex } A \textbf{ being } \text{POINT} \textbf{ of } S \textbf{ st } A \text{ on } P)\ \& \\
(\textbf{for } A,B,C \textbf{ being } \text{POINT} \textbf{ of } S \textbf{ ex } P \textbf{ being } \text{PLANE} \textbf{ of } S \textbf{ st } \{A,B,C\} \text{ on } P) \\
\& \\
(\textbf{for } A,B,C \textbf{ being } \text{POINT} \textbf{ of } S, P,Q \textbf{ being } \text{PLANE} \textbf{ of } S \\
\textbf{st not } \{A,B,C\} \text{ is_linear } \& \{A,B,C\} \text{ on } P \ \& \{A,B,C\} \text{ on } Q \textbf{ holds } P = Q) \\
\& \\
(\textbf{for } L \textbf{ being } \text{LINE} \textbf{ of } S, P \textbf{ being } \text{PLANE} \textbf{ of } S \textbf{ st} \\
\textbf{ex } A,B \textbf{ being } \text{POINT} \textbf{ of } S \textbf{ st } A \neq B \ \& \{A,B\} \text{ on } L \ \& \{A,B\} \text{ on } P \\
\textbf{holds } L \text{ on } P) \\
\& \\
(\textbf{for } A \textbf{ being } \text{POINT} \textbf{ of } S, P,Q \textbf{ being } \text{PLANE} \textbf{ of } S \\
\textbf{st } A \text{ on } P \ \& \ A \text{ on } Q \textbf{ ex } B \textbf{ being } \text{POINT} \textbf{ of } S \textbf{ st } A \neq B \ \& \ B \text{ on } P \ \& \ B \text{ on } Q) \\
\& \ (\textbf{ex } A,B,C,D \textbf{ being } \text{POINT} \textbf{ of } S \textbf{ st not } \{A,B,C,D\} \text{ is_planar})\ \& \ (\\
\textbf{for } A \textbf{ being } \text{POINT} \textbf{ of } S, L \textbf{ being } \text{LINE} \textbf{ of } S, P \textbf{ being } \text{PLANE} \textbf{ of } S \\
\textbf{st } A \text{ on } L \ \& \ L \text{ on } P \textbf{ holds } A \text{ on } P) \\
\textbf{implies } S \textbf{ is } \text{IncSpace}.
\end{gathered}$$

For simplicity we adopt the following convention: S will denote an object of the type IncSpace; A, B, C, D will denote objects of the type POINT **of** S; K, L, $L1$, $L2$ will denote objects of the type LINE **of** S; P, Q will denote objects of the type PLANE **of** S; F will denote an object of the type Subset **of the** Points **of** S. The following propositions are true:

(25) $$\mathbf{ex}\, A,B\ \mathbf{st}\ A \neq B\ \&\ \{A, B\}\ \text{on}\ L,$$

(26) $$\mathbf{ex}\, L\ \mathbf{st}\ \{A, B\}\ \text{on}\ L,$$

(27) $$A \neq B\ \&\ \{A, B\}\ \text{on}\ K\ \&\ \{A, B\}\ \text{on}\ L\ \mathbf{implies}\ K = L,$$

(28) $$\mathbf{ex}\, A\ \mathbf{st}\ A\ \text{on}\ P,$$

(29) $$\mathbf{ex}\, P\ \mathbf{st}\ \{A, B, C\}\ \text{on}\ P,$$

(30) $$\mathbf{not}\,\{A, B, C\}\,\text{is\_linear}\ \&\ \{A, B, C\}\ \text{on}\ P\ \&\ \{A, B, C\}\ \text{on}\ Q\ \mathbf{implies}\ P = Q,$$

(31) $$(\mathbf{ex}\, A,B\ \mathbf{st}\ A \neq B\ \&\ \{A, B\}\ \text{on}\ L\ \&\ \{A, B\}\ \text{on}\ P)\ \mathbf{implies}\ L\ \text{on}\ P,$$

(32) $$A\ \text{on}\ P\ \&\ A\ \text{on}\ Q\ \mathbf{implies}\ \mathbf{ex}\, B\ \mathbf{st}\ A \neq B\ \&\ B\ \text{on}\ P\ \&\ B\ \text{on}\ Q,$$

(33) $$\mathbf{ex}\, A,B,C,D\ \mathbf{st}\ \mathbf{not}\,\{A, B, C, D\}\,\text{is\_planar}\,,$$

(34) $$A\ \text{on}\ L\ \&\ L\ \text{on}\ P\ \mathbf{implies}\ A\ \text{on}\ P,$$

(35) $$F\ \text{on}\ L\ \&\ L\ \text{on}\ P\ \mathbf{implies}\ F\ \text{on}\ P,$$

(36) $$\{A, A, B\}\,\text{is\_linear}\,,$$

(37) $$\{A, A, B, C\}\,\text{is\_planar}\,,$$

(38) $$\{A, B, C\}\,\text{is\_linear}\ \mathbf{implies}\ \{A, B, C, D\}\,\text{is\_planar}\,,$$

(39) $$A \neq B\ \&\ \{A, B\}\ \text{on}\ L\ \&\ \mathbf{not}\, C\ \text{on}\ L\ \mathbf{implies}\ \mathbf{not}\,\{A, B, C\}\,\text{is\_linear}\,,$$

(40) $$\begin{gathered}\mathbf{not}\,\{A, B, C\}\,\text{is\_linear}\ \&\ \{A, B, C\}\ \text{on}\ P\ \&\ \mathbf{not}\, D\ \text{on}\ P\\ \mathbf{implies}\ \mathbf{not}\,\{A, B, C, D\}\,\text{is\_planar}\,,\end{gathered}$$

(41) $$\mathbf{not}\,(\mathbf{ex}\, P\ \mathbf{st}\ K\ \text{on}\ P\ \&\ L\ \text{on}\ P)\ \mathbf{implies}\ K \neq L,$$

(42) $$\begin{gathered}\mathbf{not}\,(\mathbf{ex}\, P\ \mathbf{st}\ L\ \text{on}\ P\ \&\ L1\ \text{on}\ P\ \&\ L2\ \text{on}\ P)\\ \&\ (\mathbf{ex}\, A\ \mathbf{st}\ A\ \text{on}\ L\ \&\ A\ \text{on}\ L1\ \&\ A\ \text{on}\ L2)\\ \mathbf{implies}\ L \neq L1,\end{gathered}$$

(43) $$\begin{gathered}L1\ \text{on}\ P\ \&\ L2\ \text{on}\ P\ \&\ \mathbf{not}\, L\ \text{on}\ P\ \&\ L1 \neq L2\\ \mathbf{implies}\ \mathbf{not}\ \mathbf{ex}\, Q\ \mathbf{st}\ L\ \text{on}\ Q\ \&\ L1\ \text{on}\ Q\ \&\ L2\ \text{on}\ Q,\end{gathered}$$

$$\textbf{ex}\, P \textbf{ st } A \text{ on } P \;\&\; L \text{ on } P, \tag{44}$$

$$(\textbf{ex}\, A \textbf{ st } A \text{ on } K \;\&\; A \text{ on } L) \textbf{ implies ex}\, P \textbf{ st } K \text{ on } P \;\&\; L \text{ on } P, \tag{45}$$

$$A \neq B \textbf{ implies ex}\, L \textbf{ st for}\, K \textbf{ holds } \{A, B\} \text{ on } K \textbf{ iff } K = L, \tag{46}$$

$$\begin{gathered} \textbf{not}\, \{A, B, C\} \text{ is\_linear} \\ \textbf{implies ex}\, P \textbf{ st for}\, Q \textbf{ holds } \{A, B, C\} \text{ on } Q \textbf{ iff } P = Q, \end{gathered} \tag{47}$$

$$\textbf{not}\, A \text{ on } L \textbf{ implies ex}\, P \textbf{ st for}\, Q \textbf{ holds } A \text{ on } Q \;\&\; L \text{ on } Q \textbf{ iff } P = Q, \tag{48}$$

$$\begin{gathered} K \neq L \;\&\; (\textbf{ex}\, A \textbf{ st } A \text{ on } K \;\&\; A \text{ on } L) \\ \textbf{implies ex}\, P \textbf{ st for}\, Q \textbf{ holds } K \text{ on } Q \;\&\; L \text{ on } Q \textbf{ iff } P = Q. \end{gathered} \tag{49}$$

Let us consider S, A, B. Assume that the following holds

$$A \neq B.$$

The functor

$$\text{Line}\,(A, B),$$

with values of the type LINE **of** S, is defined by

$$\{A, B\} \text{ on } \textbf{it}\,.$$

Let us consider S, A, B, C. Assume that the following holds

$$\textbf{not}\, \{A, B, C\} \text{ is\_linear}\,.$$

The functor

$$\text{Plane}\,(A, B, C),$$

yields the type PLANE **of** S and is defined by

$$\{A, B, C\} \text{ on } \textbf{it}\,.$$

Let us consider S, A, L. Assume that the following holds

$$\textbf{not}\, A \text{ on } L.$$

The functor

$$\text{Plane}\,(A, L),$$

with values of the type PLANE **of** S, is defined by

$$A \text{ on } \textbf{it} \;\&\; L \text{ on } \textbf{it}\,.$$

Let us consider S, K, L. Assume that the following holds

$$K \neq L.$$

Moreover we assume that

$$\mathbf{ex}\, A\ \mathbf{st}\ A \text{ on } K \ \&\ A \text{ on } L.$$

The functor

$$\text{Plane}\,(K, L),$$

with values of the type PLANE **of** S, is defined by

$$K \text{ on } \mathbf{it} \ \&\ L \text{ on } \mathbf{it}\,.$$

Next we state a number of propositions:

$$A \neq B \ \mathbf{implies}\ \{A, B\} \text{ on Line}\,(A, B), \tag{50}$$

$$A \neq B \ \&\ \{A, B\} \text{ on } K \ \mathbf{implies}\ K = \text{Line}\,(A, B), \tag{51}$$

$$\mathbf{not}\ \{A, B, C\}\, \text{is\_linear}\ \mathbf{implies}\ \{A, B, C\} \text{ on Plane}\,(A, B, C), \tag{52}$$

$$\mathbf{not}\ \{A, B, C\}\, \text{is\_linear} \ \&\ \{A, B, C\} \text{ on } Q \ \mathbf{implies}\ Q = \text{Plane}\,(A, B, C), \tag{53}$$

$$\mathbf{not}\ A \text{ on } L \ \mathbf{implies}\ A \text{ on Plane}\,(A, L) \ \&\ L \text{ on Plane}\,(A, L), \tag{54}$$

$$\mathbf{not}\ A \text{ on } L \ \&\ A \text{ on } Q \ \&\ L \text{ on } Q \ \mathbf{implies}\ Q = \text{Plane}\,(A, L), \tag{55}$$

$$\begin{gathered} K \neq L \ \&\ (\mathbf{ex}\, A\ \mathbf{st}\ A \text{ on } K \ \&\ A \text{ on } L) \\ \mathbf{implies}\ K \text{ on Plane}\,(K, L) \ \&\ L \text{ on Plane}\,(K, L), \end{gathered} \tag{56}$$

$$A \neq B \ \mathbf{implies}\ \text{Line}\,(A, B) = \text{Line}\,(B, A), \tag{57}$$

$$\mathbf{not}\ \{A, B, C\}\, \text{is\_linear}\ \mathbf{implies}\ \text{Plane}\,(A, B, C) = \text{Plane}\,(A, C, B), \tag{58}$$

$$\mathbf{not}\ \{A, B, C\}\, \text{is\_linear}\ \mathbf{implies}\ \text{Plane}\,(A, B, C) = \text{Plane}\,(B, A, C), \tag{59}$$

$$\mathbf{not}\ \{A, B, C\}\, \text{is\_linear}\ \mathbf{implies}\ \text{Plane}\,(A, B, C) = \text{Plane}\,(B, C, A), \tag{60}$$

$$\mathbf{not}\ \{A, B, C\}\, \text{is\_linear}\ \mathbf{implies}\ \text{Plane}\,(A, B, C) = \text{Plane}\,(C, A, B), \tag{61}$$

$$\mathbf{not}\ \{A, B, C\}\, \text{is\_linear}\ \mathbf{implies}\ \text{Plane}\,(A, B, C) = \text{Plane}\,(C, B, A), \tag{62}$$

$$\begin{gathered} K \neq L \ \&\ (\mathbf{ex}\, A\ \mathbf{st}\ A \text{ on } K \ \&\ A \text{ on } L) \ \&\ K \text{ on } Q \ \&\ L \text{ on } Q \\ \mathbf{implies}\ Q = \text{Plane}\,(K, L), \end{gathered} \tag{63}$$

$$K \neq L \ \&\ (\mathbf{ex}\, A\ \mathbf{st}\ A \text{ on } K \ \&\ A \text{ on } L)\ \mathbf{implies}\ \text{Plane}\,(K, L) = \text{Plane}\,(L, K), \tag{64}$$

$$A \neq B \ \&\ C \text{ on Line}\,(A, B)\ \mathbf{implies}\ \{A, B, C\}\, \text{is\_linear}\,, \tag{65}$$

$$A \neq B \ \&\ A \neq C \ \&\ \{A, B, C\}\, \text{is\_linear}\ \mathbf{implies}\ \text{Line}\,(A, B) = \text{Line}\,(A, C), \tag{66}$$

$$\mathbf{not}\ \{A, B, C\}\, \text{is\_linear}\ \mathbf{implies}\ \text{Plane}\,(A, B, C) = \text{Plane}\,(C, \text{Line}\,(A, B)), \tag{67}$$

$$\textbf{not}\,\{A,B,C\}\,\mathrm{is\_linear}\ \&\ D\ \mathrm{on}\ \mathrm{Plane}\,(A,B,C)\quad \textbf{implies}\ \{A,B,C,D\}\,\mathrm{is\_planar}\,, \tag{68}$$

$$\textbf{not}\,C\ \mathrm{on}\ L\ \&\ \{A,B\}\ \mathrm{on}\ L\ \&\ A\neq B\ \textbf{implies}\ \mathrm{Plane}\,(C,L)=\mathrm{Plane}\,(A,B,C), \tag{69}$$

$$\textbf{not}\,\{A,B,C\}\,\mathrm{is\_linear}\quad \textbf{implies}\ \mathrm{Plane}\,(A,B,C)=\mathrm{Plane}\,(\mathrm{Line}\,(A,B),\mathrm{Line}\,(A,C)), \tag{70}$$

$$\textbf{ex}\,A,B,C\ \textbf{st}\ \{A,B,C\}\ \mathrm{on}\ P\ \&\ \textbf{not}\,\{A,B,C\}\,\mathrm{is\_linear}\,, \tag{71}$$

$$\textbf{ex}\,A,B,C,D\ \textbf{st}\ A\ \mathrm{on}\ P\ \&\ \textbf{not}\,\{A,B,C,D\}\,\mathrm{is\_planar}\,, \tag{72}$$

$$\textbf{ex}\,B\ \textbf{st}\ A\neq B\ \&\ B\ \mathrm{on}\ L, \tag{73}$$

$$A\neq B\ \textbf{implies}\ \textbf{ex}\,C\ \textbf{st}\ C\ \mathrm{on}\ P\ \&\ \textbf{not}\,\{A,B,C\}\,\mathrm{is\_linear}\,, \tag{74}$$

$$\textbf{not}\,\{A,B,C\}\,\mathrm{is\_linear}\ \textbf{implies}\ \textbf{ex}\,D\ \textbf{st}\ \textbf{not}\,\{A,B,C,D\}\,\mathrm{is\_planar}\,, \tag{75}$$

