FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Properties of Fields

Józef Białas[1]
University of Łódź

Summary. The second part of considerations concerning groups and fields. It includes a definition and properties of commutative field F as a structure defined by: the set, a support of F, containing two different elements, by two binary operations $+_F$, $\cdot_F$ on this set, called addition and multiplication, and by two elements from the support of F, $\mathbf{0}_F$ being neutral for addition and $\mathbf{1}_F$ being neutral for multiplication. This structure is named a field if $\langle$the support of F, $+_F$, $\mathbf{0}_F\rangle$ and $\langle$the support of F, $\cdot_F$, $\mathbf{1}_F\rangle$ are commutative groups and multiplication has the property of left-hand and right-hand distributivity with respect to addition. It is demonstrated that the field F satisfies the definition of a field in the axiomatic approach.

MML Identifier: `REALSET2`.

The articles [4], [2], [3], and [1] provide the notation and terminology for this paper. A field structure is said to be a field if:

(Def.1) there exists an at least 2-elements set A and there exists a binary operation o_1 of A and there exists an element n_1 of A and there exists a binary operation o_2 of A preserving $A \setminus \{n_1\}$ and there exists an element n_2 of $A \setminus \mathrm{single}(n_1)$ such that it $= \mathrm{field}(A, o_1, o_2, n_1, n_2)$ and $\mathrm{group}(A, o_1, n_1)$ is a group and for every non-empty set B and for every binary operation P of B and for every element e of B such that $B = A \setminus \mathrm{single}(n_1)$ and $e = n_2$ and $P = o_2 \restriction_{n_1} A$ holds $\mathrm{group}(B, P, e)$ is a group and for all elements x, y, z of A holds $o_2(\langle x, o_1(\langle y, z\rangle)\rangle) = o_1(\langle o_2(\langle x, y\rangle), o_2(\langle x, z\rangle)\rangle)$ and $o_2(\langle o_1(\langle x, y\rangle), z\rangle) = o_1(\langle o_2(\langle x, z\rangle), o_2(\langle y, z\rangle)\rangle)$.

Next we state the proposition

(1) Let F be a field structure. Then F is a field if and only if there exists an at least 2-elements set A and there exists a binary operation o_1 of A and there exists an element n_1 of A and there exists a binary operation o_2 of A preserving $A \setminus \{n_1\}$ and there exists an element n_2 of $A \setminus \mathrm{single}(n_1)$ such

[1]Supported by RPBP.III-24.

ISSN 0777–4028

that $F = \mathrm{field}(A, o_1, o_2, n_1, n_2)$ and $\mathrm{group}(A, o_1, n_1)$ is a group and for every non-empty set B and for every binary operation P of B and for every element e of B such that $B = A \setminus \mathrm{single}(n_1)$ and $e = n_2$ and $P = o_2 \restriction_{n_1} A$ holds $\mathrm{group}(B, P, e)$ is a group and for all elements x, y, z of A holds $o_2(\langle x, o_1(\langle y, z\rangle)\rangle) = o_1(\langle o_2(\langle x, y\rangle), o_2(\langle x, z\rangle)\rangle)$ and $o_2(\langle o_1(\langle x, y\rangle), z\rangle) = o_1(\langle o_2(\langle x, z\rangle), o_2(\langle y, z\rangle)\rangle)$.

Let F be a field. The support of F yielding an at least 2-elements set is defined by:

(Def.2) there exists a binary operation o_1 of the support of F and there exists an element n_1 of the support of F and there exists a binary operation o_2 of the support of F preserving the support of $F \setminus \{n_1\}$ and there exists an element n_2 of (the support of F) $\setminus$ single(n_1) such that $F = \mathrm{field}(\text{the support of } F, o_1, o_2, n_1, n_2)$.

The following proposition is true

(2) For every field F and for every at least 2-elements set A holds $A =$ the support of F if and only if there exists a binary operation o_1 of A and there exists an element n_1 of A and there exists a binary operation o_2 of A preserving $A \setminus \{n_1\}$ and there exists an element n_2 of $A \setminus \mathrm{single}(n_1)$ such that $F = \mathrm{field}(A, o_1, o_2, n_1, n_2)$.

Let F be a field. The functor $+_F$ yielding a binary operation of the support of F
is defined as follows:

(Def.3) there exists an element n_1 of the support of F and there exists a binary operation o_2 of the support of F preserving the support of $F \setminus \{n_1\}$ and there exists an element n_2 of the support of $F \setminus \mathrm{single}(n_1)$ such that $F = \mathrm{field}(\text{the support of } F, +_F, o_2, n_1, n_2)$.

Next we state the proposition

(3) For every field F and for every binary operation o_1 of the support of F holds $o_1 = +_F$ if and only if there exists an element n_1 of the support of F and there exists a binary operation o_2 of the support of F preserving the support of $F \setminus \{n_1\}$ and there exists an element n_2 of the support of $F \setminus \mathrm{single}(n_1)$ such that $F = \mathrm{field}(\text{the support of } F, o_1, o_2, n_1, n_2)$.

Let F be a field. The functor $\mathbf{0}_F$ yielding an element of the support of F is defined by:

(Def.4) there exists a binary operation o_2 of the support of F preserving the support of $F \setminus \{\mathbf{0}_F\}$ and there exists an element n_2 of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ such that $F = \mathrm{field}(\text{the support of } F, +_F, o_2, \mathbf{0}_F, n_2)$.

Next we state the proposition

(4) For every field F and for every element n_1 of the support of F holds $n_1 = \mathbf{0}_F$ if and only if there exists a binary operation o_2 of the support of F preserving the support of $F \setminus \{n_1\}$ and there exists an element n_2 of

the support of $F \setminus \mathrm{single}(n_1)$ such that $F = \mathrm{field}(\text{the support of } F, +_F, o_2, n_1, n_2)$.

Let F be a field. The functor $\cdot_F$ yields a binary operation of the support of F preserving the support of $F \setminus \{\mathbf{0}_F\}$and is defined as follows:

(Def.5) there exists an element n_2 of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ such that $F = \mathrm{field}(\text{the support of } F, +_F, \cdot_F, \mathbf{0}_F, n_2)$.

We now state the proposition

(5) For every field F and for every binary operation o_2 of the support of F preserving the support of $F \setminus \{\mathbf{0}_F\}$ holds $o_2 = \cdot_F$ if and only if there exists an element n_2 of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ such that $F = \mathrm{field}(\text{the support of } F, +_F, o_2, \mathbf{0}_F, n_2)$.

Let F be a field. The functor $\mathbf{1}_F$ yielding an element of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ is defined as follows:

(Def.6) $F = \mathrm{field}(\text{the support of } F, +_F, \cdot_F, \mathbf{0}_F, \mathbf{1}_F)$.

The following propositions are true:

(6) For every field F and for every element n_2 of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $n_2 = \mathbf{1}_F$ if and only if $F = \mathrm{field}(\text{the support of } F, +_F, \cdot_F, \mathbf{0}_F, n_2)$.

(7) For every field F holds $F = \mathrm{field}(\text{the support of } F, +_F, \cdot_F, \mathbf{0}_F, \mathbf{1}_F)$.

(8) For every field F holds group(the support of $F, +_F, \mathbf{0}_F$) is a group.

(9) For every field F and for every non-empty set B and for every binary operation P of B and for every element e of B such that $B =$ the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ and $e = \mathbf{1}_F$ and $P = \cdot_F \restriction_{\mathbf{0}_F}$ the support of F holds group(B, P, e) is a group.

(10) Let F be a field. Let x, y, z be elements of the support of F. Then
  (i) $\cdot_F(\langle x, +_F(\langle y, z\rangle)\rangle) = +_F(\langle \cdot_F(\langle x, y\rangle), \cdot_F(\langle x, z\rangle)\rangle)$,
  (ii) $\cdot_F(\langle +_F(\langle x, y\rangle), z\rangle) = +_F(\langle \cdot_F(\langle x, z\rangle), \cdot_F(\langle y, z\rangle)\rangle)$.

(11) For every field F and for all elements a, b, c of the support of F holds $+_F(\langle +_F(\langle a, b\rangle), c\rangle) = +_F(\langle a, +_F(\langle b, c\rangle)\rangle)$.

(12) For every field F and for all elements a, b of the support of F holds $+_F(\langle a, b\rangle) = +_F(\langle b, a\rangle)$.

(13) For every field F and for every element a of the support of F holds $+_F(\langle a, \mathbf{0}_F\rangle) = a$ and $+_F(\langle \mathbf{0}_F, a\rangle) = a$.

(14) For every field F and for every element a of the support of F there exists an element b of the support of F such that $+_F(\langle a, b\rangle) = \mathbf{0}_F$ and $+_F(\langle b, a\rangle) = \mathbf{0}_F$.

Let F be an at least 2-elements set. A set is said to be a one-element subset of F if:

(Def.7) there exists an element x of F such that it $= \mathrm{single}(x)$.

We now state the proposition

(15) For every at least 2-elements set F and for every one-element subset A of F holds $F \setminus A$ is a non-empty set.

Let F be an at least 2-elements set, and let A be a one-element subset of F. Then $F \setminus A$ is a non-empty set.

The following proposition is true

(16) For every at least 2-elements set F and for every element x of F holds $\mathrm{single}(x)$ is a one-element subset of F.

Let F be an at least 2-elements set, and let x be an element of F. Then $\mathrm{single}(x)$ is a one-element subset of F.

The following propositions are true:

(20)[2] For every field F and for all elements a, b, c of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\cdot_F(\langle \cdot_F(\langle a, b \rangle), c \rangle) = \cdot_F(\langle a, \cdot_F(\langle b, c \rangle) \rangle)$.

(21) For every field F and for all elements a, b of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\cdot_F(\langle a, b \rangle) = \cdot_F(\langle b, a \rangle)$.

(22) For every field F and for every element a of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\cdot_F(\langle a, \mathbf{1}_F \rangle) = a$ and $\cdot_F(\langle \mathbf{1}_F, a \rangle) = a$.