$$\textbf{ex}\,B,C\ \textbf{st}\ \{B,C\}\ \mathrm{on}\ P\ \&\ \textbf{not}\,\{A,B,C\}\,\mathrm{is\_linear}\,, \tag{76}$$

$$A\neq B\ \textbf{implies}\ \textbf{ex}\,C,D\ \textbf{st}\ \textbf{not}\,\{A,B,C,D\}\,\mathrm{is\_planar}\,, \tag{77}$$

$$\textbf{ex}\,B,C,D\ \textbf{st}\ \textbf{not}\,\{A,B,C,D\}\,\mathrm{is\_planar}\,, \tag{78}$$

$$\textbf{ex}\,L\ \textbf{st}\ \textbf{not}\,A\ \mathrm{on}\ L\ \&\ L\ \mathrm{on}\ P, \tag{79}$$

$$A\ \mathrm{on}\ P\ \textbf{implies}\ \textbf{ex}\,L,L1,L2\ \textbf{st}\ L1\neq L2\quad \&\ L1\ \mathrm{on}\ P\ \&\ L2\ \mathrm{on}\ P\ \&\ \textbf{not}\,L\ \mathrm{on}\ P\ \&\ A\ \mathrm{on}\ L\ \&\ A\ \mathrm{on}\ L1\ \&\ A\ \mathrm{on}\ L2, \tag{80}$$

$$\textbf{ex}\,L,L1,L2\quad \textbf{st}\ A\ \mathrm{on}\ L\ \&\ A\ \mathrm{on}\ L1\ \&\ A\ \mathrm{on}\ L2\ \&\ \textbf{not}\,\textbf{ex}\,P\ \textbf{st}\ L\ \mathrm{on}\ P\ \&\ L1\ \mathrm{on}\ P\ \&\ L2\ \mathrm{on}\ P, \tag{81}$$

$$\textbf{ex}\,P\ \textbf{st}\ A\ \mathrm{on}\ P\ \&\ \textbf{not}\,L\ \mathrm{on}\ P, \tag{82}$$

$$\textbf{ex}\,A\ \textbf{st}\ A\ \mathrm{on}\ P\ \&\ \textbf{not}\,A\ \mathrm{on}\ L, \tag{83}$$

$$\textbf{ex}\,K\ \textbf{st}\ \textbf{not}\,\textbf{ex}\,P\ \textbf{st}\ L\ \mathrm{on}\ P\ \&\ K\ \mathrm{on}\ P, \tag{84}$$

$$\textbf{ex}\,P,Q\ \textbf{st}\ P\neq Q\ \&\ L\ \mathrm{on}\ P\ \&\ L\ \mathrm{on}\ Q, \tag{85}$$

$$K\neq L\ \&\ \{A,B\}\ \mathrm{on}\ K\ \&\ \{A,B\}\ \mathrm{on}\ L\ \textbf{implies}\ A=B, \tag{86}$$

$$\textbf{not}\,L\ \mathrm{on}\ P\ \&\ \{A,B\}\ \mathrm{on}\ L\ \&\ \{A,B\}\ \mathrm{on}\ P\ \textbf{implies}\ A=B, \tag{87}$$

$$P\neq Q\ \textbf{implies}\ \textbf{not}\,(\textbf{ex}\,A\ \textbf{st}\ A\ \mathrm{on}\ P\ \&\ A\ \mathrm{on}\ Q)\quad \textbf{or}\ \textbf{ex}\,L\ \textbf{st}\ \textbf{for}\,B\ \textbf{holds}\ B\ \mathrm{on}\ P\ \&\ B\ \mathrm{on}\ Q\ \textbf{iff}\ B\ \mathrm{on}\ L. \tag{88}$$

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Introduction to Lattice Theory

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Summary. A lattice is defined as an algebra on a nonempty set with binary operations join and meet which are commutative and associative, and satisfy the absorption identities. The following kinds of lattices are considered: distributive, modular, bounded (with zero and unit elements), complemented, and Boolean (with complement). The article includes also theorems which immediately follow from definitions.

The terminology and notation used in this paper are introduced in the papers [1] and [2]. The scheme *BooleDomBinOpLam* deals with a constant $\mathcal{A}$ that has the type BOOLE_DOMAIN and a binary functor $\mathcal{F}$ yielding values of the type Element **of** $\mathcal{A}$ and states that the following holds

$$\textbf{ex}\, o \,\textbf{being}\, \text{Binary\_Operation} \,\textbf{of}\, \mathcal{A}$$
$$\textbf{st for}\, a,b \,\textbf{being}\, \text{Element} \,\textbf{of}\, \mathcal{A} \,\textbf{holds}\, o.(a,b) = \mathcal{F}(a,b)$$

for all values of the parameters.

We consider structures LattStr, which are systems

$$\langle\langle \text{carrier}\,, \text{join}\,, \text{meet} \rangle\rangle$$

where carrier has the type DOMAIN, and join, meet have the type Binary_Operation **of the** carrier. In the sequel G has the type LattStr; p, q, r have the type Element **of the** carrier **of** G. We now define two new functors. Let us consider G, p, q. The functor

$$p \sqcup q,$$

yields the type Element **of the** carrier **of** G and is defined by

$$\textbf{it} = (\textbf{the}\, \text{join} \,\textbf{of}\, G).(p,q).$$

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The functor

$$p \sqcap q,$$

with values of the type Element **of the** carrier **of** G, is defined by

$$\textbf{it} = (\textbf{the}\,\text{meet}\ \textbf{of}\ G).(p,q).$$

The following propositions are true:

$$p \sqcup q = (\textbf{the}\,\text{join}\ \textbf{of}\ G).(p,q), \tag{1}$$

$$p \sqcap q = (\textbf{the}\,\text{meet}\ \textbf{of}\ G).(p,q). \tag{2}$$

Let us consider G, p, q. The predicate

$$p \sqsubseteq q \qquad \text{is defined by} \qquad p \sqcup q = q.$$

We now state a proposition

$$p \sqsubseteq q\ \textbf{iff}\ p \sqcup q = q. \tag{3}$$

The mode

$$\text{Lattice}\,,$$

which widens to the type LattStr, is defined by

$$\begin{gathered}
(\textbf{for}\ a,b\ \textbf{being}\ \text{Element}\ \textbf{of the}\ \text{carrier}\ \textbf{of it holds}\ a \sqcup b = b \sqcup a)\ \&\\
(\textbf{for}\ a,b,c\ \textbf{being}\ \text{Element}\ \textbf{of the}\ \text{carrier}\ \textbf{of it holds}\ a \sqcup (b \sqcup c) = (a \sqcup b) \sqcup c)\ \&\\
(\textbf{for}\ a,b\ \textbf{being}\ \text{Element}\ \textbf{of the}\ \text{carrier}\ \textbf{of it holds}\ (a \sqcap b) \sqcup b = b)\ \&\\
(\textbf{for}\ a,b\ \textbf{being}\ \text{Element}\ \textbf{of the}\ \text{carrier}\ \textbf{of it holds}\ a \sqcap b = b \sqcap a)\ \&\\
(\textbf{for}\ a,b,c\ \textbf{being}\ \text{Element}\ \textbf{of the}\ \text{carrier}\ \textbf{of it holds}\ a \sqcap (b \sqcap c) = (a \sqcap b) \sqcap c)\\
\&\ \textbf{for}\ a,b\ \textbf{being}\ \text{Element}\ \textbf{of the}\ \text{carrier}\ \textbf{of it holds}\ a \sqcap (a \sqcup b) = a.
\end{gathered}$$

One can prove the following proposition

$$\begin{gathered}
(\textbf{for}\ p,q\ \textbf{holds}\ p \sqcup q = q \sqcup p)\ \&\ (\textbf{for}\ p,q,r\ \textbf{holds}\ p \sqcup (q \sqcup r) = (p \sqcup q) \sqcup r)\ \&\\
(\textbf{for}\ p,q\ \textbf{holds}\ (p \sqcap q) \sqcup q = q)\ \&\ (\textbf{for}\ p,q\ \textbf{holds}\ p \sqcap q = q \sqcap p)\\
\&\ (\textbf{for}\ p,q,r\ \textbf{holds}\ p \sqcap (q \sqcap r) = (p \sqcap q) \sqcap r)\ \&\ (\textbf{for}\ p,q\ \textbf{holds}\ p \sqcap (p \sqcup q) = p)\\
\textbf{implies}\ G\ \textbf{is}\ \text{Lattice}\,.
\end{gathered} \tag{4}$$

In the sequel L has the type Lattice; a, b, c have the type Element **of the** carrier **of** L. One can prove the following propositions:

$$a \sqcup b = b \sqcup a, \tag{5}$$

$$a \sqcap b = b \sqcap a, \tag{6}$$

$$a \sqcup (b \sqcup c) = (a \sqcup b) \sqcup c, \tag{7}$$

(8) $$a \sqcap (b \sqcap c) = (a \sqcap b) \sqcap c,$$

(9) $$(a \sqcap b) \sqcup b = b \;\&\; b \sqcup (a \sqcap b) = b \;\&\; b \sqcup (b \sqcap a) = b \;\&\; (b \sqcap a) \sqcup b = b,$$

(10) $$a \sqcap (a \sqcup b) = a \;\&\; (a \sqcup b) \sqcap a = a \;\&\; (b \sqcup a) \sqcap a = a \;\&\; a \sqcap (b \sqcup a) = a.$$

The mode

Distributive_Lattice ,

which widens to the type Lattice, is defined by

for a,b,c **being** Element **of the** carrier **of it holds** $a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$.

Next we state a proposition

(11) (**for** a,b,c **holds** $a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$)
implies L **is** Distributive_Lattice .

The mode

Modular_Lattice ,

which widens to the type Lattice, is defined by

for a,b,c **being** Element **of the** carrier **of it st** $a \sqsubseteq c$ **holds** $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap c$.

One can prove the following proposition

(12) (**for** a,b,c **st** $a \sqsubseteq c$ **holds** $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap c$)
implies L **is** Modular_Lattice .

The mode

Lower_Bound_Lattice ,

which widens to the type Lattice, is defined by

ex c **being** Element **of the** carrier **of it**
st for a **being** Element **of the** carrier **of it holds** $c \sqcap a = c$.

Next we state a proposition

(13) (**ex** c **st for** a **holds** $c \sqcap a = c$) **implies** L **is** Lower_Bound_Lattice .

The mode

Upper_Bound_Lattice ,

which widens to the type Lattice, is defined by

ex c **being** Element **of the** carrier **of it**
st for a **being** Element **of the** carrier **of it holds** $c \sqcup a = c$.

One can prove the following proposition

(14) $\quad$ (**ex** c **st for** a **holds** $c \sqcup a = c$) **implies** L **is** Upper_Bound_Lattice .

The mode

Bound_Lattice ,

which widens to the type Lattice, is defined by

it is Lower_Bound_Lattice & **it is** Upper_Bound_Lattice .

Next we state a proposition

(15) $\quad$ L **is** Lower_Bound_Lattice & L **is** Upper_Bound_Lattice
implies L **is** Bound_Lattice .

Let us consider L. Assume that the following holds

ex c **st for** a **holds** $c \sqcap a = c$.

The functor

$\perp L$,

yields the type Element **of the** carrier **of** L and is defined by

it $\sqcap\, a =$ **it** .

Let L have the type Lower_Bound_Lattice. Let us note that it makes sense to consider the following functor on a restricted area. Then

$\perp L$ $\quad$ is $\quad$ Element **of the** carrier **of** L.

Let us consider L. Assume that the following holds

ex c **st for** a **holds** $c \sqcup a = c$.

The functor

$\top L$,

with values of the type Element **of the** carrier **of** L, is defined by

it $\sqcup\, a =$ **it** .

Let L have the type Upper_Bound_Lattice. Let us note that it makes sense to consider the following functor on a restricted area. Then

$\top L$ $\quad$ is $\quad$ Element **of the** carrier **of** L.

Let L have the type Bound_Lattice. Let us note that it makes sense to consider the following functors on restricted areas. Then

$\perp L$ $\quad$ is $\quad$ Element **of the** carrier **of** L,

$$\top L \quad \text{is} \quad \text{Element } \textbf{of the} \text{ carrier } \textbf{of } L.$$

Let us consider L, a, b. Assume that the following holds

$$L \textbf{ is } \text{Bound\_Lattice}.$$

The predicate

$$a \text{ is\_a\_complement\_of } b \quad \text{is defined by} \quad a \sqcup b = \top L \;\&\; a \sqcap b = \bot L.$$

The mode

$$\text{Lattice\_with\_Complement},$$

which widens to the type Bound_Lattice, is defined by

$$\textbf{for } b \textbf{ being } \text{Element } \textbf{of the} \text{ carrier } \textbf{of it}$$
$$\textbf{ex } a \textbf{ being } \text{Element } \textbf{of the} \text{ carrier } \textbf{of it st } a \text{ is\_a\_complement\_of } b.$$

The mode

$$\text{Boolean\_Lattice},$$

which widens to the type Lattice_with_Complement, is defined by

$$\textbf{it is } \text{Distributive\_Lattice}.$$

The following propositions are true:

(16) $$a \sqcup b = b \textbf{ iff } a \sqcap b = a,$$

(17) $$a \sqcup a = a,$$

(18) $$a \sqcap a = a,$$

(19) $$\textbf{for } L \textbf{ holds } (\textbf{for } a,b,c \textbf{ holds } a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c))$$
$$\textbf{iff for } a,b,c \textbf{ holds } a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c),$$

(20) $$a \sqsubseteq b \textbf{ iff } a \sqcup b = b,$$

(21) $$a \sqsubseteq b \textbf{ iff } a \sqcap b = a,$$

(22) $$a \sqsubseteq a \sqcup b,$$

(23) $$a \sqcap b \sqsubseteq a,$$

(24) $$a \sqsubseteq a,$$

(25) $$a \sqsubseteq b \;\&\; b \sqsubseteq c \textbf{ implies } a \sqsubseteq c,$$

(26) $$a \sqsubseteq b \;\&\; b \sqsubseteq a \textbf{ implies } a = b,$$

(27) $$a \sqsubseteq b \textbf{ implies } a \sqcap c \sqsubseteq b \sqcap c,$$

$$a \sqsubseteq b \textbf{ implies } c \sqcap a \sqsubseteq c \sqcap b, \tag{28}$$

$$(\textbf{for } a,b,c \textbf{ holds } (a \sqcap b) \sqcup (b \sqcap c) \sqcup (c \sqcap a) = (a \sqcup b) \sqcap (b \sqcup c) \sqcap (c \sqcup a)) \\ \textbf{implies } L \textbf{ is } \text{Distributive\_Lattice}\,. \tag{29}$$

In the sequel L denotes an object of the type Distributive_Lattice; a, b, c denote objects of the type Element **of the** carrier **of** L. One can prove the following propositions:

$$\textbf{for } L \textbf{ holds } (\textbf{for } a,b,c \textbf{ holds } a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)) \\ \& \textbf{ for } a,b,c \textbf{ holds } (b \sqcup c) \sqcap a = (b \sqcap a) \sqcup (c \sqcap a), \tag{30}$$

$$\textbf{for } L \textbf{ holds } (\textbf{for } a,b,c \textbf{ holds } a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)) \\ \& \textbf{ for } a,b,c \textbf{ holds } (b \sqcap c) \sqcup a = (b \sqcup a) \sqcap (c \sqcup a), \tag{31}$$

$$c \sqcap a = c \sqcap b \;\&\; c \sqcup a = c \sqcup b \textbf{ implies } a = b, \tag{32}$$

$$a \sqcap c = b \sqcap c \;\&\; a \sqcup c = b \sqcup c \textbf{ implies } a = b, \tag{33}$$

$$(a \sqcup b) \sqcap (b \sqcup c) \sqcap (c \sqcup a) = (a \sqcap b) \sqcup (b \sqcap c) \sqcup (c \sqcap a), \tag{34}$$

$$L \textbf{ is } \text{Modular\_Lattice}\,. \tag{35}$$

In the sequel L has the type Modular_Lattice; a, b, c have the type Element **of the** carrier **of** L. One can prove the following two propositions:

$$a \sqsubseteq c \textbf{ implies } a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap c, \tag{36}$$

$$c \sqsubseteq a \textbf{ implies } a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup c. \tag{37}$$

In the sequel L has the type Lower_Bound_Lattice; a, c have the type Element **of the** carrier **of** L. We now state four propositions:

$$\textbf{ex } c \textbf{ st for } a \textbf{ holds } c \sqcap a = c, \tag{38}$$

$$\perp L \sqcup a = a \;\&\; a \sqcup \perp L = a, \tag{39}$$

$$\perp L \sqcap a = \perp L \;\&\; a \sqcap \perp L = \perp L, \tag{40}$$

$$\perp L \sqsubseteq a. \tag{41}$$

In the sequel L denotes an object of the type Upper_Bound_Lattice; a, c denote objects of the type Element **of the** carrier **of** L. The following four propositions are true:

$$\textbf{ex } c \textbf{ st for } a \textbf{ holds } c \sqcup a = c, \tag{42}$$

$$\top L \sqcap a = a \;\&\; a \sqcap \top L = a, \tag{43}$$

(44) $$\top L \sqcup a = \top L \;\&\; a \sqcup \top L = \top L,$$

(45) $$a \sqsubseteq \top L.$$

In the sequel L has the type Lattice_with_Complement; a, b have the type Element **of the** carrier **of** L. One can prove the following proposition

(46) $$\textbf{ex}\, a \;\textbf{st}\; a \text{ is\_a\_complement\_of } b.$$

In the sequel L has the type Lattice. The arguments of the notions defined below are the following: L which is an object of the type reserved above; x which is an object of the type Element **of the** carrier **of** L. Assume that the following holds

$$L \;\textbf{is}\; \text{Boolean\_Lattice}.$$

The functor

$$x^{\,\mathrm{c}},$$

yields the type Element **of the** carrier **of** L and is defined by

$$\textbf{it} \text{ is\_a\_complement\_of } x.$$

The arguments of the notions defined below are the following: L which is an object of the type Boolean_Lattice; x which is an object of the type Element **of the** carrier **of** L. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$x^{\,\mathrm{c}} \quad \text{is} \quad \text{Element } \textbf{of the} \text{ carrier } \textbf{of } L.$$

In the sequel L will denote an object of the type Boolean_Lattice; a, b will denote objects of the type Element **of the** carrier **of** L. We now state several propositions:

(47) $$a^{\,\mathrm{c}} \sqcap a = \bot L \;\&\; a \sqcap a^{\,\mathrm{c}} = \bot L,$$

(48) $$a^{\,\mathrm{c}} \sqcup a = \top L \;\&\; a \sqcup a^{\,\mathrm{c}} = \top L,$$

(49) $$a^{\,\mathrm{c}\,\mathrm{c}} = a,$$

(50) $$(a \sqcap b)^{\,\mathrm{c}} = a^{\,\mathrm{c}} \sqcup b^{\,\mathrm{c}},$$

(51) $$(a \sqcup b)^{\,\mathrm{c}} = a^{\,\mathrm{c}} \sqcap b^{\,\mathrm{c}},$$

(52) $$b \sqcap a = \bot L \;\textbf{iff}\; b \sqsubseteq a^{\,\mathrm{c}},$$

(53) $$a \sqsubseteq b \;\textbf{implies}\; b^{\,\mathrm{c}} \sqsubseteq a^{\,\mathrm{c}}.$$

In the sequel L will have the type Bound_Lattice; a, b will have the type Element **of the** carrier **of** L. We now state three propositions:

(54) $$L \;\textbf{is}\; \text{Lower\_Bound\_Lattice} \;\&\; L \;\textbf{is}\; \text{Upper\_Bound\_Lattice},$$

(55) a is_a_complement_of b **iff** $a \sqcup b = \top L$ & $a \sqcap b = \perp L$,

(56) (**for** b **ex** a **st** a is_a_complement_of b) **implies** L **is** Lattice_with_Complement .

In the sequel L has the type Lattice_with_Complement. One can prove the following proposition

(57) L **is** Distributive_Lattice **implies** L **is** Boolean_Lattice .

In the sequel L has the type Boolean_Lattice. The following two propositions are true:

(58) L **is** Lattice_with_Complement ,

(59) L **is** Distributive_Lattice .

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Topological Spaces and Continuous Functions

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Summary. The paper contains a definition of topological space. The following notions are defined: point of topological space, subset of topological space, subspace of topological space, and continuous function.

The articles [5], [7], [6], [1], [4], [2], and [3] provide the terminology and notation for this paper. We consider structures TopStruct, which are systems

$$\langle\langle \text{carrier}, \text{topology} \rangle\rangle$$

where carrier has the type DOMAIN, and topology has the type Subset-Family **of the** carrier. In the sequel T has the type TopStruct. The mode

$$\text{TopSpace},$$

which widens to the type TopStruct, is defined by

$$\begin{gathered}\textbf{the}\,\text{carrier}\ \textbf{of it} \in \textbf{the}\,\text{topology}\ \textbf{of it}\ \&\\ (\textbf{for}\ a\ \textbf{being}\ \text{Subset-Family}\ \textbf{of the}\,\text{carrier}\ \textbf{of it}\\ \textbf{st}\ a \subseteq \textbf{the}\,\text{topology}\ \textbf{of it holds} \bigcup a \in \textbf{the}\,\text{topology}\ \textbf{of it})\\ \&\ \textbf{for}\ a{,}b\ \textbf{being}\ \text{Subset}\ \textbf{of the}\,\text{carrier}\ \textbf{of it}\\ \textbf{st}\ a \in \textbf{the}\,\text{topology}\ \textbf{of it}\ \&\ b \in \textbf{the}\,\text{topology}\ \textbf{of it holds}\ a \cap b \in \textbf{the}\,\text{topology}\ \textbf{of it}\,.\end{gathered}$$

We now state a proposition

(1) $$\begin{gathered}\textbf{the}\,\text{carrier}\ \textbf{of}\ T \in \textbf{the}\,\text{topology}\ \textbf{of}\ T\ \&\\ (\textbf{for}\ a\ \textbf{being}\ \text{Subset-Family}\ \textbf{of the}\,\text{carrier}\ \textbf{of}\ T\end{gathered}$$

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$$\textbf{st } a \subseteq \textbf{the} \text{ topology } \textbf{of } T \textbf{ holds} \bigcup a \in \textbf{the} \text{ topology } \textbf{of } T)$$
$$\& \ (\textbf{for } p,q \textbf{ being} \text{ Subset } \textbf{of the} \text{ carrier } \textbf{of } T \textbf{ st}$$
$$p \in \textbf{the} \text{ topology } \textbf{of } T \ \& \ q \in \textbf{the} \text{ topology } \textbf{of } T$$
$$\textbf{holds } p \cap q \in \textbf{the} \text{ topology } \textbf{of } T)$$
$$\textbf{implies } T \textbf{ is} \text{ TopSpace}.$$

In the sequel T, S, GX will have the type TopSpace. Let us consider T.

$$\text{Point } \textbf{of } T \quad \text{stands for} \quad \text{Element } \textbf{of the} \text{ carrier } \textbf{of } T.$$

The following proposition is true

(2) $$\textbf{for } x \textbf{ being} \text{ Element } \textbf{of the} \text{ carrier } \textbf{of } T \textbf{ holds } x \textbf{ is} \text{ Point } \textbf{of } T.$$

Let us consider T.

$$\text{Subset } \textbf{of } T \quad \text{stands for} \quad \textbf{set of} \text{ Point } \textbf{of } T.$$

We now state a proposition

(3) $$\textbf{for } P \textbf{ being} \text{ Subset } \textbf{of the} \text{ carrier } \textbf{of } T \textbf{ holds } P \textbf{ is} \text{ Subset } \textbf{of } T.$$

In the sequel P, Q will have the type Subset **of** T; p will have the type Point **of** T. Let us consider T.

$$\text{Subset-Family } \textbf{of } T \quad \text{stands for} \quad \text{Subset-Family } \textbf{of the} \text{ carrier } \textbf{of } T.$$

Next we state a proposition

(4) $$\textbf{for } F \textbf{ being} \text{ Subset-Family } \textbf{of the} \text{ carrier } \textbf{of } T$$
$$\textbf{holds } F \textbf{ is} \text{ Subset-Family } \textbf{of } T.$$

In the sequel F will denote an object of the type Subset-Family **of** T. The scheme *SubFamEx1* concerns a constant $\mathcal{A}$ that has the type TopSpace and a unary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex } F \textbf{ being} \text{ Subset-Family } \textbf{of } \mathcal{A} \textbf{ st for } B \textbf{ being} \text{ Subset } \textbf{of } \mathcal{A} \textbf{ holds } B \in F \textbf{ iff } \mathcal{P}[B]$$

for all values of the parameters.

One can prove the following propositions:

(5) $$\emptyset \in \textbf{the} \text{ topology } \textbf{of } T,$$

(6) $$\textbf{the} \text{ carrier } \textbf{of } T \in \textbf{the} \text{ topology } \textbf{of } T,$$

(7) $$\textbf{for } a \textbf{ being} \text{ Subset-Family } \textbf{of } T$$
$$\textbf{st } a \subseteq \textbf{the} \text{ topology } \textbf{of } T \textbf{ holds} \bigcup a \in \textbf{the} \text{ topology } \textbf{of } T,$$

$$P \in \textbf{the}\,\text{topology}\,\textbf{of}\,T \;\&\; Q \in \textbf{the}\,\text{topology}\,\textbf{of}\,T \quad \textbf{implies}\; P \cap Q \in \textbf{the}\,\text{topology}\,\textbf{of}\,T. \tag{8}$$

We now define two new functors. Let us consider T. The functor

$$\emptyset T,$$

with values of the type Subset **of** T, is defined by

$$\textbf{it} = \emptyset\, \textbf{the}\,\text{carrier}\,\textbf{of}\,T.$$

The functor

$$\Omega\, T,$$

with values of the type Subset **of** T, is defined by

$$\textbf{it} = \Omega\, \textbf{the}\,\text{carrier}\,\textbf{of}\,T.$$

One can prove the following four propositions:

$$\emptyset T = \emptyset\, \textbf{the}\,\text{carrier}\,\textbf{of}\,T, \tag{9}$$

$$\Omega\, T = \Omega\, \textbf{the}\,\text{carrier}\,\textbf{of}\,T, \tag{10}$$

$$\emptyset(T) = \emptyset, \tag{11}$$

$$\Omega\,(T) = \textbf{the}\,\text{carrier}\,\textbf{of}\,T. \tag{12}$$

Let us consider T, P. The functor

$$P^{\,\mathrm{c}},$$

yields the type Subset **of** T and is defined by

$$\textbf{it} = P^{\,\mathrm{c}}.$$

Let us consider T, P, Q. Let us note that it makes sense to consider the following functors on restricted areas. Then

$P \cup Q$	is	Subset **of** T,
$P \cap Q$	is	Subset **of** T,
$P \setminus Q$	is	Subset **of** T,
$P \mathbin{\dot-} Q$	is	Subset **of** T.

The following propositions are true:

$$p \in \Omega\,(T), \tag{13}$$

$$P \subseteq \Omega\,(T), \tag{14}$$

(15) $$P \cap \Omega(T) = P,$$

(16) $$\textbf{for } A \textbf{ being} \text{ set } \textbf{holds } A \subseteq \Omega(T) \textbf{ implies } A \textbf{ is} \text{ Subset } \textbf{of } T,$$

(17) $$P^{\,c} = \Omega(T) \setminus P,$$

(18) $$P \cup P^{\,c} = \Omega(T),$$

(19) $$P \subseteq Q \textbf{ iff } Q^{\,c} \subseteq P^{\,c},$$

(20) $$P = P^{\,c\,c},$$

(21) $$P \subseteq Q^{\,c} \textbf{ iff } P \cap Q = \emptyset,$$

(22) $$\Omega(T) \setminus (\Omega(T) \setminus P) = P,$$

(23) $$P \neq \Omega(T) \textbf{ iff } \Omega(T) \setminus P \neq \emptyset,$$

(24) $$\Omega(T) \setminus P = Q \textbf{ implies } \Omega(T) = P \cup Q,$$

(25) $$\Omega(T) = P \cup Q \mathbin{\&} P \cap Q = \emptyset \textbf{ implies } Q = \Omega(T) \setminus P,$$

(26) $$P \cap P^{\,c} = \emptyset(T),$$

(27) $$\Omega(T) = (\emptyset T)^{\,c},$$

(28) $$P \setminus Q = P \cap Q^{\,c},$$

(29) $$P = Q \textbf{ implies } \Omega(T) \setminus P = \Omega(T) \setminus Q.$$

Let us consider T, P. The predicate

$$P \text{ is\_open} \qquad \text{is defined by} \qquad P \in \textbf{the} \text{ topology } \textbf{of } T.$$

One can prove the following proposition

(30) $$P \text{ is\_open } \textbf{iff } P \in \textbf{the} \text{ topology } \textbf{of } T.$$

Let us consider T, P. The predicate

$$P \text{ is\_closed} \qquad \text{is defined by} \qquad \Omega(T) \setminus P \text{ is\_open}.$$

One can prove the following proposition

(31) $$P \text{ is\_closed } \textbf{iff } \Omega(T) \setminus P \text{ is\_open}.$$

Let us consider T, P. The predicate

$$P \text{ is\_open\_closed} \qquad \text{is defined by} \qquad P \text{ is\_open} \mathbin{\&} P \text{ is\_closed}.$$

We now state a proposition

(32) $$P \text{ is\_open\_closed } \textbf{iff } P \text{ is\_open} \ \& \ P \text{ is\_closed}\,.$$

Let us consider T, F. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\bigcup F \qquad \text{is} \qquad \text{Subset } \textbf{of } T.$$

Let us consider T, F. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\bigcap F \qquad \text{is} \qquad \text{Subset } \textbf{of } T.$$

Let us consider T, F. The predicate

$$F \text{ is\_a\_cover\_of } T \qquad \text{is defined by} \qquad \Omega\,(T) = \bigcup F.$$

The following proposition is true

(33) $$F \text{ is\_a\_cover\_of } T \textbf{ iff } \Omega\,(T) = \bigcup F.$$

Let us consider T. The mode

$$\text{SubSpace } \textbf{of } T,$$

which widens to the type TopSpace, is defined by

$$\Omega\,(\textbf{it}) \subseteq \Omega\,(T) \ \& \ \textbf{for } P \textbf{ being } \text{Subset } \textbf{of it holds } P \in \textbf{the } \text{topology } \textbf{of it}$$
$$\textbf{iff ex } Q \textbf{ being } \text{Subset } \textbf{of } T \textbf{ st } Q \in \textbf{the } \text{topology } \textbf{of } T \ \& \ P = Q \cap \Omega\,(\textbf{it}).$$

Next we state two propositions:

(34) $$\Omega\,(S) \subseteq \Omega\,(T) \ \& \ (\textbf{for } P \textbf{ being } \text{Subset } \textbf{of } S \textbf{ holds } P \in \textbf{the } \text{topology } \textbf{of } S$$
$$\textbf{iff ex } Q \textbf{ being } \text{Subset } \textbf{of } T \textbf{ st } Q \in \textbf{the } \text{topology } \textbf{of } T \ \& \ P = Q \cap \Omega\,(S))$$
$$\textbf{implies } S \textbf{ is } \text{SubSpace } \textbf{of } T,$$

(35) $$\textbf{for } V \textbf{ being } \text{SubSpace } \textbf{of } T \textbf{ holds } \Omega\,(V) \subseteq \Omega\,(T) \ \& \ \textbf{for } P \textbf{ being } \text{Subset } \textbf{of } V$$
$$\textbf{holds } P \in \textbf{the } \text{topology } \textbf{of } V$$
$$\textbf{iff ex } Q \textbf{ being } \text{Subset } \textbf{of } T \textbf{ st } Q \in \textbf{the } \text{topology } \textbf{of } T \ \& \ P = Q \cap \Omega\,(V).$$

Let us consider T, P. Assume that the following holds

$$P \neq \emptyset(T).$$

The functor

$$T \mid P,$$

with values of the type SubSpace **of** T, is defined by

$$\Omega\,(\textbf{it}) = P.$$

One can prove the following proposition

(36) $P \neq \emptyset(T)$ **implies for** S **being** SubSpace **of** T **holds** $S = T \mid P$ **iff** $\Omega\,(S) = P$.