(23) For every field F and for every element a of
the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$
there exists an element b of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ such that $\cdot_F(\langle a, b \rangle) = \mathbf{1}_F$ and $\cdot_F(\langle b, a \rangle) = \mathbf{1}_F$.

Let F be a field. The functor $-_F$ yielding a function from the support of F into the support of F is defined by:

(Def.8) for every element x of the support of F holds $+_F(\langle x, -_F(x) \rangle) = \mathbf{0}_F$.

One can prove the following propositions:

(24) For every field F and for every element x of the support of F holds $+_F(\langle x, -_F(x) \rangle) = \mathbf{0}_F$.

(25) For every field F and for every function S from the support of F into the support of F holds $S = -_F$ if and only if for every element x of the support of F holds $+_F(\langle x, S(x) \rangle) = \mathbf{0}_F$.

(26) For every field F and for every element x of the support of F and for every element y of the support of F such that $+_F(\langle x, y \rangle) = \mathbf{0}_F$ holds $y = -_F(x)$.

(27) For every field F and for every element x of the support of F holds $x = -_F(-_F(x))$.

(28) For every field F and for all elements a, b of the support of F holds $+_F(\langle a, b \rangle)$ is an element of the support of F and $\cdot_F(\langle a, b \rangle)$ is an element of the support of F and $-_F(a)$ is an element of the support of F.

(29) For every field F and for all elements a, b, c of the support of F holds $\cdot_F(\langle a, +_F(\langle b, -_F(c) \rangle) \rangle) = +_F(\langle \cdot_F(\langle a, b \rangle), -_F(\cdot_F(\langle a, c \rangle)) \rangle)$.

(30) For every field F and for all elements a, b, c of the support of F holds $\cdot_F(\langle +_F(\langle a, -_F(b) \rangle), c \rangle) = +_F(\langle \cdot_F(\langle a, c \rangle), -_F(\cdot_F(\langle b, c \rangle)) \rangle)$.

[2]The propositions (17)–(19) became obvious.

(31) For every field F and for every element a of the support of F holds $\cdot_F(\langle a, \mathbf{0}_F\rangle) = \mathbf{0}_F$.

(32) For every field F and for every element a of the support of F holds $\cdot_F(\langle \mathbf{0}_F, a\rangle) = \mathbf{0}_F$.

(33) For every field F and for all elements a, b of the support of F holds $-_F(\cdot_F(\langle a, b\rangle)) = \cdot_F(\langle a, -_F(b)\rangle)$.

(34) For every field F holds $\cdot_F(\langle \mathbf{1}_F, \mathbf{0}_F\rangle) = \mathbf{0}_F$.

(35) For every field F holds $\cdot_F(\langle \mathbf{0}_F, \mathbf{1}_F\rangle) = \mathbf{0}_F$.

(36) For every field F and for all elements a, b of the support of F holds $\cdot_F(\langle a, b\rangle)$ is an element of the support of F.

(37) For every field F and for all elements a, b, c of the support of F holds $\cdot_F(\langle \cdot_F(\langle a, b\rangle), c\rangle) = \cdot_F(\langle a, \cdot_F(\langle b, c\rangle)\rangle)$.

(38) For every field F and for all elements a, b of the support of F holds $\cdot_F(\langle a, b\rangle) = \cdot_F(\langle b, a\rangle)$.

(39) For every field F and for every element a of the support of F holds $\cdot_F(\langle a, \mathbf{1}_F\rangle) = a$ and $\cdot_F(\langle \mathbf{1}_F, a\rangle) = a$.

Let F be a field. The functor $\overset{-1}{F}$ yielding a function from the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ into the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ is defined by:

(Def.9) for every element x of the support of $F\setminus\mathrm{single}(\mathbf{0}_F)$ holds $\cdot_F(\langle x, \overset{-1}{F}(x)\rangle) = \mathbf{1}_F$.

One can prove the following propositions:

(40) For every field F and for every element x of the support of $F\setminus\mathrm{single}(\mathbf{0}_F)$ holds $\cdot_F(\langle x, \overset{-1}{F}(x)\rangle) = \mathbf{1}_F$.

(41) For every field F and for every function S from the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ into the support of $F\setminus\mathrm{single}(\mathbf{0}_F)$ holds $S = \overset{-1}{F}$ if and only if for every element x of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\cdot_F(\langle x, S(x)\rangle) = \mathbf{1}_F$.

(42) For every field F and for every element x of
the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$
and for every element y of the support of $F\setminus\mathrm{single}(\mathbf{0}_F)$ such that $\cdot_F(\langle x, y\rangle) = \mathbf{1}_F$ holds $y = \overset{-1}{F}(x)$.

(43) For every field F and for every element x of the support of $F\setminus\mathrm{single}(\mathbf{0}_F)$ holds $x = \overset{-1}{F}(\overset{-1}{F}(x))$.

(44) For every field F and for all elements a, b of the support of $F\setminus\mathrm{single}(\mathbf{0}_F)$ holds $\cdot_F(\langle a, b\rangle)$ is an element of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ and $\overset{-1}{F}(a)$ is an element of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$.

(45) For every field F and for all elements a, b, c of the support of F such that $+_F(\langle a, b\rangle) = +_F(\langle a, c\rangle)$ holds $b = c$.

(46) For every field F and for every element a of the support of $F\setminus\mathrm{single}(\mathbf{0}_F)$ and for all elements b, c of the support of F such that $\cdot_F(\langle a, b\rangle) = \cdot_F(\langle a, c\rangle)$ holds $b = c$.

References

[1] Józef Białas. Group and field definitions. *Formalized Mathematics*, 1(**3**):433–439, 1990.

[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[4] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

Received June 20, 1990

FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Filters - Part I

Grzegorz Bancerek[1]
Warsaw University
Białystok

Summary. Filters of a lattice, maximal filters (ultrafilters), operation to create a filter generating by an element or by a nonempty set of elements of the lattice are discussed. Besides, there are introduced implicative lattices such that for every two elements there is an element being pseudo-complement of them. Some facts concerning these concepts are presented too, i.e. for any proper filter there exists an ultrafilter consists it.

MML Identifier: `FILTER_0`.

The articles [3], [1], [4], [7], [5], [6], and [2] provide the notation and terminology for this paper. We adopt the following convention: L is a lattice, p, p_1, q, q_1, r, r_1 are elements of the carrier of L, and x is arbitrary. Let E be a non-empty set, and let p be an element of E. Then $\{p\}$ is a non-empty subset of E.

Let E be a non-empty set, and let D_1, D_2 be non-empty subsets of E. Then $D_1 \cup D_2$ is a non-empty subset of E.

The following propositions are true:

(1) If $p \sqsubseteq q$, then $r \sqcup p \sqsubseteq r \sqcup q$ and $p \sqcup r \sqsubseteq q \sqcup r$ and $p \sqcup r \sqsubseteq r \sqcup q$ and $r \sqcup p \sqsubseteq q \sqcup r$.

(2) If $p \sqsubseteq r$, then $p \sqcap q \sqsubseteq r$ and $q \sqcap p \sqsubseteq r$.

(3) If $p \sqsubseteq r$, then $p \sqsubseteq q \sqcup r$ and $p \sqsubseteq r \sqcup q$.

(4) If $p \sqsubseteq p_1$ and $q \sqsubseteq q_1$, then $p \sqcup q \sqsubseteq p_1 \sqcup q_1$ and $p \sqcup q \sqsubseteq q_1 \sqcup p_1$.

(5) If $p \sqsubseteq p_1$ and $q \sqsubseteq q_1$, then $p \sqcap q \sqsubseteq p_1 \sqcap q_1$ and $p \sqcap q \sqsubseteq q_1 \sqcap p_1$.

(6) If $p \sqsubseteq r$ and $q \sqsubseteq r$, then $p \sqcup q \sqsubseteq r$.

(7) If $r \sqsubseteq p$ and $r \sqsubseteq q$, then $r \sqsubseteq p \sqcap q$.

Let us consider L. A non-empty subset of the carrier of L is said to be a filter of L if:

[1]Supported by RPBP III-24.C1.

ISSN 0777–4028

(Def.1) $p \in$ it and $q \in$ it if and only if $p \sqcap q \in$ it.

One can prove the following two propositions:

(8) For every non-empty subset D of the carrier of L holds D is a filter of L if and only if for all p, q holds $p \in D$ and $q \in D$ if and only if $p \sqcap q \in D$.

(9) For every non-empty subset D of the carrier of L holds D is a filter of L if and only if for all p, q such that $p \in D$ and $q \in D$ holds $p \sqcap q \in D$ and for all p, q such that $p \in D$ and $p \sqsubseteq q$ holds $q \in D$.

In the sequel H, F are filters of L. We now state several propositions:

(10) If $p \in H$, then $p \sqcup q \in H$ and $q \sqcup p \in H$.

(11) There exists p such that $p \in H$.

(12) If L is an upper bound lattice, then $\top_L \in H$.

(13) If L is an upper bound lattice, then $\{\top_L\}$ is a filter of L.

(14) If $\{p\}$ is a filter of L, then L is an upper bound lattice.

(15) The carrier of L is a filter of L.

Let us consider L. The functor $[L]$ yields a filter of L and is defined by:

(Def.2) $[L] =$ the carrier of L.

One can prove the following proposition

(16) $[L] =$ the carrier of L.

Let us consider L, p. The functor $[p]$ yields a filter of L and is defined as follows:

(Def.3) $[p] = \{q : p \sqsubseteq q\}$.

One can prove the following four propositions:

(17) $[p] = \{q : p \sqsubseteq q\}$.

(18) $q \in [p]$ if and only if $p \sqsubseteq q$.

(19) $p \in [p]$ and $p \sqcup q \in [p]$ and $q \sqcup p \in [p]$.

(20) If L is a lower bound lattice, then $[L] = [\bot_L]$.