Let us consider T, S.

map **of** T, S stands for Function **of** (**the** carrier **of** T),(**the** carrier **of** S).

Next we state a proposition

(37) **for** f **being** Function **of the** carrier **of** T,**the** carrier **of** S **holds** f **is** map **of** T, S.

In the sequel f has the type map **of** T, S; $P1$ has the type Subset **of** S. Let us consider T, S, f, P. Let us note that it makes sense to consider the following functor on a restricted area. Then

$f\ ^{\circ}\ P$ is Subset **of** S.

Let us consider T, S, f, $P1$. Let us note that it makes sense to consider the following functor on a restricted area. Then

$f\ ^{-1}\ P1$ is Subset **of** T.

Let us consider T, S, f. The predicate

f is_continuous

is defined by

for $P1$ **holds** $P1$ is_closed **implies** $f\ ^{-1}\ P1$ is_closed .

The following proposition is true

(38) f is_continuous **iff for** $P1$ **holds** $P1$ is_closed **implies** $f\ ^{-1}\ P1$ is_closed .

The scheme *TopAbstr* concerns a constant $\mathcal{A}$ that has the type TopSpace and a unary predicate $\mathcal{P}$ and states that the following holds

ex P **being** Subset **of** $\mathcal{A}$ **st for** x **being** Point **of** $\mathcal{A}$ **holds** $x \in P$ **iff** $\mathcal{P}[x]$

for all values of the parameters.

One can prove the following propositions:

(39) **for** X' **being** SubSpace **of** GX **for** A **being** Subset **of** X' **holds** A **is** Subset **of** GX,

(40) **for** A **being** Subset **of** GX, x **being** Any **st** $x \in A$ **holds** x **is** Point **of** GX,

(41) **for** A **being** Subset **of** GX **st** $A \neq \emptyset(GX)$ **ex** x **being** Point **of** GX **st** $x \in A$,

(42) $\Omega\,(GX)$ is_closed ,

(43) $$\textbf{for } X' \textbf{ being } \text{SubSpace } \textbf{of } GX, B \textbf{ being } \text{Subset } \textbf{of } X' \textbf{ holds}$$
$$B \text{ is\_closed } \textbf{iff } \textbf{ex } C \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ st } C \text{ is\_closed } \& \ C \cap (\Omega\,(X')) = B,$$

(44) $$\textbf{for } F \textbf{ being } \text{Subset-Family } \textbf{of } GX \textbf{ st}$$
$$F \neq \emptyset \ \& \ \textbf{for } A \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ st } A \in F \textbf{ holds } A \text{ is\_closed}$$
$$\textbf{holds } \bigcap F \text{ is\_closed}\,.$$

The arguments of the notions defined below are the following: GX which is an object of the type TopSpace; A which is an object of the type Subset **of** GX. The functor

$$\text{Cl}\, A,$$

yields the type Subset **of** GX and is defined by

$$\textbf{for } p \textbf{ being } \text{Point } \textbf{of } GX \textbf{ holds } p \in \textbf{it}$$
$$\textbf{iff for } G \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ st } G \text{ is\_open } \textbf{holds } p \in G \textbf{ implies } A \cap G \neq \emptyset(GX).$$

We now state a number of propositions:

(45) $$\textbf{for } A \textbf{ being } \text{Subset } \textbf{of } GX, p \textbf{ being } \text{Point } \textbf{of } GX \textbf{ holds } p \in \text{Cl}\, A$$
$$\textbf{iff for } C \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ st } C \text{ is\_closed } \textbf{holds } A \subseteq C \textbf{ implies } p \in C,$$

(46) $$\textbf{for } A \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ ex } F \textbf{ being } \text{Subset-Family } \textbf{of } GX \textbf{ st}$$
$$(\textbf{for } C \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ holds } C \in F \textbf{ iff } C \text{ is\_closed } \& \ A \subseteq C)$$
$$\& \ \text{Cl}\, A = \bigcap F,$$

(47) $$\textbf{for}$$
$$X' \textbf{ being } \text{SubSpace } \textbf{of } GX, A \textbf{ being } \text{Subset } \textbf{of } GX, A1 \textbf{ being } \text{Subset } \textbf{of } X'$$
$$\textbf{st } A = A1 \textbf{ holds } \text{Cl}\, A1 = (\text{Cl}\, A) \cap (\Omega\,(X')),$$

(48) $$\textbf{for } A \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ holds } A \subseteq \text{Cl}\, A,$$

(49) $$\textbf{for } A,B \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ st } A \subseteq B \textbf{ holds } \text{Cl}\, A \subseteq \text{Cl}\, B,$$

(50) $$\textbf{for } A,B \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ holds } \text{Cl}\,(A \cup B) = \text{Cl}\, A \cup \text{Cl}\, B,$$

(51) $$\textbf{for } A,B \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ holds } \text{Cl}\,(A \cap B) \subseteq (\text{Cl}\, A) \cap \text{Cl}\, B,$$

(52) $$\textbf{for } A \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ holds } A \text{ is\_closed } \textbf{iff } \text{Cl}\, A = A,$$

(53) $$\textbf{for } A \textbf{ being } \text{Subset } \textbf{of } GX$$
$$\textbf{holds } A \text{ is\_open } \textbf{iff } \text{Cl}\,(\Omega\,(GX) \setminus A) = \Omega\,(GX) \setminus A,$$

(54) $$\textbf{for } A \textbf{ being } \text{Subset } \textbf{of } GX, p \textbf{ being } \text{Point } \textbf{of } GX \textbf{ holds } p \in \text{Cl}\, A \textbf{ iff}$$
$$\textbf{for } G \textbf{ being } \text{Subset } \textbf{of } GX$$
$$\textbf{st } G \text{ is\_open } \textbf{holds } p \in G \textbf{ implies } A \cap G \neq \emptyset(GX).$$

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Subsets of Topological Spaces

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Summary. The article contains some theorems about open and closed sets. The following topological operations on sets are defined: closure, interior and frontier. The following notions are introduced: dense set, boundary set, nowheredense set and set being domain (closed domain and open domain), and some basic facts concerning them are proved.

The papers [4], [5], [3], [1], and [2] provide the notation and terminology for this paper. For simplicity we adopt the following convention: TS denotes an object of the type TopSpace; x denotes an object of the type Any; P, Q, G denote objects of the type Subset **of** TS; p denotes an object of the type Point **of** TS. One can prove the following propositions:

(1) $$x \in P \textbf{ implies } x \textbf{ is } \mathrm{Point} \textbf{ of } TS,$$

(2) $$P \cup \Omega\, TS = \Omega\, TS \;\&\; \Omega\, TS \cup P = \Omega\, TS,$$

(3) $$P \cap \Omega\, TS = P \;\&\; \Omega\, TS \cap P = P,$$

(4) $$P \cap \emptyset TS = \emptyset TS \;\&\; \emptyset TS \cap P = \emptyset TS,$$

(5) $$P^{\,\mathrm{c}} = \Omega\, TS \setminus P,$$

(6) $$P^{\,\mathrm{c}} = (P \textbf{ qua } \mathrm{Subset} \textbf{ of the } \mathrm{carrier} \textbf{ of } TS)^{\,\mathrm{c}},$$

(7) $$p \in P^{\,\mathrm{c}} \textbf{ iff not } p \in P,$$

(8) $$(\Omega\, TS)^{\,\mathrm{c}} = \emptyset TS,$$

(9) $$\Omega\, TS = (\emptyset TS)^{\,\mathrm{c}},$$

(10) $$(P^{\,\mathrm{c}})^{\,\mathrm{c}} = P,$$

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$$P \cup P^{\,\mathrm{c}} = \Omega\, TS \mathbin{\&} P^{\,\mathrm{c}} \cup P = \Omega\, TS, \tag{11}$$

$$P \cap P^{\,\mathrm{c}} = \emptyset TS \mathbin{\&} P^{\,\mathrm{c}} \cap P = \emptyset TS, \tag{12}$$

$$(P \cup Q)^{\,\mathrm{c}} = (P^{\,\mathrm{c}}) \cap (Q^{\,\mathrm{c}}), \tag{13}$$

$$(P \cap Q)^{\,\mathrm{c}} = (P^{\,\mathrm{c}}) \cup (Q^{\,\mathrm{c}}), \tag{14}$$

$$P \subseteq Q \text{ \textbf{iff} } Q^{\,\mathrm{c}} \subseteq P^{\,\mathrm{c}}, \tag{15}$$

$$P \setminus Q = P \cap Q^{\,\mathrm{c}}, \tag{16}$$

$$(P \setminus Q)^{\,\mathrm{c}} = P^{\,\mathrm{c}} \cup Q, \tag{17}$$

$$P \subseteq Q^{\,\mathrm{c}} \text{ \textbf{implies} } Q \subseteq P^{\,\mathrm{c}}, \tag{18}$$

$$P^{\,\mathrm{c}} \subseteq Q \text{ \textbf{implies} } Q^{\,\mathrm{c}} \subseteq P, \tag{19}$$

$$P \subseteq Q \text{ \textbf{iff} } P \cap Q^{\,\mathrm{c}} = \emptyset, \tag{20}$$

$$P^{\,\mathrm{c}} = Q^{\,\mathrm{c}} \text{ \textbf{implies} } P = Q, \tag{21}$$

$$\emptyset TS \text{ is\_closed}, \tag{22}$$

$$\mathrm{Cl}\,(\emptyset TS) = \emptyset TS, \tag{23}$$

$$P \subseteq \mathrm{Cl}\, P, \tag{24}$$

$$P \subseteq Q \text{ \textbf{implies} } \mathrm{Cl}\, P \subseteq \mathrm{Cl}\, Q, \tag{25}$$

$$\mathrm{Cl}\,(\mathrm{Cl}\, P) = \mathrm{Cl}\, P, \tag{26}$$

$$\mathrm{Cl}\,(\Omega\, TS) = \Omega\, TS, \tag{27}$$

$$\Omega\, TS \text{ is\_closed}, \tag{28}$$

$$P \text{ is\_closed \textbf{iff} } P^{\,\mathrm{c}} \text{ is\_open}, \tag{29}$$

$$P \text{ is\_open \textbf{iff} } P^{\,\mathrm{c}} \text{ is\_closed}, \tag{30}$$

$$Q \text{ is\_closed} \mathbin{\&} P \subseteq Q \text{ \textbf{implies} } \mathrm{Cl}\, P \subseteq Q, \tag{31}$$

$$\mathrm{Cl}\, P \setminus \mathrm{Cl}\, Q \subseteq \mathrm{Cl}\,(P \setminus Q), \tag{32}$$

$$\mathrm{Cl}\,(P \cap Q) \subseteq \mathrm{Cl}\, P \cap \mathrm{Cl}\, Q, \tag{33}$$

$$P \text{ is\_closed} \mathbin{\&} Q \text{ is\_closed \textbf{implies} } \mathrm{Cl}\,(P \cap Q) = \mathrm{Cl}\, P \cap \mathrm{Cl}\, Q, \tag{34}$$

$$P \text{ is\_closed} \mathbin{\&} Q \text{ is\_closed \textbf{implies} } P \cap Q \text{ is\_closed}, \tag{35}$$

$$P \text{ is\_closed} \ \& \ Q \text{ is\_closed } \mathbf{implies} \ P \cup Q \text{ is\_closed}, \tag{36}$$

$$P \text{ is\_open} \ \& \ Q \text{ is\_open } \mathbf{implies} \ P \cup Q \text{ is\_open}, \tag{37}$$

$$P \text{ is\_open} \ \& \ Q \text{ is\_open } \mathbf{implies} \ P \cap Q \text{ is\_open}, \tag{38}$$

$$p \in \mathrm{Cl}\, P \ \mathbf{iff} \ \mathbf{for} \ G \ \mathbf{st} \ G \text{ is\_open } \mathbf{holds} \ p \in G \ \mathbf{implies} \ P \cap G \neq \emptyset, \tag{39}$$

$$Q \text{ is\_open } \mathbf{implies} \ Q \cap \mathrm{Cl}\, P \subseteq \mathrm{Cl}\,(Q \cap P), \tag{40}$$

$$Q \text{ is\_open } \mathbf{implies} \ \mathrm{Cl}\,(Q \cap \mathrm{Cl}\, P) = \mathrm{Cl}\,(Q \cap P). \tag{41}$$

Let us consider TS, P. The functor

$$\mathrm{Int}\, P,$$

yields the type Subset **of** TS and is defined by

$$\mathbf{it} = (\mathrm{Cl}\,(P^{\mathrm{c}}))^{\mathrm{c}}.$$

One can prove the following propositions:

$$\mathrm{Int}\, P = (\mathrm{Cl}\, P^{\mathrm{c}})^{\mathrm{c}}, \tag{42}$$

$$\mathrm{Int}\,(\Omega\, TS) = \Omega\, TS, \tag{43}$$

$$\mathrm{Int}\, P \subseteq P, \tag{44}$$

$$\mathrm{Int}\,(\mathrm{Int}\, P) = \mathrm{Int}\, P, \tag{45}$$

$$\mathrm{Int}\, P \cap \mathrm{Int}\, Q = \mathrm{Int}\,(P \cap Q), \tag{46}$$

$$\mathrm{Int}\,(\emptyset TS) = \emptyset TS, \tag{47}$$

$$P \subseteq Q \ \mathbf{implies} \ \mathrm{Int}\, P \subseteq \mathrm{Int}\, Q, \tag{48}$$

$$\mathrm{Int}\, P \cup \mathrm{Int}\, Q \subseteq \mathrm{Int}\,(P \cup Q), \tag{49}$$

$$\mathrm{Int}\,(P \setminus Q) \subseteq \mathrm{Int}\, P \setminus \mathrm{Int}\, Q, \tag{50}$$

$$\mathrm{Int}\, P \text{ is\_open}, \tag{51}$$

$$\emptyset TS \text{ is\_open}, \tag{52}$$

$$\Omega\, TS \text{ is\_open}, \tag{53}$$

$$x \in \mathrm{Int}\, P \ \mathbf{iff} \ \mathbf{ex} \ Q \ \mathbf{st} \ Q \text{ is\_open} \ \& \ Q \subseteq P \ \& \ x \in Q, \tag{54}$$

$$P \text{ is\_open } \mathbf{iff} \ \mathrm{Int}\, P = P, \tag{55}$$

$$Q \text{ is\_open} \ \& \ Q \subseteq P \ \mathbf{implies} \ Q \subseteq \mathrm{Int}\, P, \tag{56}$$

$$P \text{ is\_open } \textbf{iff for } x \textbf{ holds } x \in P \textbf{ iff ex } Q \textbf{ st } Q \text{ is\_open } \& \ Q \subseteq P \ \& \ x \in Q, \tag{57}$$

$$\mathrm{Cl}\,(\mathrm{Int}\,P) = \mathrm{Cl}\,(\mathrm{Int}\,(\mathrm{Cl}\,(\mathrm{Int}\,P))), \tag{58}$$

$$P \text{ is\_open } \textbf{implies } \mathrm{Cl}\,(\mathrm{Int}\,(\mathrm{Cl}\,P)) = \mathrm{Cl}\,P. \tag{59}$$