Let us consider L, F. We say that F is ultrafilter if and only if:

(Def.4) $F \neq$ the carrier of L and for every H such that $F \subseteq H$ and $H \neq$ the carrier of L holds $F = H$.

One can prove the following four propositions:

(21) F is ultrafilter if and only if $F \neq$ the carrier of L and for every H such that $F \subseteq H$ and $H \neq$ the carrier of L holds $F = H$.

(22) If L is a lower bound lattice, then for every F such that $F \neq$ the carrier of L there exists H such that $F \subseteq H$ and H is ultrafilter.

(23) If there exists r such that $p \sqcap r \neq p$, then $[p] \neq$ the carrier of L.

(24) If L is a lower bound lattice and $p \neq \bot_L$, then there exists H such that $p \in H$ and H is ultrafilter.

In the sequel D is a non-empty subset of the carrier of L. Let us consider L, D. The functor $[D]$ yields a filter of L and is defined by:

(Def.5) $D \subseteq [D]$ and for every F such that $D \subseteq F$ holds $[D] \subseteq F$.

One can prove the following two propositions:

(25) $D \subseteq [D]$ and for every F such that $D \subseteq F$ holds $[D] \subseteq F$.

(26) $[F] = F$.

In the sequel D_1, D_2 will be non-empty subsets of the carrier of L. We now state several propositions:

(27) If $D_1 \subseteq D_2$, then $[D_1] \subseteq [D_2]$.

(28) $[[D]] \subseteq [D]$.

(29) If $p \in D$, then $[p] \subseteq [D]$.

(30) If $D = \{p\}$, then $[D] = [p]$.

(31) If L is a lower bound lattice and $\perp_L \in D$, then $[D] = [L]$ and $[D] =$ the carrier of L.

(32) If L is a lower bound lattice and $\perp_L \in F$, then $F = [L]$ and $F =$ the carrier of L.

Let us consider L, F. We say that F is prime if and only if:

(Def.6) $p \sqcup q \in F$ if and only if $p \in F$ or $q \in F$.

One can prove the following two propositions:

(33) F is prime if and only if for all p, q holds $p \sqcup q \in F$ if and only if $p \in F$ or $q \in F$.

(34) If L is a boolean lattice, then for all p, q holds $p \sqcap (p^{\mathrm{c}} \sqcup q) \sqsubseteq q$ and for every r such that $p \sqcap r \sqsubseteq q$ holds $r \sqsubseteq p^{\mathrm{c}} \sqcup q$.

A lattice is called a implicative lattice if:

(Def.7) for every elements p, q of the carrier of it there exists an element r of the carrier of it such that $p \sqcap r \sqsubseteq q$ and for every element r_1 of the carrier of it such that $p \sqcap r_1 \sqsubseteq q$ holds $r_1 \sqsubseteq r$.

One can prove the following proposition

(35) L is a implicative lattice if and only if for every p, q there exists r such that $p \sqcap r \sqsubseteq q$ and for every r_1 such that $p \sqcap r_1 \sqsubseteq q$ holds $r_1 \sqsubseteq r$.

Let us consider L, p, q. Let us assume that L is a implicative lattice. The functor $p \Rightarrow q$ yields an element of the carrier of L and is defined as follows:

(Def.8) $p \sqcap (p \Rightarrow q) \sqsubseteq q$ and for every r such that $p \sqcap r \sqsubseteq q$ holds $r \sqsubseteq p \Rightarrow q$.

The following proposition is true

(36) If L is a implicative lattice, then for all p, q, r holds $r = p \Rightarrow q$ if and only if $p \sqcap r \sqsubseteq q$ and for every r_1 such that $p \sqcap r_1 \sqsubseteq q$ holds $r_1 \sqsubseteq r$.

In the sequel I will denote a implicative lattice and i will denote an element of the carrier of I. The following three propositions are true:

(37) I is an upper bound lattice.

(38) $i \Rightarrow i = \top_I$.

(39) I is a distributive lattice.

In the sequel B is a boolean lattice and F_1, H_1 are filters of B. Next we state the proposition

(40) B is a implicative lattice.

We see that the implicative lattice is a distributive lattice.

For simplicity we follow the rules: I will be a implicative lattice, i, j, k will be elements of the carrier of I, D_3 will be a non-empty subset of the carrier of I, and F_2 will be a filter of I. The following propositions are true:

(41) If $i \in F_2$ and $i \Rightarrow j \in F_2$, then $j \in F_2$.

(42) If $j \in F_2$, then $i \Rightarrow j \in F_2$.

Let us consider L, D_1, D_2. The functor $D_1 \sqcap D_2$ yielding a non-empty subset of the carrier of L is defined as follows:

(Def.9) $D_1 \sqcap D_2 = \{p \sqcap q : p \in D_1 \wedge q \in D_2\}$.

Next we state four propositions:

(43) $D_1 \sqcap D_2 = \{p \sqcap q : p \in D_1 \wedge q \in D_2\}$.

(44) If $p \in D_1$ and $q \in D_2$, then $p \sqcap q \in D_1 \sqcap D_2$ and $q \sqcap p \in D_1 \sqcap D_2$.

(45) If $x \in D_1 \sqcap D_2$, then there exist p, q such that $x = p \sqcap q$ and $p \in D_1$ and $q \in D_2$.

(46) $D_1 \sqcap D_2 = D_2 \sqcap D_1$.

Let L be a distributive lattice, and let F_3, F_4 be filters of L. Then $F_3 \sqcap F_4$ is a filter of L.

Let L be a boolean lattice, and let F_3, F_4 be filters of L. Then $F_3 \sqcap F_4$ is a filter of L.

One can prove the following propositions:

(47) $[D_1 \cup D_2] = [[D_1] \cup D_2]$ and $[D_1 \cup D_2] = [D_1 \cup [D_2]]$.

(48) $[F \cup H] = \{r : \bigvee_{pq}[p \sqcap q \sqsubseteq r \wedge p \in F \wedge q \in H]\}$.

(49) $F \subseteq F \sqcap H$ and $H \subseteq F \sqcap H$.

(50) $[F \cup H] = [F \sqcap H]$.

In the sequel F_3, F_4 are filters of I. The following four propositions are true:

(51) $[F_3 \cup F_4] = F_3 \sqcap F_4$.

(52) $[F_1 \cup H_1] = F_1 \sqcap H_1$.

(53) If $j \in [D_3 \cup \{i\}]$, then $i \Rightarrow j \in [D_3]$.

(54) If $i \Rightarrow j \in F_2$ and $j \Rightarrow k \in F_2$, then $i \Rightarrow k \in F_2$.

In the sequel a, b, c will denote elements of the carrier of B. One can prove the following propositions:

(55) $a \Rightarrow b = a^{\rm c} \sqcup b$.

(56) $a \sqsubseteq b$ if and only if $a \sqcap b^{\rm c} = \bot_B$.

(57) F_1 is ultrafilter if and only if $F_1 \neq$ the carrier of B and for every a holds $a \in F_1$ or $a^{\rm c} \in F_1$.

(58) $F_1 \neq [B]$ and F_1 is prime if and only if F_1 is ultrafilter.

(59) If F_1 is ultrafilter, then for every a holds $a \in F_1$ if and only if $a^{\rm c} \notin F_1$.

(60) If $a \neq b$, then there exists F_1 such that F_1 is ultrafilter but $a \in F_1$ and $b \notin F_1$ or $a \notin F_1$ and $b \in F_1$.

In the sequel o_1, o_2 are binary operations on F. Let us consider L, F. The functor $\mathbb{L}_F$ yielding a lattice is defined as follows:

(Def.10) there exist o_1, o_2 such that $o_1 =$ (the join operation of L) $\restriction [\![F, F]\!]$ and $o_2 =$ (the meet operation of L) $\restriction [\![F, F]\!]$ and $\mathbb{L}_F = \langle F, o_1, o_2 \rangle$.

In the sequel K is a lattice. Next we state a number of propositions:

(61) $K = \mathbb{L}_F$ if and only if there exist o_1, o_2 such that $o_1 =$ (the join operation of L) $\restriction [\![F, F]\!]$ and $o_2 =$ (the meet operation of L) $\restriction [\![F, F]\!]$ and $K = \langle F, o_1, o_2 \rangle$.

(62) $\mathbb{L}_{[L]} = L$.

(63) The carrier of $\mathbb{L}_F = F$ and the join operation of $\mathbb{L}_F =$ (the join operation of L) $\restriction [\![F, F]\!]$ and the meet operation of $\mathbb{L}_F =$ (the meet operation of L) $\restriction [\![F, F]\!]$.

(64) For all p, q and for all elements p', q' of the carrier of $\mathbb{L}_F$ such that $p = p'$ and $q = q'$ holds $p \sqcup q = p' \sqcup q'$ and $p \sqcap q = p' \sqcap q'$.

(65) For all p, q and for all elements p', q' of the carrier of $\mathbb{L}_F$ such that $p = p'$ and $q = q'$ holds $p \sqsubseteq q$ if and only if $p' \sqsubseteq q'$.

(66) If L is an upper bound lattice, then $\mathbb{L}_F$ is an upper bound lattice.

(67) If L is a modular lattice, then $\mathbb{L}_F$ is a modular lattice.

(68) If L is a distributive lattice, then $\mathbb{L}_F$ is a distributive lattice.

(69) If L is a implicative lattice, then $\mathbb{L}_F$ is a implicative lattice.

(70) $\mathbb{L}_{[p]}$ is a lower bound lattice.

(71) $\perp_{\mathbb{L}_{[p]}} = p$.

(72) If L is an upper bound lattice, then $\top_{\mathbb{L}_{[p]}} = \top_L$.

(73) If L is an upper bound lattice, then $\mathbb{L}_{[p]}$ is a bound lattice.

(74) If L is a complemented lattice and L is a modular lattice, then $\mathbb{L}_{[p]}$ is a complemented lattice.

(75) If L is a boolean lattice, then $\mathbb{L}_{[p]}$ is a boolean lattice.