Let us consider TS, P. The functor

$$\mathrm{Fr}\,P,$$

yields the type Subset **of** TS and is defined by

$$\textbf{it} = \mathrm{Cl}\,P \cap \mathrm{Cl}\,(P^{\,\mathrm{c}}).$$

We now state a number of propositions:

$$\mathrm{Fr}\,P = \mathrm{Cl}\,P \cap \mathrm{Cl}\,(P^{\,\mathrm{c}}), \tag{60}$$

$$p \in \mathrm{Fr}\,P \textbf{ iff for } Q \textbf{ st } Q \text{ is\_open } \& \ p \in Q \textbf{ holds } P \cap Q \neq \emptyset \ \& \ P^{\,\mathrm{c}} \cap Q \neq \emptyset, \tag{61}$$

$$\mathrm{Fr}\,P = \mathrm{Fr}\,(P^{\,\mathrm{c}}), \tag{62}$$

$$\mathrm{Fr}\,P \subseteq \mathrm{Cl}\,P, \tag{63}$$

$$\mathrm{Fr}\,P = \mathrm{Cl}\,(P^{\,\mathrm{c}}) \cap P \cup (\mathrm{Cl}\,P \setminus P), \tag{64}$$

$$\mathrm{Cl}\,P = P \cup \mathrm{Fr}\,P, \tag{65}$$

$$\mathrm{Fr}\,(P \cap Q) \subseteq \mathrm{Fr}\,P \cup \mathrm{Fr}\,Q, \tag{66}$$

$$\mathrm{Fr}\,(P \cup Q) \subseteq \mathrm{Fr}\,P \cup \mathrm{Fr}\,Q, \tag{67}$$

$$\mathrm{Fr}\,(\mathrm{Fr}\,P) \subseteq \mathrm{Fr}\,P, \tag{68}$$

$$P \text{ is\_closed } \textbf{implies } \mathrm{Fr}\,P \subseteq P, \tag{69}$$

$$\mathrm{Fr}\,P \cup \mathrm{Fr}\,Q = \mathrm{Fr}\,(P \cup Q) \cup \mathrm{Fr}\,(P \cap Q) \cup (\mathrm{Fr}\,P \cap \mathrm{Fr}\,Q), \tag{70}$$

$$\mathrm{Fr}\,(\mathrm{Int}\,P) \subseteq \mathrm{Fr}\,P, \tag{71}$$

$$\mathrm{Fr}\,(\mathrm{Cl}\,P) \subseteq \mathrm{Fr}\,P, \tag{72}$$

$$\mathrm{Int}\,P \cap \mathrm{Fr}\,P = \emptyset, \tag{73}$$

$$\mathrm{Int}\,P = P \setminus \mathrm{Fr}\,P, \tag{74}$$

$$\mathrm{Fr}\,(\mathrm{Fr}\,(\mathrm{Fr}\,P)) = \mathrm{Fr}\,(\mathrm{Fr}\,P), \tag{75}$$

$$P \text{ is\_open } \textbf{iff } \mathrm{Fr}\,P = \mathrm{Cl}\,P \setminus P, \tag{76}$$

$$P \text{ is\_closed } \textbf{iff } \mathrm{Fr}\,P = P \setminus \mathrm{Int}\,P. \tag{77}$$

Let us consider TS, P. The predicate

$$P \text{ is\_dense} \qquad \text{is defined by} \qquad \mathrm{Cl}\, P = \Omega\, TS.$$

We now state several propositions:

(78) $$P \text{ is\_dense } \textbf{iff } \mathrm{Cl}\, P = \Omega\, TS,$$

(79) $$P \text{ is\_dense } \& \ P \subseteq Q \ \textbf{implies } Q \text{ is\_dense}\,,$$

(80) $$P \text{ is\_dense } \textbf{iff for } Q \ \textbf{st } Q \neq \emptyset \ \& \ Q \text{ is\_open } \textbf{holds } P \cap Q \neq \emptyset,$$

(81) $$P \text{ is\_dense } \textbf{implies for } Q \ \textbf{holds } Q \text{ is\_open } \textbf{implies } \mathrm{Cl}\, Q = \mathrm{Cl}\,(Q \cap P),$$

(82) $$P \text{ is\_dense } \& \ Q \text{ is\_dense } \& \ Q \text{ is\_open } \textbf{implies } P \cap Q \text{ is\_dense}\,.$$

Let us consider TS, P. The predicate

$$P \text{ is\_boundary} \qquad \text{is defined by} \qquad P^{\,\mathrm{c}} \text{ is\_dense}\,.$$

Next we state several propositions:

(83) $$P \text{ is\_boundary } \textbf{iff } P^{\,\mathrm{c}} \text{ is\_dense}\,,$$

(84) $$P \text{ is\_boundary } \textbf{iff } \mathrm{Int}\, P = \emptyset,$$

(85) $$P \text{ is\_boundary } \& \ Q \text{ is\_boundary } \& \ Q \text{ is\_closed } \textbf{implies } P \cup Q \text{ is\_boundary}\,,$$

(86) $$P \text{ is\_boundary } \textbf{iff for } Q \ \textbf{st } Q \subseteq P \ \& \ Q \text{ is\_open } \textbf{holds } Q = \emptyset,$$

(87) $$P \text{ is\_closed } \textbf{implies } (P \text{ is\_boundary } \textbf{iff for } Q$$
$$\textbf{st } Q \neq \emptyset \ \& \ Q \text{ is\_open } \textbf{ex } G \ \textbf{st } G \subseteq Q \ \& \ G \neq \emptyset \ \& \ G \text{ is\_open } \& \ P \cap G = \emptyset),$$

(88) $$P \text{ is\_boundary } \textbf{iff } P \subseteq \mathrm{Fr}\, P.$$

Let us consider TS, P. The predicate

$$P \text{ is\_nowheredense} \qquad \text{is defined by} \qquad \mathrm{Cl}\, P \text{ is\_boundary}\,.$$

One can prove the following propositions:

(89) $$P \text{ is\_nowheredense } \textbf{iff } \mathrm{Cl}\, P \text{ is\_boundary}\,,$$

(90) $$P \text{ is\_nowheredense } \& \ Q \text{ is\_nowheredense } \textbf{implies } P \cup Q \text{ is\_nowheredense}\,,$$

(91) $$P \text{ is\_nowheredense } \textbf{implies } P^{\,\mathrm{c}} \text{ is\_dense}\,,$$

(92) $$P \text{ is\_nowheredense } \textbf{implies } P \text{ is\_boundary}\,,$$

(93) $$Q \text{ is\_boundary } \& \ Q \text{ is\_closed } \textbf{implies } Q \text{ is\_nowheredense}\,,$$

(94) $$P \text{ is\_closed } \textbf{implies } (P \text{ is\_nowheredense } \textbf{iff } P = \mathrm{Fr}\, P),$$

(95) $$P \text{ is\_open } \textbf{implies } \mathrm{Fr}\, P \text{ is\_nowheredense}\,,$$

(96) $$P \text{ is\_closed } \textbf{implies } \mathrm{Fr}\, P \text{ is\_nowheredense}\,,$$

(97) $$P \text{ is\_open } \& \ P \text{ is\_nowheredense } \textbf{implies } P = \emptyset.$$

We now define three new predicates. Let us consider TS, P. The predicate

$$P \text{ is\_domain} \qquad \text{is defined by} \qquad \mathrm{Int}\,(\mathrm{Cl}\, P) \subseteq P \ \& \ P \subseteq \mathrm{Cl}\,(\mathrm{Int}\, P).$$

The predicate

$$P \text{ is\_closed\_domain} \qquad \text{is defined by} \qquad P = \mathrm{Cl}\,(\mathrm{Int}\, P).$$

The predicate

$$P \text{ is\_open\_domain} \qquad \text{is defined by} \qquad P = \mathrm{Int}\,(\mathrm{Cl}\, P).$$

The following propositions are true:

(98) $$P \text{ is\_domain } \textbf{iff } \mathrm{Int}\,(\mathrm{Cl}\, P) \subseteq P \ \& \ P \subseteq \mathrm{Cl}\,(\mathrm{Int}\, P),$$

(99) $$P \text{ is\_closed\_domain } \textbf{iff } P = \mathrm{Cl}\,(\mathrm{Int}\, P),$$

(100) $$P \text{ is\_open\_domain } \textbf{iff } P = \mathrm{Int}\,(\mathrm{Cl}\, P),$$

(101) $$P \text{ is\_open\_domain } \textbf{iff } P^{\mathrm{c}} \text{ is\_closed\_domain}\,,$$

(102) $$P \text{ is\_closed\_domain } \textbf{implies } \mathrm{Fr}\,(\mathrm{Int}\, P) = \mathrm{Fr}\, P,$$

(103) $$P \text{ is\_closed\_domain } \textbf{implies } \mathrm{Fr}\, P \subseteq \mathrm{Cl}\,(\mathrm{Int}\, P),$$

(104) $$P \text{ is\_open\_domain } \textbf{implies } \mathrm{Fr}\, P = \mathrm{Fr}\,(\mathrm{Cl}\, P) \ \& \ \mathrm{Fr}\,(\mathrm{Cl}\, P) = \mathrm{Cl}\, P \setminus P,$$

(105) $$P \text{ is\_open } \& \ P \text{ is\_closed } \textbf{implies } (P \text{ is\_closed\_domain } \textbf{iff } P \text{ is\_open\_domain}),$$

(106) $$P \text{ is\_closed } \& \ P \text{ is\_domain } \textbf{iff } P \text{ is\_closed\_domain}\,,$$

(107) $$P \text{ is\_open } \& \ P \text{ is\_domain } \textbf{iff } P \text{ is\_open\_domain}\,,$$

(108) $$P \text{ is\_closed\_domain } \& \ Q \text{ is\_closed\_domain } \textbf{implies } P \cup Q \text{ is\_closed\_domain}\,,$$

(109) $$P \text{ is\_open\_domain } \& \ Q \text{ is\_open\_domain } \textbf{implies } P \cap Q \text{ is\_open\_domain}\,,$$

(110) $$P \text{ is\_domain } \textbf{implies } \mathrm{Int}\,(\mathrm{Fr}\, P) = \emptyset,$$

(111) $$P \text{ is\_domain } \textbf{implies } \mathrm{Int}\, P \text{ is\_domain } \& \ \mathrm{Cl}\, P \text{ is\_domain}\,.$$

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Connected Spaces

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Summary. The following notions are defined: separated sets, connected spaces, connected sets, components of a topological space, the component of a point. The definition of the boundary of a set is also included. The singleton of a point of a topological space is redefined as a subset of the space. Some theorems about these notions are proved.

The articles [3], [4], [1], [2], and [5] provide the notation and terminology for this paper. For simplicity we adopt the following convention: GX, GY will have the type TopSpace; A, $A1$, B, $B1$, C will have the type Subset **of** GX. The arguments of the notions defined below are the following: GX which is an object of the type TopSpace; A, B which are objects of the type Subset **of** GX. The predicate

A, B are_separated is defined by $\mathrm{Cl}\, A \cap B = \emptyset(GX)$ & $A \cap \mathrm{Cl}\, B = \emptyset(GX)$.

The following propositions are true:

(1) A, B are_separated **implies** B, A are_separated ,

(2) A, B are_separated **implies** $A \cap B = \emptyset(GX)$,

(3) $\Omega\,(GX) = A \cup B$ & A is_closed & B is_closed & $A \cap B = \emptyset(GX)$ **implies** A, B are_separated ,

(4) $\Omega\,(GX) = A \cup B$ & A is_open & B is_open & $A \cap B = \emptyset(GX)$ **implies** A, B are_separated ,

(5) $\Omega\,(GX) = A \cup B$ & A, B are_separated **implies** A is_open_closed & B is_open_closed ,

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(6) $$\textbf{for } X' \textbf{ being } \text{SubSpace } \textbf{of } GX,\ P1,Q1 \textbf{ being } \text{Subset } \textbf{of } GX,\ P,Q \textbf{ being } \text{Subset } \textbf{of } X' \textbf{ st } P = P1 \ \&\ Q = Q1 \textbf{ holds } P,Q \text{ are\_separated } \textbf{implies } P1,Q1 \text{ are\_separated},$$

(7) $$\textbf{for } X' \textbf{ being } \text{SubSpace } \textbf{of } GX,\ P,Q \textbf{ being } \text{Subset } \textbf{of } GX,\ P1,Q1 \textbf{ being } \text{Subset } \textbf{of } X' \textbf{ st } P = P1 \ \&\ Q = Q1 \ \&\ P \cup Q \subseteq \Omega\,(X') \textbf{ holds } P,Q \text{ are\_separated } \textbf{implies } P1,Q1 \text{ are\_separated},$$

(8) $$A,B \text{ are\_separated } \&\ A1 \subseteq A \ \&\ B1 \subseteq B \textbf{ implies } A1,B1 \text{ are\_separated},$$

(9) $$A,B \text{ are\_separated } \&\ A,C \text{ are\_separated } \textbf{implies } A,B \cup C \text{ are\_separated},$$

(10) $$A \text{ is\_closed } \&\ B \text{ is\_closed } \textbf{or } A \text{ is\_open } \&\ B \text{ is\_open } \textbf{implies } A \setminus B, B \setminus A \text{ are\_separated}.$$

Let GX have the type TopSpace. The predicate

$$GX \text{ is\_connected}$$

is defined by

$$\textbf{for } A,B \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ st } \Omega\,(GX) = A \cup B \ \&\ A,B \text{ are\_separated } \textbf{holds } A = \emptyset(GX) \textbf{ or } B = \emptyset(GX).$$

One can prove the following propositions:

(11) $$GX \text{ is\_connected } \textbf{iff for } A,B \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ st } \Omega\,(GX) = A \cup B \ \&\ A \neq \emptyset(GX) \ \&\ B \neq \emptyset(GX) \ \&\ A \text{ is\_closed } \&\ B \text{ is\_closed } \textbf{holds } A \cap B \neq \emptyset(GX),$$

(12) $$GX \text{ is\_connected } \textbf{iff for } A,B \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ st } \Omega\,(GX) = A \cup B \ \&\ A \neq \emptyset(GX) \ \&\ B \neq \emptyset(GX) \ \&\ A \text{ is\_open } \&\ B \text{ is\_open } \textbf{holds } A \cap B \neq \emptyset(GX),$$

(13) $$GX \text{ is\_connected } \textbf{iff for } A \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ st } A \neq \emptyset(GX) \ \&\ A \neq \Omega\,(GX) \textbf{ holds } (\mathrm{Cl}\,A) \cap \mathrm{Cl}\,(\Omega\,(GX) \setminus A) \neq \emptyset(GX),$$

(14) $$GX \text{ is\_connected } \textbf{iff for } A \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ st } A \text{ is\_open\_closed } \textbf{holds } A = \emptyset(GX) \textbf{ or } A = \Omega\,(GX),$$

(15) $$\textbf{for } F \textbf{ being } \text{map } \textbf{of } GX,GY \textbf{ st } F \text{ is\_continuous } \&\ F^{\circ}\,(\Omega\,(GX)) = \Omega\,(GY) \ \&\ GX \text{ is\_connected } \textbf{holds } GY \text{ is\_connected}.$$

The arguments of the notions defined below are the following: GX which is an object of the type TopSpace; A which is an object of the type Subset **of** GX. The predicate

A is_connected is defined by $GX \mid A$ is_connected .