Let us consider L, p, q. The functor $p \Leftrightarrow q$ yielding an element of the carrier of L is defined by:

(Def.11) $p \Leftrightarrow q = p \Rightarrow q \sqcap q \Rightarrow p$.

Next we state three propositions:

(76) $p \Leftrightarrow q = p \Rightarrow q \sqcap q \Rightarrow p$.

(77) $p \Leftrightarrow q = q \Leftrightarrow p$.

(78) If $i \Leftrightarrow j \in F_2$ and $j \Leftrightarrow k \in F_2$, then $i \Leftrightarrow k \in F_2$.

Let us consider L, F. The functor $\equiv_F$ yielding a binary relation is defined as follows:

(Def.12) field $\equiv_F \subseteq$ the carrier of L and for all p, q holds $\langle p, q \rangle \in \equiv_F$ if and only if $p \Leftrightarrow q \in F$.

In the sequel R will denote a binary relation. We now state several propositions:

(79) $R = \equiv_F$ if and only if field $R \subseteq$ the carrier of L and for all p, q holds $\langle p, q \rangle \in R$ if and only if $p \Leftrightarrow q \in F$.

(80) $\equiv_F$ is a binary relation on the carrier of L.

(81) If L is a implicative lattice, then $\equiv_F$ is reflexive in the carrier of L.

(82) $\equiv_F$ is symmetric in the carrier of L.

(83) If L is a implicative lattice, then $\equiv_F$ is transitive in the carrier of L.

(84) If L is a implicative lattice, then $\equiv_F$ is an equivalence relation of the carrier of L.

(85) If L is a implicative lattice, then field $\equiv_F =$ the carrier of L.

Let us consider I, F_2. Then $\equiv_{F_2}$ is an equivalence relation of the carrier of I.

Let us consider B, F_1. Then $\equiv_{F_1}$ is an equivalence relation of the carrier of B.

Let us consider L, F, p, q. The predicate $p \equiv_F q$ is defined by:

(Def.13) $p \Leftrightarrow q \in F$.

Next we state several propositions:

(86) $p \equiv_F q$ if and only if $p \Leftrightarrow q \in F$.

(87) $p \equiv_F q$ if and only if $\langle p, q \rangle \in \equiv_F$.

(88) $i \equiv_{F_2} i$ and $a \equiv_{F_1} a$.

(89) If $p \equiv_F q$, then $q \equiv_F p$.

(90) If $i \equiv_{F_2} j$ and $j \equiv_{F_2} k$, then $i \equiv_{F_2} k$ but if $a \equiv_{F_1} b$ and $b \equiv_{F_1} c$, then $a \equiv_{F_1} c$.

References

[1] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[2] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Formalized Mathematics*, 1(**3**):441–444, 1990.

[3] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[4] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

[5] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[6] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Formalized Mathematics*, 1(**1**):85–89, 1990.

[7] Stanisław Żukowski. Introduction to lattice theory. *Formalized Mathematics*, 1(**1**):215–222, 1990.

Received July 3, 1990

FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Groups

Wojciech A. Trybulec
Warsaw University

Summary. Notions of group and abelian group are introduced. The power of an element of a group, order of group and order of an element of a group are defined. Basic theorems concerning those notions are presented.

MML Identifier: GROUP_1.

The notation and terminology used in this paper are introduced in the following articles: [6], [7], [9], [2], [3], [5], [12], [11], [1], [8], [4], [10], and [13]. We follow the rules: x is arbitrary, m, n are natural numbers, and i, j are integers. Let N be a non-empty subset of $\mathbb{R}$, and let D be a non-empty set, and let f be a function from N into D, and let n be an element of N. Then $f(n)$ is an element of D.

Let D be a non-empty set, and let N be a non-empty subset of $\mathbb{R}$, and let E be a non-empty set, and let f be a function from $[: D, N :]$ into E, and let h be an element of D, and let n be an element of N. Then $f(h, n)$ is an element of E.

Let us consider i. Then $|i|$ is a natural number.

We consider half group structures which are systems
$\langle$a carrier, an operation$\rangle$,
where the carrier is a non-empty set and the operation is a binary operation on the carrier. In the sequel S denotes a half group structure. Let us consider S. An element of S is an element of the carrier of S.

In the sequel r, s, s_1, s_2, t will be elements of S. Let us consider S, x. The predicate $x \in S$ is defined as follows:

(Def.1) $x \in$ the carrier of S.

The following propositions are true:

(1) $x \in S$ if and only if $x \in$ the carrier of S.

(2) $s \in S$.

ISSN 0777–4028

(3) If $x \in S$, then x is an element of S.

Let us consider S, s_1, s_2. The functor $s_1 \cdot s_2$ yielding an element of S is defined by:

(Def.2) $s_1 \cdot s_2 =$ (the operation of S)(s_1, s_2).

One can prove the following proposition

(4) $s_1 \cdot s_2 =$ (the operation of S)(s_1, s_2).

A half group structure is called a group if:

(Def.3) (i) for all elements f, g, h of it holds $(f \cdot g) \cdot h = f \cdot (g \cdot h)$,

(ii) there exists an element e of it such that for every element h of it holds $h \cdot e = h$ and $e \cdot h = h$ and there exists an element g of it such that $h \cdot g = e$ and $g \cdot h = e$.

We now state three propositions:

(5) If for all r, s, t holds $(r \cdot s) \cdot t = r \cdot (s \cdot t)$ and there exists t such that for every s_1 holds $s_1 \cdot t = s_1$ and $t \cdot s_1 = s_1$ and there exists s_2 such that $s_1 \cdot s_2 = t$ and $s_2 \cdot s_1 = t$, then S is a group.

(6) If for all r, s, t holds $(r \cdot s) \cdot t = r \cdot (s \cdot t)$ and for all r, s holds there exists t such that $r \cdot t = s$ and there exists t such that $t \cdot r = s$, then S is a group.

(7) $\langle \mathbb{R}, +_{\mathbb{R}} \rangle$ is a group.

We follow a convention: G denotes a group and e, f, g, h denote elements of G. Next we state two propositions:

(8) $(h \cdot g) \cdot f = h \cdot (g \cdot f)$.

(9) There exists e such that for every h holds $h \cdot e = h$ and $e \cdot h = h$ and there exists g such that $h \cdot g = e$ and $g \cdot h = e$.

Let us consider G. The functor 1_G yielding an element of G is defined by:

(Def.4) $h \cdot (1_G) = h$ and $(1_G) \cdot h = h$.

One can prove the following two propositions:

(10) If for every h holds $h \cdot e = h$ and $e \cdot h = h$, then $e = 1_G$.

(11) $h \cdot (1_G) = h$ and $(1_G) \cdot h = h$.

Let us consider G, h. The functor h^{-1} yields an element of G and is defined as follows:

(Def.5) $h \cdot (h^{-1}) = 1_G$ and $(h^{-1}) \cdot h = 1_G$.

One can prove the following propositions:

(12) If $h \cdot g = 1_G$ and $g \cdot h = 1_G$, then $g = h^{-1}$.

(13) $h \cdot h^{-1} = 1_G$ and $h^{-1} \cdot h = 1_G$.

(14) If $h \cdot g = h \cdot f$ or $g \cdot h = f \cdot h$, then $g = f$.

(15) If $h \cdot g = h$ or $g \cdot h = h$, then $g = 1_G$.

(16) $(1_G)^{-1} = 1_G$.

(17) If $h^{-1} = g^{-1}$, then $h = g$.

(18) If $h^{-1} = 1_G$, then $h = 1_G$.

(19) $(h^{-1})^{-1} = h$.

(20) If $h \cdot g = 1_G$ or $g \cdot h = 1_G$, then $h = g^{-1}$ and $g = h^{-1}$.

(21) $h \cdot f = g$ if and only if $f = h^{-1} \cdot g$.

(22) $f \cdot h = g$ if and only if $f = g \cdot h^{-1}$.

(23) There exists f such that $g \cdot f = h$.

(24) There exists f such that $f \cdot g = h$.

(25) $(h \cdot g)^{-1} = g^{-1} \cdot h^{-1}$.

(26) $g \cdot h = h \cdot g$ if and only if $(g \cdot h)^{-1} = g^{-1} \cdot h^{-1}$.

(27) $g \cdot h = h \cdot g$ if and only if $g^{-1} \cdot h^{-1} = h^{-1} \cdot g^{-1}$.

(28) $g \cdot h = h \cdot g$ if and only if $g \cdot h^{-1} = h^{-1} \cdot g$.

In the sequel u is a unary operation on the carrier of G. Let us consider G. The functor $\cdot_G^{-1}$ yields a unary operation on the carrier of G and is defined by:

(Def.6) $\cdot_G^{-1}(h) = h^{-1}$.

We now state several propositions:

(29) If for every h holds $u(h) = h^{-1}$, then $u = \cdot_G^{-1}$.

(30) $\cdot_G^{-1}(h) = h^{-1}$.

(31) The operation of G is associative.

(32) 1_G is a unity w.r.t. the operation of G.

(33) $\mathbf{1}_{\text{the operation of } G} = 1_G$.

(34) The operation of G has a unity.

(35) $\cdot_G^{-1}$ is an inverse operation w.r.t. the operation of G.

(36) The operation of G has an inverse operation.

(37) The inverse operation w.r.t. (the operation of G) $= \cdot_G^{-1}$.

Let us consider G. The functor power_G yields a function from $[\!:$ the carrier of G, $\mathbb{N}\,:\!]$ into the carrier of G and is defined by:

(Def.7) $\mathrm{power}_G(h, 0) = 1_G$ and for every n holds $\mathrm{power}_G(h, n+1) = \mathrm{power}_G(h, n) \cdot h$.

In the sequel H is a function from $[\!:$ the carrier of G, $\mathbb{N}\,:\!]$ into the carrier of G. We now state three propositions:

(38) If for every h holds $H(h, 0) = 1_G$ and for every n holds $H(h, n+1) = H(h, n) \cdot h$, then $H = \mathrm{power}_G$.

(39) $\mathrm{power}_G(h, 0) = 1_G$.

(40) $\mathrm{power}_G(h, n+1) = \mathrm{power}_G(h, n) \cdot h$.

Let us consider G, n, h. The functor h^n yields an element of G and is defined as follows:

(Def.8) $h^n = \mathrm{power}_G(h, n)$.

We now state a number of propositions:

(41) $h^n = \mathrm{power}_G(h, n)$.

(42) $(1_G)^n = 1_G$.

(43) $h^0 = 1_G$.

(44) $h^1 = h$.

(45) $h^2 = h \cdot h$.

(46) $h^3 = (h \cdot h) \cdot h$.

(47) $h^2 = 1_G$ if and only if $h^{-1} = h$.

(48) $h^{n+m} = h^n \cdot h^m$ and $h^{m+n} = h^n \cdot h^m$.

(49) $h^{n+1} = h^n \cdot h$ and $h^{n+1} = h \cdot h^n$ and $h^{1+n} = h^n \cdot h$ and $h^{1+n} = h \cdot h^n$.

(50) $h^{n \cdot m} = (h^n)^m$.

(51) $(h^{-1})^n = (h^n)^{-1}$.

(52) If $g \cdot h = h \cdot g$, then $g \cdot h^n = h^n \cdot g$.

(53) If $g \cdot h = h \cdot g$, then $g^n \cdot h^m = h^m \cdot g^n$.

(54) If $g \cdot h = h \cdot g$, then $(g \cdot h)^n = g^n \cdot h^n$.

Let us consider G, i, h. The functor h^i yielding an element of G is defined by:

(Def.9) $h^i = h^{|i|}$ if $0 \leq i$, $h^i = (h^{|i|})^{-1}$, otherwise.

The following propositions are true:

(55) If $0 \leq i$, then $h^i = h^{|i|}$.

(56) If $0 \not\leq i$, then $h^i = (h^{|i|})^{-1}$.

(57) If $i < 0$, then $h^i = (h^{|i|})^{-1}$.

(58) If $i = n$, then $h^i = h^n$.

(59) If $i = 0$, then $h^i = 1_G$.

(60) If $i \leq 0$, then $h^i = (h^{|i|})^{-1}$.

(61) $(1_G)^i = 1_G$.

(62) $h^{-1} = h^{-1}$.

(63) $h^{i+j} = h^i \cdot h^j$.

(64) $h^{n+j} = h^n \cdot h^j$.

(65) $h^{i+m} = h^i \cdot h^m$.

(66) $h^{j+1} = h^j \cdot h$ and $h^{j+1} = h \cdot h^j$ and $h^{1+j} = h^j \cdot h$ and $h^{1+j} = h \cdot h^j$.

(67) $h^{i \cdot j} = (h^i)^j$.

(68) $h^{n \cdot j} = (h^n)^j$.

(69) $h^{i \cdot m} = (h^i)^m$.

(70) $h^{-i} = (h^i)^{-1}$.

(71) $h^{-n} = (h^n)^{-1}$.

(72) $(h^{-1})^i = (h^i)^{-1}$.

(73) If $g \cdot h = h \cdot g$, then $(g \cdot h)^i = g^i \cdot h^i$.

(74) If $g \cdot h = h \cdot g$, then $g^i \cdot h^j = h^j \cdot g^i$.

(75) If $g \cdot h = h \cdot g$, then $g^n \cdot h^j = h^j \cdot g^n$.

(76) If $g \cdot h = h \cdot g$, then $g^i \cdot h^m = h^m \cdot g^i$.

(77) If $g \cdot h = h \cdot g$, then $g \cdot h^i = h^i \cdot g$.

Let us consider G, h. We say that h is of order 0 if and only if:

(Def.10) if $h^n = 1_G$, then $n = 0$.

We now state two propositions:

(78) h is of order 0 if and only if for every n such that $h^n = 1_G$ holds $n = 0$.

(79) 1_G is not of order 0.

Let us consider G, h. The functor $\mathrm{ord}(h)$ yields a natural number and is defined by:

(Def.11) $\mathrm{ord}(h) = 0$ if h is of order 0, $h^{\mathrm{ord}(h)} = 1_G$ and $\mathrm{ord}(h) \neq 0$ and for every m such that $h^m = 1_G$ and $m \neq 0$ holds $\mathrm{ord}(h) \leq m$, otherwise.

One can prove the following propositions:

(80) If h is not of order 0 and $h^m = 1_G$ and $m \neq 0$ and for every n such that $h^n = 1_G$ and $n \neq 0$ holds $m \leq n$, then $m = \mathrm{ord}(h)$.

(81) h is of order 0 if and only if $\mathrm{ord}(h) = 0$.

(82) $h^{\mathrm{ord}(h)} = 1_G$.

(83) If h is not of order 0 and $h^m = 1_G$ and $m \neq 0$, then $\mathrm{ord}(h) \leq m$.

(84) $\mathrm{ord}(1_G) = 1$.

(85) If $\mathrm{ord}(h) = 1$, then $h = 1_G$.

(86) If $h^n = 1_G$, then $\mathrm{ord}(h) \mid n$.

Let us consider G. The functor $\mathrm{Ord}(G)$ yielding a cardinal number is defined as follows:

(Def.12) $\mathrm{Ord}(G) = \overline{\overline{\text{the carrier of } G}}$.

We now state the proposition

(87) $\mathrm{Ord}(G) = \overline{\overline{\text{the carrier of } G}}$.

We now define two new predicates. Let us consider G. We say that G is finite if and only if:

(Def.13) the carrier of G is finite.

We say that G is infinite if and only if G is not finite.

The following proposition is true

(88) G is finite if and only if the carrier of G is finite.

Let us consider G. Let us assume that G is finite. The functor $\mathrm{ord}(G)$ yielding a natural number is defined by:

(Def.14) $\mathrm{ord}(G) = \mathrm{card}\,(\text{the carrier of } G)$.

Next we state two propositions:

(89) If G is finite, then $\mathrm{ord}(G) = \mathrm{card}\,(\text{the carrier of } G)$.

(90) If G is finite, then $\mathrm{ord}(G) \geq 1$.

A group is called an Abelian group if:

(Def.15) for all elements a, b of it holds $a \cdot b = b \cdot a$.

We now state two propositions:

(91) If for all h, g holds $h \cdot g = g \cdot h$, then G is an Abelian group.

(92) $\langle \mathbb{R}, +_{\mathbb{R}} \rangle$ is an Abelian group.

In the sequel A is an Abelian group and a, b are elements of A. One can prove the following propositions:

(93) $a \cdot b = b \cdot a$.

(94) $(a \cdot b)^{-1} = a^{-1} \cdot b^{-1}$.

(95) $(a \cdot b)^n = a^n \cdot b^n$.

(96) $(a \cdot b)^i = a^i \cdot b^i$.

(97) $\langle$The carrier of A, the operation of $A, \cdot_A^{-1}, 1_A\rangle$ is an Abelian group.

In the sequel B denotes an Abelian group. We now state two propositions:

(98) $\langle$The carrier of B, the addition of $B\rangle$ is an Abelian group.

(99) $-1 < 0$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[4] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[5] Czesław Byliński. Binary operations applied to finite sequences. *Formalized Mathematics*, 1(**4**):643–649, 1990.

[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[8] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[9] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[10] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[11] Andrzej Trybulec. Semilattice operations on finite subsets. *Formalized Mathematics*, 1(**2**):369–376, 1990.

[12] Andrzej Trybulec and Agata Darmochwał. Boolean domains. *Formalized Mathematics*, 1(**1**):187–190, 1990.

[13] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.

Received July 3, 1990

FORMALIZED MATHEMATICS
Vol.1, No.**5**, November–December 1990
Université Catholique de Louvain

The Divisibility of Integers and Integer Relatively Primes [1]

Rafał Kwiatek
Nicolaus Copernicus University
Toruń

Grzegorz Zwara
Warsaw University
Białystok

Summary. We introduce the following notions: 1)the least common multiple of two integers (lcm(i, j)), 2)the greatest common divisor of two integers (gcd(i, j)), 3)the relative prime integer numbers, 4)the prime numbers. A few facts concerning the above items, among them a so-called Foundamental Theorem of Arithmetic, are introduced.

MML Identifier: `INT_2`.

The papers [2], [1], and [3] provide the terminology and notation for this paper. In the sequel a, b will be natural numbers. Next we state several propositions:

(1) $\mathrm{lcm}(a, b) = \mathrm{lcm}(b, a)$.

(2) $\gcd(a, b) = \gcd(b, a)$.

(3) $0 \mid a$ if and only if $a = 0$.

(4) $a = 0$ or $b = 0$ if and only if $\mathrm{lcm}(a, b) = 0$.

(5) $a = 0$ and $b = 0$ if and only if $\gcd(a, b) = 0$.

(6) $a \cdot b = \mathrm{lcm}(a, b) \cdot \gcd(a, b)$.

We follow the rules: m, n are natural numbers and a, b, c, a_1, b_1 are integers. Let us consider n. The functor $+n$ yields an integer and is defined by:

(Def.1) $+n = n$.

Next we state a number of propositions:

(7) $+n = n$.

(8) $-n$ is a natural number if and only if $n = 0$.

(9) -1 is not a natural number.

(10) $+0 \mid a$ if and only if $a = 0$.

(11) $a \mid a$ and $a \mid -a$ and $-a \mid a$.

[1]Supported by RPBP.III-24.B5.

ISSN 0777–4028

(12) If $a \mid b$, then $a \mid b \cdot c$.

(13) If $a \mid b$ and $b \mid c$, then $a \mid c$.

(14) $a \mid b$ if and only if $a \mid -b$ but $a \mid b$ if and only if $-a \mid b$ but $a \mid b$ if and only if $-a \mid -b$ but $a \mid -b$ if and only if $-a \mid b$.