One can prove the following propositions:

(16) $A \neq \emptyset(GX)$ **implies** (A is_connected **iff for** P,Q **being** Subset **of** GX **st** $A = P \cup Q$ & P, Q are_separated **holds** $P = \emptyset(GX)$ **or** $Q = \emptyset(GX)$),

(17) A is_connected & $A \subseteq B \cup C$ & B, C are_separated **implies** $A \subseteq B$ **or** $A \subseteq C$,

(18) A is_connected & B is_connected & **not** A, B are_separated **implies** $A \cup B$ is_connected ,

(19) $C \neq \emptyset(GX)$ & C is_connected & $C \subseteq A$ & $A \subseteq \mathrm{Cl}\, C$ **implies** A is_connected ,

(20) $A \neq \emptyset(GX)$ & A is_connected **implies** $\mathrm{Cl}\, A$ is_connected ,

(21) GX is_connected & $A \neq \emptyset(GX)$ & A is_connected & $\Omega\,(GX) \setminus A = B \cup C$ & B, C are_separated **implies** $A \cup B$ is_connected & $A \cup C$ is_connected ,

(22) $\Omega\,(GX) \setminus A = B \cup C$ & B, C are_separated & A is_closed **implies** $A \cup B$ is_closed & $A \cup C$ is_closed ,

(23) C is_connected & $C \cap A \neq \emptyset(GX)$ & $C \setminus A \neq \emptyset(GX)$ **implies** $C \cap \mathrm{Fr}\, A \neq \emptyset(GX)$,

(24) **for** X' **being** SubSpace **of** GX, A **being** Subset **of** GX, B **being** Subset **of** X' **st** $A \neq \emptyset(GX)$ & $A = B$ **holds** A is_connected **iff** B is_connected ,

(25) $A \cap B \neq \emptyset(GX)$ & A is_closed & B is_closed **implies** ($A \cup B$ is_connected & $A \cap B$ is_connected **implies** A is_connected & B is_connected),

(26) **for** F **being** Subset-Family **of** GX **st** (**for** A **being** Subset **of** GX **st** $A \in F$ **holds** A is_connected) & **ex** A **being** Subset **of** GX **st** $A \neq \emptyset(GX)$ & $A \in F$ & **for** B **being** Subset **of** GX **st** $B \in F$ & $B \neq A$ **holds not** A, B are_separated **holds** $\bigcup F$ is_connected ,

(27) **for** F **being** Subset-Family **of** GX **st** (**for** A **being** Subset **of** GX **st** $A \in F$ **holds** A is_connected) & $\bigcap F \neq \emptyset(GX)$ **holds** $\bigcup F$ is_connected ,

(28) $$\Omega\,(GX)\ \mathrm{is\_connected}\ \textbf{iff}\ GX\ \mathrm{is\_connected}\,.$$

The arguments of the notions defined below are the following: GX which is an object of the type TopSpace; x which is an object of the type Point **of** GX. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\{x\} \qquad \text{is} \qquad \text{Subset}\ \textbf{of}\ GX.$$

We now state a proposition

(29) $$\textbf{for}\ x\ \textbf{being}\ \text{Point}\ \textbf{of}\ GX\ \textbf{holds}\ \{x\}\ \mathrm{is\_connected}\,.$$

The arguments of the notions defined below are the following: GX which is an object of the type TopSpace; x, y which are objects of the type Point **of** GX. The predicate

$$x, y\ \mathrm{are\_joined}$$

is defined by

$$\textbf{ex}\ C\ \textbf{being}\ \text{Subset}\ \textbf{of}\ GX\ \textbf{st}\ C\ \mathrm{is\_connected}\ \&\ x \in C\ \&\ y \in C.$$

We now state four propositions:

(30) $$(\textbf{ex}\ x\ \textbf{being}\ \text{Point}\ \textbf{of}\ GX\ \textbf{st}\ \textbf{for}\ y\ \textbf{being}\ \text{Point}\ \textbf{of}\ GX\ \textbf{holds}\ x, y\ \mathrm{are\_joined})$$
$$\textbf{implies}\ GX\ \mathrm{is\_connected}\,,$$

(31) $$(\textbf{ex}\ x\ \textbf{being}\ \text{Point}\ \textbf{of}\ GX\ \textbf{st}\ \textbf{for}\ y\ \textbf{being}\ \text{Point}\ \textbf{of}\ GX\ \textbf{holds}\ x, y\ \mathrm{are\_joined})$$
$$\textbf{iff}\ \textbf{for}\ x,y\ \textbf{being}\ \text{Point}\ \textbf{of}\ GX\ \textbf{holds}\ x, y\ \mathrm{are\_joined}\,,$$

(32) $$(\textbf{for}\ x,y\ \textbf{being}\ \text{Point}\ \textbf{of}\ GX\ \textbf{holds}\ x, y\ \mathrm{are\_joined})\ \textbf{implies}\ GX\ \mathrm{is\_connected}\,,$$

(33) $$\textbf{for}\ x\ \textbf{being}\ \text{Point}\ \textbf{of}\ GX, F\ \textbf{being}\ \text{Subset-Family}\ \textbf{of}\ GX\ \textbf{st}$$
$$\textbf{for}\ A\ \textbf{being}\ \text{Subset}\ \textbf{of}\ GX\ \textbf{holds}\ A \in F\ \textbf{iff}\ A\ \mathrm{is\_connected}\ \&\ x \in A$$
$$\textbf{holds}\ F \neq \emptyset.$$

The arguments of the notions defined below are the following: GX which is an object of the type TopSpace; A which is an object of the type Subset **of** GX. The predicate

$$A\ \mathrm{is\_a\_component\_of}\ GX$$

is defined by

$$A\ \mathrm{is\_connected}$$
$$\&\ \textbf{for}\ B\ \textbf{being}\ \text{Subset}\ \textbf{of}\ GX\ \textbf{st}\ B\ \mathrm{is\_connected}\ \textbf{holds}\ A \subseteq B\ \textbf{implies}\ A = B.$$

The following propositions are true:

(34) $$A\ \mathrm{is\_a\_component\_of}\ GX\ \textbf{implies}\ A \neq \emptyset(GX),$$

(35) $$A \text{ is\_a\_component\_of } GX \textbf{ implies } A \text{ is\_closed},$$

(36) $$A \text{ is\_a\_component\_of } GX \ \& \ B \text{ is\_a\_component\_of } GX$$
$$\textbf{implies } A = B \textbf{ or } (A \neq B \textbf{ implies } A, B \text{ are\_separated}),$$

(37) $$A \text{ is\_a\_component\_of } GX \ \& \ B \text{ is\_a\_component\_of } GX$$
$$\textbf{implies } A = B \textbf{ or } (A \neq B \textbf{ implies } A \cap B = \emptyset(GX)),$$

(38) $$C \text{ is\_connected } \textbf{implies for } S \textbf{ being } \text{Subset } \textbf{of } GX$$
$$\textbf{st } S \text{ is\_a\_component\_of } GX \textbf{ holds } C \cap S = \emptyset(GX) \textbf{ or } C \subseteq S.$$

The arguments of the notions defined below are the following: GX which is an object of the type TopSpace; A, B which are objects of the type Subset **of** GX. The predicate

$$B \text{ is\_a\_component\_of } A$$

is defined by

$$\textbf{ex } B1 \textbf{ being } \text{Subset } \textbf{of } GX \mid A \textbf{ st } B1 = B \ \& \ B1 \text{ is\_a\_component\_of } (GX \mid A).$$

We now state a proposition

(39) $$GX \text{ is\_connected} \ \& \ A \neq \Omega\,(GX)$$
$$\& \ A \neq \emptyset(GX) \ \& \ A \text{ is\_connected} \ \& \ C \text{ is\_a\_component\_of } (\Omega\,(GX) \setminus A)$$
$$\textbf{implies } (\Omega\,(GX) \setminus C) \text{ is\_connected}.$$

The arguments of the notions defined below are the following: GX which is an object of the type TopSpace; x which is an object of the type Point **of** GX. The functor

$$\text{skl}\, x,$$

with values of the type Subset **of** GX, is defined by

$$\textbf{ex } F \textbf{ being } \text{Subset-Family } \textbf{of } GX$$
$$\textbf{st } (\textbf{for } A \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ holds } A \in F \textbf{ iff } A \text{ is\_connected} \ \& \ x \in A) \ \& \ \bigcup F = \textbf{it}.$$

In the sequel x has the type Point **of** GX. One can prove the following propositions:

(40) $$x \in \text{skl}\, x,$$

(41) $$\text{skl}\, x \text{ is\_connected},$$

(42) $$C \text{ is\_connected } \textbf{implies } (\text{skl}\, x \subseteq C \textbf{ implies } C = \text{skl}\, x),$$

(43) $$A \text{ is\_a\_component\_of } GX \textbf{ iff ex } x \textbf{ being } \text{Point } \textbf{of } GX \textbf{ st } A = \text{skl}\, x,$$

(44) $$A \text{ is\_a\_component\_of } GX \ \& \ x \in A \textbf{ implies } A = \text{skl}\, x,$$

$$\text{(45)}\qquad \textbf{for } S \textbf{ being } \mathrm{Subset} \textbf{ of } GX \textbf{ st } S = \mathrm{skl}\, x \textbf{ for } p \textbf{ being } \mathrm{Point} \textbf{ of } GX \textbf{ st } p \neq x \ \&\ p \in S \textbf{ holds } \mathrm{skl}\, p = S,$$

$$\text{(46)}\qquad \textbf{for } F \textbf{ being } \text{Subset-Family} \textbf{ of } GX \textbf{ st for } A \textbf{ being } \mathrm{Subset} \textbf{ of } GX \textbf{ holds } A \in F \textbf{ iff } A \ \mathrm{is\_a\_component\_of}\ GX \textbf{ holds } F \ \mathrm{is\_a\_cover\_of}\ GX,$$

$$\text{(47)}\qquad A, B \ \mathrm{are\_separated} \textbf{ iff } \mathrm{Cl}\, A \cap B = \emptyset(GX) \ \&\ A \cap \mathrm{Cl}\, B = \emptyset(GX),$$

$$\text{(48)}\qquad GX \ \mathrm{is\_connected} \textbf{ iff for } A, B \textbf{ being } \mathrm{Subset} \textbf{ of } GX \textbf{ st } \Omega\,(GX) = A \cup B \ \&\ A, B \ \mathrm{are\_separated} \textbf{ holds } A = \emptyset(GX) \textbf{ or } B = \emptyset(GX),$$

$$\text{(49)}\qquad A \ \mathrm{is\_connected} \textbf{ iff } GX \mid A \ \mathrm{is\_connected}\,,$$

$$\text{(50)}\qquad A \ \mathrm{is\_a\_component\_of}\ GX \textbf{ iff } A \ \mathrm{is\_connected} \ \&\ \textbf{for } B \textbf{ being } \mathrm{Subset} \textbf{ of } GX \textbf{ st } B \ \mathrm{is\_connected} \textbf{ holds } A \subseteq B \textbf{ implies } A = B,$$

$$\text{(51)}\qquad B \ \mathrm{is\_a\_component\_of}\ A \textbf{ iff ex } B1 \textbf{ being } \mathrm{Subset} \textbf{ of } GX \mid A \textbf{ st } B1 = B \ \&\ B1 \ \mathrm{is\_a\_component\_of}\ (GX \mid A),$$

$$\text{(52)}\qquad B = \mathrm{skl}\, x \textbf{ iff ex } F \textbf{ being } \text{Subset-Family} \textbf{ of } GX \textbf{ st } (\textbf{for } A \textbf{ being } \mathrm{Subset} \textbf{ of } GX \textbf{ holds } A \in F \textbf{ iff } A \ \mathrm{is\_connected} \ \&\ x \in A) \ \&\ \bigcup F = B.$$

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Basic Functions and Operations on Functions

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Summary. We define the following mappings: the characteristic function of a subset of a set, the inclusion function (injection or embedding), the projections from a Cartesian product onto its arguments and diagonal function (inclusion of a set into its Cartesian square). Some operations on functions are also defined: the products of two functions (the complex function and the more general product-function), the function induced on power sets by the image and inverse-image. Some simple propositions related to the introduced notions are proved.

The terminology and notation used in this paper are introduced in the following papers: [3], [4], [1], and [2]. For simplicity we adopt the following convention: x, y, z, $z1$, $z2$ denote objects of the type Any; A, B, V, X, $X1$, $X2$, Y, $Y1$, $Y2$, Z denote objects of the type set; C, $C1$, $C2$, D, $D1$, $D2$ denote objects of the type DOMAIN. We now state several propositions:

(1) $$A \subseteq Y \textbf{ implies } \mathrm{id}\, A = (\mathrm{id}\, Y) \mid A,$$

(2) $$\textbf{for } f{,}g \textbf{ being } \mathrm{Function} \textbf{ st } X \subseteq \mathrm{dom}\,(g \cdot f) \textbf{ holds } f \,{}^{\circ}\, X \subseteq \mathrm{dom}\, g,$$

(3) $$\textbf{for } f{,}g \textbf{ being } \mathrm{Function} \textbf{ st } X \subseteq \mathrm{dom}\, f \;\&\; f \,{}^{\circ}\, X \subseteq \mathrm{dom}\, g \textbf{ holds } X \subseteq \mathrm{dom}\,(g \cdot f),$$

(4) $$\textbf{for } f{,}g \textbf{ being } \mathrm{Function} \textbf{ st } Y \subseteq \mathrm{rng}\,(g \cdot f) \;\&\; g \,\mathrm{is\_one\text{-}to\text{-}one} \textbf{ holds } g^{-1}\, Y \subseteq \mathrm{rng}\, f,$$

(5) $$\textbf{for } f{,}g \textbf{ being } \mathrm{Function} \textbf{ st } Y \subseteq \mathrm{rng}\, g \;\&\; g^{-1}\, Y \subseteq \mathrm{rng}\, f \textbf{ holds } Y \subseteq \mathrm{rng}\,(g \cdot f).$$

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In the article we present several logical schemes. The scheme *FuncEx_3* concerns a constant $\mathcal{A}$ that has the type set, a constant $\mathcal{B}$ that has the type set and a ternary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex}\, f \textbf{ being } \mathrm{Function}$$
$$\textbf{st } \mathrm{dom}\, f = [:\mathcal{A},\mathcal{B}:] \,\&\, \textbf{for}\, x,y \textbf{ st } x \in \mathcal{A} \,\&\, y \in \mathcal{B} \textbf{ holds } \mathcal{P}[x, y, f\,.\langle x, y\rangle]$$

provided the parameters satisfy the following conditions:

- $\textbf{for}\, x,y,z1,z2 \textbf{ st } x \in \mathcal{A} \,\&\, y \in \mathcal{B} \,\&\, \mathcal{P}[x, y, z1] \,\&\, \mathcal{P}[x, y, z2] \textbf{ holds } z1 = z2,$
- $\textbf{for}\, x,y \textbf{ st } x \in \mathcal{A} \,\&\, y \in \mathcal{B} \textbf{ ex}\, z \textbf{ st } \mathcal{P}[x, y, z].$

The scheme *Lambda_3* concerns a constant $\mathcal{A}$ that has the type set, a constant $\mathcal{B}$ that has the type set and a binary functor $\mathcal{F}$ and states that the following holds

$$\textbf{ex}\, f \textbf{ being } \mathrm{Function}$$
$$\textbf{st } \mathrm{dom}\, f = [:\mathcal{A},\mathcal{B}:] \,\&\, \textbf{for}\, x,y \textbf{ st } x \in \mathcal{A} \,\&\, y \in \mathcal{B} \textbf{ holds } f\,.\langle x, y\rangle = \mathcal{F}(x, y)$$

for all values of the parameters.