(15) If $a \mid b$ and $b \mid a$, then $a = b$ or $a = -b$.

(16) $a \mid +0$ and $+1 \mid a$ and $-1 \mid a$.

(17) If $a \mid +1$ or $a \mid -1$, then $a = 1$ or $a = -1$.

(18) If $a = 1$ or $a = -1$, then $a \mid +1$ and $a \mid -1$.

(19) $a \equiv b (\mathrm{mod}\, c)$ if and only if $c \mid a - b$.

(20) $|a|$ is a natural number.

Let us consider a. Then $|a|$ is a natural number.

We now state the proposition

(21) $a \mid b$ if and only if $|a| \mid |b|$.

Let us consider a, b. The functor $\mathrm{lcm}(a, b)$ yields an integer and is defined as follows:

(Def.2) $\mathrm{lcm}(a, b) = \mathrm{lcm}(|a|, |b|)$.

The following propositions are true:

(22) $\mathrm{lcm}(a, b) = \mathrm{lcm}(|a|, |b|)$.

(23) $\mathrm{lcm}(a, b)$ is a natural number.

(24) $\mathrm{lcm}(a, b) = \mathrm{lcm}(b, a)$.

(25) $a \mid \mathrm{lcm}(a, b)$.

(26) $b \mid \mathrm{lcm}(a, b)$.

(27) For every c such that $a \mid c$ and $b \mid c$ holds $\mathrm{lcm}(a, b) \mid c$.

Let us consider a, b. The functor $\mathrm{gcd}(a, b)$ yields an integer and is defined by:

(Def.3) $\mathrm{gcd}(a, b) = \mathrm{gcd}(|a|, |b|)$.

One can prove the following propositions:

(28) $\mathrm{gcd}(a, b) = \mathrm{gcd}(|a|, |b|)$.

(29) $\mathrm{gcd}(a, b)$ is a natural number.

(30) $\mathrm{gcd}(a, b) = \mathrm{gcd}(b, a)$.

(31) $\mathrm{gcd}(a, b) \mid a$.

(32) $\mathrm{gcd}(a, b) \mid b$.

(33) For every c such that $c \mid a$ and $c \mid b$ holds $c \mid \mathrm{gcd}(a, b)$.

(34) $a = 0$ or $b = 0$ if and only if $\mathrm{lcm}(a, b) = 0$.

(35) $a = 0$ and $b = 0$ if and only if $\mathrm{gcd}(a, b) = 0$.

Let us consider a, b. We say that a and b are relatively prime if and only if:

(Def.4) $\mathrm{gcd}(a, b) = 1$.

Next we state several propositions:

(36) a and b are relatively prime if and only if $\mathrm{gcd}(a, b) = 1$.

(37) If a and b are relatively prime, then b and a are relatively prime.

(38) If $a \neq 0$ or $b \neq 0$, then there exist a_1, b_1 such that $a = \gcd(a, b) \cdot a_1$ and $b = \gcd(a, b) \cdot b_1$ and a_1 and b_1 are relatively prime.

(39) If a and b are relatively prime, then $\gcd(c \cdot a, c \cdot b) = |c|$ and $\gcd(c \cdot a, b \cdot c) = |c|$ and $\gcd(a \cdot c, c \cdot b) = |c|$ and $\gcd(a \cdot c, b \cdot c) = |c|$.

(40) If $c \mid a \cdot b$ and a and c are relatively prime, then $c \mid b$.

(41) If a and c are relatively prime and b and c are relatively prime, then $a \cdot b$ and c are relatively prime.

In the sequel p, q, k, l will denote natural numbers. Let us consider p. We say that p is prime if and only if:

(Def.5) $p > 1$ and for every n such that $n \mid p$ holds $n = 1$ or $n = p$.

The following proposition is true

(42) p is prime if and only if $p > 1$ and for every n such that $n \mid p$ holds $n = 1$ or $n = p$.

Let us consider m, n. We say that m and n are relatively prime if and only if:

(Def.6) $\gcd(m, n) = 1$.

We now state several propositions:

(43) m and n are relatively prime if and only if $\gcd(m, n) = 1$.

(44) 2 is prime.

(45) There exists p such that p is prime.

(46) There exists p such that p is not prime.

(47) If p is prime and q is prime, then p and q are relatively prime or $p = q$.

In this article we present several logical schemes. The scheme *Ind1* concerns a natural number $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

for every k such that $k \geq \mathcal{A}$ holds $\mathcal{P}[k]$

provided the parameters meet the following conditions:

- $\mathcal{P}[\mathcal{A}]$,
- for every k such that $k \geq \mathcal{A}$ and $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$.

The scheme *Comp_Ind1* concerns a natural number $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

for every k such that $k \geq \mathcal{A}$ holds $\mathcal{P}[k]$

provided the parameters have the following property:

- for every k such that $k \geq \mathcal{A}$ and for every n such that $n \geq \mathcal{A}$ and $n < k$ holds $\mathcal{P}[n]$ holds $\mathcal{P}[k]$.

Next we state the proposition

(48) If $l \geq 2$, then there exists p such that p is prime and $p \mid l$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[3] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.

Received July 10, 1990

FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

From Loops to Abelian Multiplicative Groups with Zero [1]

Michał Muzalewski
Warsaw University
Białystok

Wojciech Skaba
Nicolaus Copernicus University
Toruń

Summary. Elementary axioms and theorems on the theory of algebraic structures, taken from the book [4]. First a loop structure $\langle G, 0, + \rangle$ is defined and six axioms corresponding to it are given. Group is defined by extending the set of axioms with $(a+b)+c = a+(b+c)$. At the same time an alternate approach to the set of axioms is shown and both sets are proved to yield the same algebraic structure. A trivial example of loop is used to ensure the existence of the modes being constructed. A multiplicative group is contemplated, which is quite similar to the previously defined additive group (called simply a group here), but is supposed to be of greater interest in the future considerations of algebraic structures. The final section brings a slightly more sophisticated structure i.e: a multiplicative loop/group with zero: $\langle G, \cdot, 1, 0 \rangle$. Here the proofs are a more challenging and the above trivial example is replaced by a more common (and comprehensive) structure built on the foundation of real numbers.

MML Identifier: ALGSTR_1.

The notation and terminology used in this paper are introduced in the following articles: [1], [2], and [3]. We consider loop structures which are systems
⟨a carrier, an addition, a zero⟩,
where the carrier is a non-empty set, the addition is a binary operation on the carrier, and the zero is an element of the carrier. In the sequel G_1 will denote a loop structure. Let us consider G_1. An element of G_1 is an element of the carrier of G_1.

In the sequel a, b will denote elements of G_1. Let us consider G_1, a, b. The functor $a + b$ yielding an element of G_1 is defined as follows:

(Def.1) $a + b =$ (the addition of G_1)(a, b).

We now state the proposition

[1] Supported by RPBP.III-24.C6.

ISSN 0777–4028

(1) $a + b =$ (the addition of G_1)(a, b).

Let us consider G_1. The functor 0_{G_1} yielding an element of G_1 is defined as follows:

(Def.2) $0_{G_1} =$ the zero of G_1.

One can prove the following proposition

(2) $0_{G_1} =$ the zero of G_1.

Let x be arbitrary. The functor Extract(x) yielding an element of $\{x\}$ is defined by:

(Def.3) Extract(x) $= x$.

One can prove the following proposition

(3) For an arbitrary x holds Extract(x) $= x$.

The trivial loop a loop structure is defined as follows:

(Def.4) the trivial loop $= \langle\{0\}, zo, \mathrm{Extract}(0)\rangle$.

One can prove the following three propositions:

(4) The trivial loop $= \langle\{0\}, zo, \mathrm{Extract}(0)\rangle$.

(5) If a is an element of the trivial loop, then $a = 0_{\text{the trivial loop}}$.

(6) For all elements a, b of the trivial loop holds $a + b = 0_{\text{the trivial loop}}$.

A loop structure is called a loop if:

(Def.5) (i) for every element a of it holds $a + 0_{\text{it}} = a$,

(ii) for every element a of it holds $0_{\text{it}} + a = a$,

(iii) for every elements a, b of it there exists an element x of it such that $a + x = b$,

(iv) for every elements a, b of it there exists an element x of it such that $x + a = b$,

(v) for all elements a, x, y of it such that $a + x = a + y$ holds $x = y$,

(vi) for all elements a, x, y of it such that $x + a = y + a$ holds $x = y$.

The following proposition is true

(7) Let G_1 be a loop structure. Then G_1 is a loop if and only if the following conditions are satisfied:

(i) for every element a of G_1 holds $a + 0_{G_1} = a$,

(ii) for every element a of G_1 holds $0_{G_1} + a = a$,

(iii) for every elements a, b of G_1 there exists an element x of G_1 such that $a + x = b$,

(iv) for every elements a, b of G_1 there exists an element x of G_1 such that $x + a = b$,

(v) for all elements a, x, y of G_1 such that $a + x = a + y$ holds $x = y$,

(vi) for all elements a, x, y of G_1 such that $x + a = y + a$ holds $x = y$.

Let us note that it makes sense to consider the following constant. Then the trivial loop is a loop.

A loop is called a group if:

(Def.6) for all elements a, b, c of it holds $(a + b) + c = a + (b + c)$.

We now state the proposition

(8) For every loop G_1 holds G_1 is a group if and only if for all elements a, b, c of G_1 holds $(a+b)+c = a+(b+c)$.

We follow the rules: L will be a loop structure and a, b, c, x will be elements of L. We now state the proposition

(9) L is a group if and only if for every a holds $a + 0_L = a$ and for every a there exists x such that $a + x = 0_L$ and for all a, b, c holds $(a+b)+c = a+(b+c)$.

Let us note that it makes sense to consider the following constant. Then the trivial loop is a group.

A group is called an Abelian group if:

(Def.7) for all elements a, b of it holds $a + b = b + a$.

Next we state two propositions:

(10) For every group G holds G is an Abelian group if and only if for all elements a, b of G holds $a + b = b + a$.

(11) L is an Abelian group if and only if the following conditions are satisfied:
   (i) for every a holds $a + 0_L = a$,
   (ii) for every a there exists x such that $a + x = 0_L$,
   (iii) for all a, b, c holds $(a+b)+c = a+(b+c)$,
   (iv) for all a, b holds $a + b = b + a$.

Let L be a group, and let a be an element of L. The functor $-a$ yielding an element of L is defined by:

(Def.8) $a + (-a) = 0_L$.

We now state the proposition

(12) For every group L and for every element a of L holds $a + (-a) = 0_L$.

In the sequel G will denote a group and a, b will denote elements of G. One can prove the following proposition

(13) $a + (-a) = 0_G$ and $(-a) + a = 0_G$.

Let us consider G, a, b. The functor $a - b$ yields an element of G and is defined as follows:

(Def.9) $a - b = a + (-b)$.

Next we state the proposition

(14) $a - b = a + (-b)$.

We consider mutiplicative loop structures which are systems

$\langle$a carrier, a multiplication, a unity$\rangle$,

where the carrier is a non-empty set, the multiplication is a binary operation on the carrier, and the unity is an element of the carrier. In the sequel G_1 is a mutiplicative loop structure. Let us consider G_1. An element of G_1 is an element of the carrier of G_1.

In the sequel a, b are elements of G_1. Let us consider G_1, a, b. The functor $a \cdot b$ yields an element of G_1 and is defined as follows:

(Def.10) $a \cdot b =$ (the multiplication of G_1)(a, b).

One can prove the following proposition

(15) $a \cdot b =$ (the multiplication of G_1)(a, b).

Let us consider G_1. The functor 1_{G_1} yields an element of G_1 and is defined by:

(Def.11) $1_{G_1} =$ the unity of G_1.

One can prove the following proposition

(16) $1_{G_1} =$ the unity of G_1.

The trivial multiplicative loop a mutiplicative loop structure is defined as follows:

(Def.12) the trivial multiplicative loop $= \langle \{0\}, zo, \mathrm{Extract}(0) \rangle$.

The following propositions are true:

(17) The trivial multiplicative loop $= \langle \{0\}, zo, \mathrm{Extract}(0) \rangle$.

(18) If a is an element of the trivial multiplicative loop, then $a = 1_{\text{the trivial multiplicative loop}}$.

(19) For all elements a, b of the trivial multiplicative loop holds $a \cdot b = 1_{\text{the trivial multiplicative loop}}$.

A mutiplicative loop structure is said to be a multiplicative loop if:

(Def.13) (i) for every element a of it holds $a \cdot (1_{\mathrm{it}}) = a$,
(ii) for every element a of it holds $(1_{\mathrm{it}}) \cdot a = a$,
(iii) for every elements a, b of it there exists an element x of it such that $a \cdot x = b$,
(iv) for every elements a, b of it there exists an element x of it such that $x \cdot a = b$,
(v) for all elements a, x, y of it such that $a \cdot x = a \cdot y$ holds $x = y$,
(vi) for all elements a, x, y of it such that $x \cdot a = y \cdot a$ holds $x = y$.

We now state the proposition

(20) Let L be a mutiplicative loop structure. Then L is a multiplicative loop if and only if the following conditions are satisfied:
(i) for every element a of L holds $a \cdot (1_L) = a$,
(ii) for every element a of L holds $(1_L) \cdot a = a$,
(iii) for every elements a, b of L there exists an element x of L such that $a \cdot x = b$,
(iv) for every elements a, b of L there exists an element x of L such that $x \cdot a = b$,
(v) for all elements a, x, y of L such that $a \cdot x = a \cdot y$ holds $x = y$,
(vi) for all elements a, x, y of L such that $x \cdot a = y \cdot a$ holds $x = y$.

Let us note that it makes sense to consider the following constant. Then the trivial multiplicative loop is a multiplicative loop.

A multiplicative loop is said to be a multiplicative group if:

(Def.14) for all elements a, b, c of it holds $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

One can prove the following proposition

(21) For every multiplicative loop L holds L is a multiplicative group if and only if for all elements a, b, c of L holds $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

We follow the rules: L is a mutiplicative loop structure and a, b, c, x are elements of L. One can prove the following proposition

(22) L is a multiplicative group if and only if for every a holds $a \cdot (1_L) = a$ and for every a there exists x such that $a \cdot x = 1_L$ and for all a, b, c holds $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Let us note that it makes sense to consider the following constant. Then the trivial multiplicative loop is a multiplicative group.

A multiplicative group is called a multiplicative Abelian group if:

(Def.15) for all elements a, b of it holds $a \cdot b = b \cdot a$.

The following propositions are true:

(23) For every multiplicative group G holds G is a multiplicative Abelian group if and only if for all elements a, b of G holds $a \cdot b = b \cdot a$.

(24) L is a multiplicative Abelian group if and only if the following conditions are satisfied:

(i) for every a holds $a \cdot (1_L) = a$,

(ii) for every a there exists x such that $a \cdot x = 1_L$,

(iii) for all a, b, c holds $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,

(iv) for all a, b holds $a \cdot b = b \cdot a$.

Let L be a multiplicative group, and let a be an element of L. The functor a^{-1} yields an element of L and is defined by:

(Def.16) $a \cdot (a^{-1}) = 1_L$.

The following proposition is true

(25) For every multiplicative group L and for every element a of L holds $a \cdot a^{-1} = 1_L$.

In the sequel G is a multiplicative group and a, b are elements of G. The following proposition is true

(26) $a \cdot a^{-1} = 1_G$ and $a^{-1} \cdot a = 1_G$.

Let us consider G, a, b. The functor $\frac{a}{b}$ yields an element of G and is defined by:

(Def.17) $\frac{a}{b} = a \cdot b^{-1}$.

One can prove the following proposition

(27) $\frac{a}{b} = a \cdot b^{-1}$.

We consider mutiplicative loop with zero structures which are systems

⟨a carrier, a multiplication, a unity, a zero⟩,

where the carrier is a non-empty set, the multiplication is a binary operation on the carrier, the unity is an element of the carrier, and the zero is an element of the carrier. In the sequel G_1 will be a mutiplicative loop with zero structure. Let us consider G_1. An element of G_1 is an element of the carrier of G_1.

In the sequel a, b will denote elements of G_1. Let us consider G_1, a, b. The functor $a \cdot b$ yielding an element of G_1 is defined by:

(Def.18) $a \cdot b =$ (the multiplication of G_1)(a, b).

The following proposition is true

(28) $a \cdot b =$ (the multiplication of G_1)(a, b).

Let us consider G_1. The functor 1_{G_1} yields an element of G_1 and is defined as follows:

(Def.19) $1_{G_1} =$ the unity of G_1.

One can prove the following proposition

(29) $1_{G_1} =$ the unity of G_1.

Let us consider G_1. The functor 0_{G_1} yielding an element of G_1 is defined as follows:

(Def.20) $0_{G_1} =$ the zero of G_1.

One can prove the following proposition

(30) $0_{G_1} =$ the zero of G_1.

The trivial multiplicative loop$_0$ a mutiplicative loop with zero structure is defined by:

(Def.21) the trivial multiplicative loop$_0$ $= \langle \mathbb{R}, \cdot_{\mathbb{R}}, 1, 0 \rangle$.

One can prove the following three propositions:

(31) The trivial multiplicative loop$_0$ $= \langle \mathbb{R}, \cdot_{\mathbb{R}}, 1, 0 \rangle$.

(32) For all real numbers q, p such that $q \neq 0$ there exists a real number y such that $p = q \cdot y$.

(33) For all real numbers q, p such that $q \neq 0$ there exists a real number y such that $p = y \cdot q$.

A mutiplicative loop with zero structure is called a multiplicative loop with zero if:

(Def.22) (i) $0_{\rm it} \neq 1_{\rm it}$,

(ii) for every element a of it holds $a \cdot (1_{\rm it}) = a$,

(iii) for every element a of it holds $(1_{\rm it}) \cdot a = a$,

(iv) for all elements a, b of it such that $a \neq 0_{\rm it}$ there exists an element x of it such that $a \cdot x = b$,

(v) for all elements a, b of it such that $a \neq 0_{\rm it}$ there exists an element x of it such that $x \cdot a = b$,

(vi) for all elements a, x, y of it such that $a \neq 0_{\rm it}$ holds if $a \cdot x = a \cdot y$, then $x = y$,

(vii) for all elements a, x, y of it such that $a \neq 0_{\rm it}$ holds if $x \cdot a = y \cdot a$, then $x = y$,

(viii) for every element a of it holds $a \cdot 0_{\rm it} = 0_{\rm it}$,

(ix) for every element a of it holds $0_{\rm it} \cdot a = 0_{\rm it}$.

The following proposition is true

(34) Let L be a mutiplicative loop with zero structure. Then L is a multiplicative loop with zero if and only if the following conditions are satisfied:

(i) $0_L \neq 1_L$,

(ii) for every element a of L holds $a \cdot (1_L) = a$,

(iii) for every element a of L holds $(1_L) \cdot a = a$,

(iv) for all elements a, b of L such that $a \neq 0_L$ there exists an element x of L such that $a \cdot x = b$,

(v) for all elements a, b of L such that $a \neq 0_L$ there exists an element x of L such that $x \cdot a = b$,

(vi) for all elements a, x, y of L such that $a \neq 0_L$ holds if $a \cdot x = a \cdot y$, then $x = y$,

(vii) for all elements a, x, y of L such that $a \neq 0_L$ holds if $x \cdot a = y \cdot a$, then $x = y$,

(viii) for every element a of L holds $a \cdot 0_L = 0_L$,

(ix) for every element a of L holds $0_L \cdot a = 0_L$.

Let us note that it makes sense to consider the following constant. Then the trivial multiplicative loop_0 is a multiplicative loop with zero.