We now state a proposition

(6) $$\textbf{for}\, f,g \textbf{ being } \mathrm{Function} \textbf{ st}$$
$$\mathrm{dom}\, f = [:X, Y:]$$
$$\&\, \mathrm{dom}\, g = [:X, Y:] \,\&\, \textbf{for}\, x,y \textbf{ st } x \in X \,\&\, y \in Y \textbf{ holds } f\,.\langle x, y\rangle = g\,.\langle x, y\rangle$$
$$\textbf{holds } f = g.$$

Let f have the type Function. The functor

$$^{\circ} f,$$

yields the type Function and is defined by

$$\mathrm{dom}\, \textbf{it} = \mathrm{bool}\,\mathrm{dom}\, f \,\&\, \textbf{for}\, X \textbf{ st } X \in \mathrm{bool}\,\mathrm{dom}\, f \textbf{ holds } \textbf{it}.X = f\,^{\circ}\, X.$$

The following propositions are true:

(7) $$\textbf{for}\, f,g \textbf{ being } \mathrm{Function} \textbf{ holds } g = {}^{\circ} f$$
$$\textbf{iff } \mathrm{dom}\, g = \mathrm{bool}\,\mathrm{dom}\, f \,\&\, \textbf{for}\, X \textbf{ st } X \in \mathrm{bool}\,\mathrm{dom}\, f \textbf{ holds } g.X = f\,^{\circ}\, X,$$

(8) $$\textbf{for}\, f \textbf{ being } \mathrm{Function} \textbf{ st } X \in \mathrm{dom}\,(^{\circ} f) \textbf{ holds } (^{\circ} f).X = f\,^{\circ}\, X,$$

(9) $$\textbf{for}\, f \textbf{ being } \mathrm{Function} \textbf{ holds } (^{\circ} f).\emptyset = \emptyset,$$

(10) $$\textbf{for}\, f \textbf{ being } \mathrm{Function} \textbf{ holds } \mathrm{rng}\,(^{\circ} f) \subseteq \mathrm{bool}\,\mathrm{rng}\, f,$$

(11) $$\textbf{for}\, f \textbf{ being } \mathrm{Function}$$
$$\textbf{holds } Y \in (^{\circ} f)\,^{\circ}\, A \textbf{ iff ex}\, X \textbf{ st } X \in \mathrm{dom}\,(^{\circ} f) \,\&\, X \in A \,\&\, Y = (^{\circ} f).X,$$

(12) $$\textbf{for } f \textbf{ being } \text{Function} \textbf{ holds } ({}^{\circ} f)\,{}^{\circ}\, A \subseteq \text{bool}\,\text{rng}\, f,$$

(13) $$\textbf{for } f \textbf{ being } \text{Function} \textbf{ holds } ({}^{\circ} f)\,^{-1}\, B \subseteq \text{bool}\,\text{dom}\, f,$$

(14) $$\textbf{for } f \textbf{ being } \text{Function} \textbf{ of } X, D \textbf{ holds } ({}^{\circ} f)\,^{-1}\, B \subseteq \text{bool}\, X,$$

(15) $$\textbf{for } f \textbf{ being } \text{Function} \textbf{ holds } \bigcup(({}^{\circ} f)\,{}^{\circ}\, A) \subseteq f\,{}^{\circ}\,(\bigcup A),$$

(16) $$\textbf{for } f \textbf{ being } \text{Function} \textbf{ st } A \subseteq \text{bool}\,\text{dom}\, f \textbf{ holds } f\,{}^{\circ}\,(\bigcup A) = \bigcup(({}^{\circ} f)\,{}^{\circ}\, A),$$

(17) $$\textbf{for } f \textbf{ being } \text{Function} \textbf{ of } X, D \textbf{ st } A \subseteq \text{bool}\, X \textbf{ holds } f\,{}^{\circ}\,(\bigcup A) = \bigcup(({}^{\circ} f)\,{}^{\circ}\, A),$$

(18) $$\textbf{for } f \textbf{ being } \text{Function} \textbf{ holds } \bigcup(({}^{\circ} f)\,^{-1}\, B) \subseteq f\,^{-1}\,(\bigcup B),$$

(19) $$\textbf{for } f \textbf{ being } \text{Function} \textbf{ st } B \subseteq \text{bool}\,\text{rng}\, f \textbf{ holds } f\,^{-1}\,(\bigcup B) = \bigcup(({}^{\circ} f)\,^{-1}\, B),$$

(20) $$\textbf{for } f, g \textbf{ being } \text{Function} \textbf{ holds } {}^{\circ}\,(g \cdot f) = {}^{\circ}\, g \cdot {}^{\circ}\, f,$$

(21) $$\textbf{for } f \textbf{ being } \text{Function} \textbf{ holds } {}^{\circ}\, f \textbf{ is } \text{Function} \textbf{ of } \text{bool}\,\text{dom}\, f, \text{bool}\,\text{rng}\, f,$$

(22) $$\textbf{for } f \textbf{ being } \text{Function} \textbf{ of } X, Y$$
$$\textbf{st } Y = \emptyset \textbf{ implies } X = \emptyset \textbf{ holds } {}^{\circ}\, f \textbf{ is } \text{Function} \textbf{ of } \text{bool}\, X, \text{bool}\, Y.$$

The arguments of the notions defined below are the following: X, D which are objects of the type reserved above; f which is an object of the type Function **of** X, D. Let us note that it makes sense to consider the following functor on a restricted area. Then

$${}^{\circ} f \qquad \text{is} \qquad \text{Function} \textbf{ of } \text{bool}\, X, \text{bool}\, D.$$

Let f have the type Function. The functor

$$^{-1}\, f,$$

yields the type Function and is defined by

$$\text{dom}\, \textbf{it} = \text{bool}\,\text{rng}\, f \;\&\; \textbf{for } Y \textbf{ st } Y \in \text{bool}\,\text{rng}\, f \textbf{ holds } \textbf{it}.Y = f\,^{-1}\, Y.$$

We now state a number of propositions:

(23) $$\textbf{for } g, f \textbf{ being } \text{Function} \textbf{ holds}$$
$$g = {}^{-1}\, f \textbf{ iff } \text{dom}\, g = \text{bool}\,\text{rng}\, f \;\&\; \textbf{for } Y \textbf{ st } Y \in \text{bool}\,\text{rng}\, f \textbf{ holds } g.Y = f\,^{-1}\, Y,$$

(24) $$\textbf{for } f \textbf{ being } \text{Function} \textbf{ st } Y \in \text{dom}\,({}^{-1} f) \textbf{ holds } ({}^{-1} f).Y = f\,^{-1}\, Y,$$

(25) $$\textbf{for } f \textbf{ being } \text{Function} \textbf{ holds } \text{rng}\,({}^{-1} f) \subseteq \text{bool}\,\text{dom}\, f,$$

(26) $$\textbf{for } f \textbf{ being } \text{Function}$$
$$\textbf{holds } X \in ({}^{-1} f)\,{}^{\circ}\, A \textbf{ iff } \textbf{ex } Y \textbf{ st } Y \in \text{dom}\,({}^{-1} f) \;\&\; Y \in A \;\&\; X = ({}^{-1} f).Y,$$

(27) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{holds } ({}^{-1} f)^{\circ}\, B \subseteq \text{bool}\,\text{dom}\, f,$$

(28) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{holds } ({}^{-1} f)^{-1}\, A \subseteq \text{bool}\,\text{rng}\, f,$$

(29) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{holds } \bigcup(({}^{-1} f)^{\circ}\, B) \subseteq f^{-1}\,(\bigcup B),$$

(30) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{st } B \subseteq \text{bool}\,\text{rng}\, f \textbf{ holds } \bigcup(({}^{-1} f)^{\circ}\, B) = f^{-1}\,(\bigcup B),$$

(31) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{holds } \bigcup(({}^{-1} f)^{-1}\, A) \subseteq f^{\circ}\,(\bigcup A),$$

(32) $$\textbf{for } f \textbf{ being } \text{Function}$$
$$\textbf{st } A \subseteq \text{bool}\,\text{dom}\, f \;\&\; f \text{ is\_one-to-one } \textbf{holds } \bigcup(({}^{-1} f)^{-1}\, A) = f^{\circ}\,(\bigcup A),$$

(33) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{holds } ({}^{-1} f)^{\circ}\, B \subseteq ({}^{\circ} f)^{-1}\, B,$$

(34) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{st } f \text{ is\_one-to-one } \textbf{holds } ({}^{-1} f)^{\circ}\, B = ({}^{\circ} f)^{-1}\, B,$$

(35) $$\textbf{for } f \textbf{ being } \text{Function}, A \textbf{ being } \text{set}$$
$$\textbf{st } A \subseteq \text{bool}\,\text{dom}\, f \textbf{ holds } ({}^{-1} f)^{-1}\, A \subseteq ({}^{\circ} f)^{\circ}\, A,$$

(36) $$\textbf{for } f \textbf{ being } \text{Function}, A \textbf{ being } \text{set}$$
$$\textbf{st } f \text{ is\_one-to-one } \textbf{holds } ({}^{\circ} f)^{\circ}\, A \subseteq ({}^{-1} f)^{-1}\, A,$$

(37) $$\textbf{for } f \textbf{ being } \text{Function}, A \textbf{ being } \text{set}$$
$$\textbf{st } f \text{ is\_one-to-one} \;\&\; A \subseteq \text{bool}\,\text{dom}\, f \textbf{ holds } ({}^{-1} f)^{-1}\, A = ({}^{\circ} f)^{\circ}\, A,$$

(38) $$\textbf{for } f, g \textbf{ being } \text{Function } \textbf{st } g \text{ is\_one-to-one } \textbf{holds } {}^{-1}(g \cdot f) = {}^{-1} f \cdot {}^{-1} g,$$

(39) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{holds } {}^{-1} f \textbf{ is } \text{Function } \textbf{of } \text{bool}\,\text{rng}\, f, \text{bool}\,\text{dom}\, f.$$

Let us consider A, X. The functor

$$\chi\,(A, X),$$

yields the type Function and is defined by

$$\text{dom}\,\textbf{it} = X$$
$$\&\; \textbf{for } x \textbf{ st } x \in X \textbf{ holds } (x \in A \textbf{ implies } \textbf{it}.x = 1) \;\&\; (\textbf{not } x \in A \textbf{ implies } \textbf{it}.x = 0).$$

We now state a number of propositions:

(40) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{holds } f = \chi\,(A, X) \textbf{ iff } \text{dom}\, f = X \;\&\; \textbf{for } x$$
$$\textbf{st } x \in X \textbf{ holds } (x \in A \textbf{ implies } f.x = 1) \;\&\; (\textbf{not } x \in A \textbf{ implies } f.x = 0),$$

(41) $$A \subseteq X \;\&\; x \in A \textbf{ implies } \chi\,(A, X).x = 1,$$

(42) $$x \in X \;\&\; \chi\,(A, X).x = 1 \textbf{ implies } x \in A,$$

(43) $$x \in X \setminus A \textbf{ implies } \chi(A, X).x = 0,$$

(44) $$x \in X \ \& \ \chi(A, X).x = 0 \textbf{ implies not } x \in A,$$

(45) $$x \in X \textbf{ implies } \chi(\emptyset, X).x = 0,$$

(46) $$x \in X \textbf{ implies } \chi(X, X).x = 1,$$

(47) $$A \subseteq X \ \& \ B \subseteq X \ \& \ \chi(A, X) = \chi(B, X) \textbf{ implies } A = B,$$

(48) $$\operatorname{rng} \chi(A, X) \subseteq \{0,1\},$$

(49) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, \{0,1\} \textbf{ holds } f = \chi(f^{-1} \{1\}, X).$$

Let us consider A, X. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\chi(A, X) \qquad \text{is} \qquad \text{Function } \textbf{of } X, \{0,1\}.$$

One can prove the following propositions:

(50) $$\textbf{for } d \textbf{ being } \text{Element } \textbf{of } D \textbf{ holds } \chi(A, D).d = 1 \textbf{ iff } d \in A,$$

(51) $$\textbf{for } d \textbf{ being } \text{Element } \textbf{of } D \textbf{ holds } \chi(A, D).d = 0 \textbf{ iff not } d \in A.$$

The arguments of the notions defined below are the following: Y which is an object of the type reserved above; A which is an object of the type Subset **of** Y. The functor

$$\operatorname{incl} A,$$

yields the type Function **of** A, Y and is defined by

$$\textbf{it} = \operatorname{id} A.$$

We now state several propositions:

(52) $$\textbf{for } A \textbf{ being } \text{Subset } \textbf{of } Y \textbf{ holds } \operatorname{incl} A = \operatorname{id} A,$$

(53) $$\textbf{for } A \textbf{ being } \text{Subset } \textbf{of } Y \textbf{ holds } \operatorname{incl} A = (\operatorname{id} Y) \mid A,$$

(54) $$\textbf{for } A \textbf{ being } \text{Subset } \textbf{of } Y \textbf{ holds } \operatorname{dom} \operatorname{incl} A = A \ \& \ \operatorname{rng} \operatorname{incl} A = A,$$

(55) $$\textbf{for } A \textbf{ being } \text{Subset } \textbf{of } Y \textbf{ st } x \in A \textbf{ holds } (\operatorname{incl} A).x = x,$$

(56) $$\textbf{for } A \textbf{ being } \text{Subset } \textbf{of } Y \textbf{ st } x \in A \textbf{ holds } \operatorname{incl}(A).x \in Y.$$

We now define two new functors. Let us consider X, Y. The functor

$$\pi_1(X, Y),$$

with values of the type Function, is defined by

$$\operatorname{dom}\textbf{it} = [:X, Y:] \ \&\ \textbf{for}\ x,y\ \textbf{st}\ x \in X \ \&\ y \in Y\ \textbf{holds}\ \textbf{it}.\langle x, y\rangle = x.$$

The functor

$$\pi_2\,(X, Y),$$

yields the type Function and is defined by

$$\operatorname{dom}\textbf{it} = [:X, Y:] \ \&\ \textbf{for}\ x,y\ \textbf{st}\ x \in X \ \&\ y \in Y\ \textbf{holds}\ \textbf{it}.\langle x, y\rangle = y.$$

Next we state several propositions:

(57) $$\textbf{for}\ f\ \textbf{being}\ \text{Function}\ \textbf{holds}\ f = \pi_1\,(X, Y)$$
$$\textbf{iff}\ \operatorname{dom} f = [:X, Y:] \ \&\ \textbf{for}\ x,y\ \textbf{st}\ x \in X \ \&\ y \in Y\ \textbf{holds}\ f.\langle x, y\rangle = x,$$

(58) $$\textbf{for}\ f\ \textbf{being}\ \text{Function}\ \textbf{holds}\ f = \pi_2\,(X, Y)$$
$$\textbf{iff}\ \operatorname{dom} f = [:X, Y:] \ \&\ \textbf{for}\ x,y\ \textbf{st}\ x \in X \ \&\ y \in Y\ \textbf{holds}\ f.\langle x, y\rangle = y,$$

(59) $$\operatorname{rng}\pi_1\,(X, Y) \subseteq X,$$

(60) $$Y \neq \emptyset\ \textbf{implies}\ \operatorname{rng}\pi_1\,(X, Y) = X,$$

(61) $$\operatorname{rng}\pi_2\,(X, Y) \subseteq Y,$$

(62) $$X \neq \emptyset\ \textbf{implies}\ \operatorname{rng}\pi_2\,(X, Y) = Y.$$

Let us consider X, Y. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$\pi_1\,(X, Y) \qquad \text{is} \qquad \text{Function}\ \textbf{of}\ [:X, Y:],X,$$

$$\pi_2\,(X, Y) \qquad \text{is} \qquad \text{Function}\ \textbf{of}\ [:X, Y:],Y.$$

We now state two propositions:

(63) $$\textbf{for}\ d1\ \textbf{being}\ \text{Element}\ \textbf{of}\ D1$$
$$\textbf{for}\ d2\ \textbf{being}\ \text{Element}\ \textbf{of}\ D2\ \textbf{holds}\ \pi_1\,(D1,D2).\langle d1,d2\rangle = d1,$$

(64) $$\textbf{for}\ d1\ \textbf{being}\ \text{Element}\ \textbf{of}\ D1$$
$$\textbf{for}\ d2\ \textbf{being}\ \text{Element}\ \textbf{of}\ D2\ \textbf{holds}\ \pi_2\,(D1,D2).\langle d1,d2\rangle = d2.$$

Let us consider X. The functor

$$\delta\, X,$$

with values of the type Function, is defined by

$$\operatorname{dom}\textbf{it} = X \ \&\ \textbf{for}\ x\ \textbf{st}\ x \in X\ \textbf{holds}\ \textbf{it}.x = \langle x, x\rangle.$$

The following two propositions are true:

(65) $$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ holds } f = \delta\, X \textbf{ iff } \mathrm{dom}\, f = X \ \&\ \textbf{for } x \textbf{ st } x \in X \textbf{ holds } f\,.x = \langle x, x\rangle,$$

(66) $$\mathrm{rng}\,\delta\, X \subseteq [\!:X, X:\!].$$

Let us consider X. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\delta\, X \qquad \text{is} \qquad \mathrm{Function}\ \textbf{of}\ X, [\!:X, X:\!].$$

Let f, g have the type Function. The functor

$$[\!(f, g)\!],$$

with values of the type Function, is defined by

$$\mathrm{dom}\,\textbf{it} = \mathrm{dom}\, f \cap \mathrm{dom}\, g \ \&\ \textbf{for } x \textbf{ st } x \in \mathrm{dom}\,\textbf{it} \textbf{ holds } \textbf{it}.x = \langle f\,.x, g\,.x\rangle.$$

We now state a number of propositions:

(67) $$\textbf{for } f, g, fg \textbf{ being } \mathrm{Function} \textbf{ holds } fg = [\!(f, g)\!] \textbf{ iff } \mathrm{dom}\, fg = \mathrm{dom}\, f \cap \mathrm{dom}\, g \ \&\ \textbf{for } x \textbf{ st } x \in \mathrm{dom}\, fg \textbf{ holds } fg\,.x = \langle f\,.x, g\,.x\rangle,$$

(68) $$\textbf{for } f, g \textbf{ being } \mathrm{Function} \textbf{ st } x \in \mathrm{dom}\, f \cap \mathrm{dom}\, g \textbf{ holds } [\!(f, g)\!].x = \langle f\,.x, g\,.x\rangle,$$

(69) $$\textbf{for } f, g \textbf{ being } \mathrm{Function} \textbf{ st } \mathrm{dom}\, f = X \ \&\ \mathrm{dom}\, g = X \ \&\ x \in X \textbf{ holds } [\!(f, g)\!].x = \langle f\,.x, g\,.x\rangle,$$

(70) $$\textbf{for } f, g \textbf{ being } \mathrm{Function} \textbf{ st } \mathrm{dom}\, f = X \ \&\ \mathrm{dom}\, g = X \textbf{ holds } \mathrm{dom}\,[\!(f, g)\!] = X,$$

(71) $$\textbf{for } f, g \textbf{ being } \mathrm{Function} \textbf{ holds } \mathrm{rng}\,[\!(f, g)\!] \subseteq [\!:\mathrm{rng}\, f, \mathrm{rng}\, g:\!],$$

(72) $$\textbf{for } f, g \textbf{ being } \mathrm{Function} \textbf{ st } \mathrm{dom}\, f = \mathrm{dom}\, g \ \&\ \mathrm{rng}\, f \subseteq Y \ \&\ \mathrm{rng}\, g \subseteq Z \textbf{ holds } \pi_1\,(Y, Z) \cdot [\!(f, g)\!] = f \ \&\ \pi_2\,(Y, Z) \cdot [\!(f, g)\!] = g,$$

(73) $$[\!(\pi_1\,(X, Y), \pi_2\,(X, Y))\!] = \mathrm{id}\,[\!:X, Y:\!],$$

(74) $$\textbf{for } f, g, h, k \textbf{ being } \mathrm{Function} \textbf{ st } \mathrm{dom}\, f = \mathrm{dom}\, g \ \&\ \mathrm{dom}\, k = \mathrm{dom}\, h \ \&\ [\!(f, g)\!] = [\!(k, h)\!] \textbf{ holds } f = k \ \&\ g = h,$$

(75) $$\textbf{for } f, g, h \textbf{ being } \mathrm{Function} \textbf{ holds } [\!(f \cdot h, g \cdot h)\!] = [\!(f, g)\!] \cdot h,$$

(76) $$\textbf{for } f, g \textbf{ being } \mathrm{Function} \textbf{ holds } [\!(f, g)\!] \circ A \subseteq [\!:f \circ A, g \circ A:\!],$$

(77) $$\textbf{for } f, g \textbf{ being } \mathrm{Function} \textbf{ holds } [\!(f, g)\!]^{-1}\,[\!:B, C:\!] = f^{-1}\, B \cap g^{-1}\, C,$$

(78) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ for } g \textbf{ being } \text{Function } \textbf{of } X, Z \textbf{ st } (Y = \emptyset \textbf{ implies } X = \emptyset) \ \& \ (Z = \emptyset \textbf{ implies } X = \emptyset) \textbf{ holds } [\![f, g]\!] \textbf{ is } \text{Function } \textbf{of } X, [\!: Y, Z :\!].$$

The arguments of the notions defined below are the following: X, $D1$, $D2$ which are objects of the type reserved above; $f1$ which is an object of the type Function **of** X, $D1$; $f2$ which is an object of the type Function **of** X, $D2$. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$[\![f1,f2]\!] \qquad \text{is} \qquad \text{Function } \textbf{of } X, [\!: D1,D2 :\!].$$

We now state several propositions:

(79) $$\textbf{for } f1 \textbf{ being } \text{Function } \textbf{of } C, D1 \textbf{ for } f2 \textbf{ being } \text{Function } \textbf{of } C, D2 \textbf{ for } c \textbf{ being } \text{Element } \textbf{of } C \textbf{ holds } [\![f1,f2]\!] . c = \langle f1.c, f2.c \rangle,$$

(80) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ for } g \textbf{ being } \text{Function } \textbf{of } X, Z \textbf{ st } (Y = \emptyset \textbf{ implies } X = \emptyset) \ \& \ (Z = \emptyset \textbf{ implies } X = \emptyset) \textbf{ holds } \operatorname{rng} [\![f, g]\!] \subseteq [\!: Y, Z :\!],$$

(81) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ for } g \textbf{ being } \text{Function } \textbf{of } X, Z \textbf{ st } (Y = \emptyset \textbf{ implies } X = \emptyset) \ \& \ (Z = \emptyset \textbf{ implies } X = \emptyset) \textbf{ holds } \pi_1 (Y, Z) \cdot [\![f, g]\!] = f \ \& \ \pi_2 (Y, Z) \cdot [\![f, g]\!] = g,$$

(82) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D1 \textbf{ for } g \textbf{ being } \text{Function } \textbf{of } X, D2 \textbf{ holds } \pi_1 (D1,D2) \cdot [\![f, g]\!] = f \ \& \ \pi_2 (D1,D2) \cdot [\![f, g]\!] = g,$$

(83) $$\textbf{for } f1,f2 \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ for } g1,g2 \textbf{ being } \text{Function } \textbf{of } X, Z \textbf{ st } (Y = \emptyset \textbf{ implies } X = \emptyset) \ \& \ (Z = \emptyset \textbf{ implies } X = \emptyset) \ \& \ [\![f1,g1]\!] = [\![f2,g2]\!] \textbf{ holds } f1 = f2 \ \& \ g1 = g2,$$

(84) $$\textbf{for } f1,f2 \textbf{ being } \text{Function } \textbf{of } X, D1 \textbf{ for } g1,g2 \textbf{ being } \text{Function } \textbf{of } X, D2 \textbf{ st } [\![f1,g1]\!] = [\![f2,g2]\!] \textbf{ holds } f1 = f2 \ \& \ g1 = g2.$$

Let f, g have the type Function. The functor

$$[\!: f, g :\!],$$

yields the type Function and is defined by

$$\operatorname{dom} \textbf{it} = [\!: \operatorname{dom} f, \operatorname{dom} g :\!] \ \& \ \textbf{for } x,y \textbf{ st } x \in \operatorname{dom} f \ \& \ y \in \operatorname{dom} g \textbf{ holds } \textbf{it} . \langle x, y \rangle = \langle f.x, g.y \rangle.$$

The following propositions are true:

(85) $$\textbf{for } f,g,fg \textbf{ being } \text{Function } \textbf{holds } fg = [\!: f, g :\!] \textbf{ iff } \operatorname{dom} fg = [\!: \operatorname{dom} f, \operatorname{dom} g :\!] \ \& \ \textbf{for } x,y \textbf{ st } x \in \operatorname{dom} f \ \& \ y \in \operatorname{dom} g \textbf{ holds } fg . \langle x, y \rangle = \langle f.x, g.y \rangle,$$

(86) $$\textbf{for } f,g \textbf{ being } \mathrm{Function}, x,y \;\; \textbf{st } \langle x,y\rangle \in [:\mathrm{dom}\, f,\mathrm{dom}\, g:] \textbf{ holds } [:f,g:].\langle x,y\rangle = \langle f.x,g.y\rangle,$$

(87) $$\textbf{for } f,g \textbf{ being } \mathrm{Function} \;\; \textbf{holds } [:f,g:] = [\!\langle f\cdot\pi_1(\mathrm{dom}\, f,\mathrm{dom}\, g), g\cdot\pi_2(\mathrm{dom}\, f,\mathrm{dom}\, g)\rangle\!],$$

(88) $$\textbf{for } f,g \textbf{ being } \mathrm{Function} \textbf{ holds } \mathrm{rng}\,[:f,g:] = [:\mathrm{rng}\, f,\mathrm{rng}\, g:],$$

(89) $$\textbf{for } f,g \textbf{ being } \mathrm{Function} \;\; \textbf{st } \mathrm{dom}\, f = X \;\&\; \mathrm{dom}\, g = X \textbf{ holds } [\!\langle f,g\rangle\!] = [:f,g:]\cdot(\delta\, X),$$

(90) $$[:\mathrm{id}\, X,\mathrm{id}\, Y:] = \mathrm{id}\,[:X,Y:],$$

(91) $$\textbf{for } f,g,h,k \textbf{ being } \mathrm{Function} \textbf{ holds } [:f,h:]\cdot[\!\langle g,k\rangle\!] = [\!\langle f\cdot g,h\cdot k\rangle\!],$$

(92) $$\textbf{for } f,g,h,k \textbf{ being } \mathrm{Function} \textbf{ holds } [:f,h:]\cdot[:g,k:] = [:f\cdot g,h\cdot k:],$$

(93) $$\textbf{for } f,g \textbf{ being } \mathrm{Function} \textbf{ holds } [:f,g:]\,^{\circ}\,[:B,C:] = [:f\,^{\circ}\,B,g\,^{\circ}\,C:],$$

(94) $$\textbf{for } f,g \textbf{ being } \mathrm{Function} \textbf{ holds } [:f,g:]^{-1}\,[:B,C:] = [:f^{-1}\,B,g^{-1}\,C:],$$

(95) $$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ of } X,Y \textbf{ for } g \textbf{ being } \mathrm{Function} \textbf{ of } V,Z \textbf{ st } (Y=\emptyset \textbf{ implies } X=\emptyset) \;\&\; (Z=\emptyset \textbf{ implies } V=\emptyset) \;\; \textbf{holds } [:f,g:] \textbf{ is } \mathrm{Function} \textbf{ of } [:X,V:],[:Y,Z:].$$

The arguments of the notions defined below are the following: $X1$, $X2$, $D1$, $D2$ which are objects of the type reserved above; $f1$ which is an object of the type Function **of** $X1$, $D1$; $f2$ which is an object of the type Function **of** $X2$, $D2$. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$[:f1,f2:] \quad \text{is} \quad \mathrm{Function} \textbf{ of } [:X1,X2:],[:D1,D2:].$$

One can prove the following propositions:

(96) $$\textbf{for } f1 \textbf{ being } \mathrm{Function} \textbf{ of } C1,D1 \textbf{ for } f2 \textbf{ being } \mathrm{Function} \textbf{ of } C2,D2 \;\; \textbf{for } c1 \textbf{ being } \mathrm{Element} \textbf{ of } C1 \;\; \textbf{for } c2 \textbf{ being } \mathrm{Element} \textbf{ of } C2 \textbf{ holds } [:f1,f2:].\langle c1,c2\rangle = \langle f1.c1,f2.c2\rangle,$$

(97) $$\textbf{for } f1 \textbf{ being } \mathrm{Function} \textbf{ of } X1,Y1 \textbf{ for } f2 \textbf{ being } \mathrm{Function} \textbf{ of } X2,Y2 \textbf{ st } (Y1=\emptyset \textbf{ implies } X1=\emptyset) \;\&\; (Y2=\emptyset \textbf{ implies } X2=\emptyset) \;\; \textbf{holds } [:f1,f2:] = [\!\langle f1\cdot\pi_1(X1,X2), f2\cdot\pi_2(X1,X2)\rangle\!],$$

(98) $$\textbf{for } f1 \textbf{ being } \mathrm{Function} \textbf{ of } X1,D1 \textbf{ for } f2 \textbf{ being } \mathrm{Function} \textbf{ of } X2,D2 \;\; \textbf{holds } [:f1,f2:] = [\!\langle f1\cdot\pi_1(X1,X2), f2\cdot\pi_2(X1,X2)\rangle\!],$$

(99) $$\textbf{for } f1 \textbf{ being } \text{Function} \textbf{ of } X, Y1 \textbf{ for } f2 \textbf{ being } \text{Function} \textbf{ of } X, Y2 \textbf{ st}$$
$$(Y1 = \emptyset \textbf{ implies } X = \emptyset) \mathrel{\&} (Y2 = \emptyset \textbf{ implies } X = \emptyset)$$
$$\textbf{holds } [\![f1, f2]\!] = [: f1, f2 :] \cdot (\delta\, X),$$

(100) $$\textbf{for } f1 \textbf{ being } \text{Function} \textbf{ of } X, D1$$
$$\textbf{for } f2 \textbf{ being } \text{Function} \textbf{ of } X, D2 \textbf{ holds } [\![f1, f2]\!] = [: f1, f2 :] \cdot (\delta\, X).$$

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