A multiplicative loop with zero is called a multiplicative group with zero if:

(Def.23) for all elements a, b, c of it holds $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

One can prove the following proposition

(35) For every multiplicative loop L with zero holds L is a multiplicative group with zero if and only if for all elements a, b, c of L holds $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

We follow a convention: L denotes a mutiplicative loop with zero structure and a, b, c, x denote elements of L. One can prove the following proposition

(36) L is a multiplicative group with zero if and only if the following conditions are satisfied:

(i) $0_L \neq 1_L$,

(ii) for every a holds $a \cdot (1_L) = a$,

(iii) for every a such that $a \neq 0_L$ there exists x such that $a \cdot x = 1_L$,

(iv) for all a, b, c holds $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,

(v) for every a holds $a \cdot 0_L = 0_L$,

(vi) for every a holds $0_L \cdot a = 0_L$.

Let us note that it makes sense to consider the following constant. Then the trivial multiplicative loop_0 is a multiplicative group with zero.

A multiplicative group with zero is said to be a multiplicative commutative group with zero if:

(Def.24) for all elements a, b of it holds $a \cdot b = b \cdot a$.

We now state two propositions:

(37) For every multiplicative group L with zero holds L is a multiplicative commutative group with zero if and only if for all elements a, b of L holds $a \cdot b = b \cdot a$.

(38) L is a multiplicative commutative group with zero if and only if the following conditions are satisfied:

(i) $0_L \neq 1_L$,

(ii) for every a holds $a \cdot (1_L) = a$,

(iii) for every a such that $a \neq 0_L$ there exists x such that $a \cdot x = 1_L$,

(iv) for all a, b, c holds $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,

(v) for every a holds $a \cdot 0_L = 0_L$,

(vi) for every a holds $0_L \cdot a = 0_L$,

(vii) for all a, b holds $a \cdot b = b \cdot a$.

Let L be a multiplicative group with zero, and let a be an element of L. Let us assume that $a \neq 0_L$. The functor a^{-1} yielding an element of L is defined as follows:

(Def.25) $a \cdot (a^{-1}) = 1_L$.

We now state the proposition

(39) For every multiplicative group L with zero and for every element a of L such that $a \neq 0_L$ holds $a \cdot a^{-1} = 1_L$.

In the sequel G will be a multiplicative group with zero and a, b will be elements of G. One can prove the following proposition

(40) If $a \neq 0_G$, then $a \cdot a^{-1} = 1_G$ and $a^{-1} \cdot a = 1_G$.

Let us consider G, a, b. Let us assume that $b \neq 0_G$. The functor $\frac{a}{b}$ yields an element of G and is defined by:

(Def.26) $\frac{a}{b} = a \cdot b^{-1}$.

We now state the proposition

(41) If $b \neq 0_G$, then $\frac{a}{b} = a \cdot b^{-1}$.

References

[1] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[2] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[3] Michał Muzalewski. Midpoint algebras. *Formalized Mathematics*, 1(**3**):483–488, 1990.

[4] Wanda Szmielew. *From Affine to Euclidean Geometry.* Volume 27, PWN – D.Reidel Publ. Co., Warszawa – Dordrecht, 1983.

Received July 10, 1990

FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Basic Properties of Rational Numbers

Andrzej Kondracki[1]
Warsaw University

Summary. A definition of rational numbers and some basic properties of them. Operations of addition, substraction, multiplication are redefined for rational numbers. Functors numerator (num p) and denominator (den p) (p is rational) are defined and some properties of them are presented. Density of rational numbers is also given.

MML Identifier: `RAT_1`.

The notation and terminology used here are introduced in the following papers: [4], [2], [1], [3], and [5]. For simplicity we follow the rules: x is arbitrary, a, b are real numbers, k, k_1, l, l_1 are natural numbers, m, m_1, n, n_1 are integers, and D is a non-empty set. Let us consider m. Then $|m|$ is a natural number.

Let us consider k. Then $|k|$ is a natural number.

The non-empty set $\mathbb{Q}$ is defined by:

(Def.1) $x \in \mathbb{Q}$ if and only if there exist m, n such that $n \neq 0$ and $x = \frac{m}{n}$.

One can prove the following proposition

(1) $D = \mathbb{Q}$ if and only if for every x holds $x \in D$ if and only if there exist m, n such that $n \neq 0$ and $x = \frac{m}{n}$.

A real number is said to be a rational number if:

(Def.2) it is an element of $\mathbb{Q}$.

We now state a number of propositions:

(2) For every real number x holds x is a rational number if and only if x is an element of $\mathbb{Q}$.

(4)[2] If $x \in \mathbb{Q}$, then $x \in \mathbb{R}$.

(5) x is a rational number if and only if $x \in \mathbb{Q}$.

(6) x is a rational number if and only if there exist m, n such that $n \neq 0$ and $x = \frac{m}{n}$.

[1]Supported by RPBP.III-24.C1.

[2]The proposition (3) became obvious.

ISSN 0777–4028

(7) For every integer x holds x is a rational number.

(8) For every natural number x holds x is a rational number.

(9) 1 is a rational number and 0 is a rational number.

(10) $\mathbb{Q} \subseteq \mathbb{R}$.

(11) $\mathbb{Z} \subseteq \mathbb{Q}$.

(12) $\mathbb{N} \subseteq \mathbb{Q}$.

In the sequel p, q denote rational numbers. Next we state three propositions:

(13) If $x = \frac{k}{l}$ and $l \neq 0$, then x is a rational number.

(14) If $x = \frac{m}{k}$ and $k \neq 0$, then x is a rational number.

(15) If $x = \frac{k}{m}$ and $m \neq 0$, then x is a rational number.

Let us consider p, q. Then $p \cdot q$ is a rational number. Then $p + q$ is a rational number. Then $p - q$ is a rational number.

Let us consider p, m. Then $p + m$ is a rational number. Then $p - m$ is a rational number. Then $p \cdot m$ is a rational number.

Let us consider m, p. Then $m + p$ is a rational number. Then $m - p$ is a rational number. Then $m \cdot p$ is a rational number.

Let us consider p, k. Then $p+k$ is a rational number. Then $p-k$ is a rational number. Then $p \cdot k$ is a rational number.

Let us consider k, p. Then $k+p$ is a rational number. Then $k-p$ is a rational number. Then $k \cdot p$ is a rational number.

Let us consider p. Then $-p$ is a rational number. Then $|p|$ is a rational number.

One can prove the following propositions:

(16) For all p, q such that $q \neq 0$ holds $\frac{p}{q}$ is a rational number.

(17) If $k \neq 0$, then $\frac{p}{k}$ is a rational number.

(18) If $m \neq 0$, then $\frac{p}{m}$ is a rational number.

(19) If $p \neq 0$, then $\frac{k}{p}$ is a rational number and $\frac{m}{p}$ is a rational number.

(20) For every p such that $p \neq 0$ holds $\frac{1}{p}$ is a rational number.

(21) For every p such that $p \neq 0$ holds p^{-1} is a rational number.

(22) For all a, b such that $a < b$ there exists p such that $a < p$ and $p < b$.

(23) $a < b$ if and only if there exists p such that $a < p$ and $p < b$.

(24) For every p there exist m, k such that $k \neq 0$ and $p = \frac{m}{k}$.

(25) For every p there exist m, k such that $k \neq 0$ and $p = \frac{m}{k}$ and for all n, l such that $l \neq 0$ and $p = \frac{n}{l}$ holds $k \leq l$.

Let us consider p. The functor $\operatorname{den} p$ yielding a natural number is defined by:

(Def.3) $\operatorname{den} p \neq 0$ and there exists m such that $p = \frac{m}{\operatorname{den} p}$ and for all n, k such that $k \neq 0$ and $p = \frac{n}{k}$ holds $\operatorname{den} p \leq k$.

We now state the proposition

(26) $\operatorname{den} p \neq 0$ and there exists m such that $p = \frac{m}{\operatorname{den} p}$ and for all n, k such that $k \neq 0$ and $p = \frac{n}{k}$ holds $\operatorname{den} p \leq k$.

Let us consider p. The functor $\operatorname{num} p$ yields an integer and is defined by:

(Def.4) $\operatorname{num} p = \operatorname{den} p \cdot p$.

One can prove the following propositions:

(27) $0 < \operatorname{den} p$.

(28) $0 \neq \operatorname{den} p$.

(29) $1 \leq \operatorname{den} p$.

(30) $0 < \operatorname{den} p^{-1}$.

(31) $0 \leq \operatorname{den} p$.

(32) $0 \leq \operatorname{den} p^{-1}$.

(33) $0 \neq \operatorname{den} p^{-1}$.

(34) $1 \geq \operatorname{den} p^{-1}$.

(35) $\operatorname{num} p = \operatorname{den} p \cdot p$ and $\operatorname{num} p = p \cdot \operatorname{den} p$.

(36) $\operatorname{num} p = 0$ if and only if $p = 0$.

(37) $p = \frac{\operatorname{num} p}{\operatorname{den} p}$ and $p = \operatorname{num} p \cdot \operatorname{den} p^{-1}$ and $p = \operatorname{den} p^{-1} \cdot \operatorname{num} p$.

(38) If $p \neq 0$, then $\operatorname{den} p = \frac{\operatorname{num} p}{p}$.

(39) If $p = \frac{m}{k}$ and $k \neq 0$, then $\operatorname{den} p \leq k$.

(40) If p is an integer, then $\operatorname{den} p = 1$ and $\operatorname{num} p = p$.

(41) If $\operatorname{num} p = p$ or $\operatorname{den} p = 1$, then p is an integer.

(42) $\operatorname{num} p = p$ if and only if $\operatorname{den} p = 1$.

(43) If p is a natural number, then $\operatorname{den} p = 1$ and $\operatorname{num} p = p$.

(44) If $\operatorname{num} p = p$ or $\operatorname{den} p = 1$ but $0 \leq p$, then p is a natural number.

(45) $1 < \operatorname{den} p$ if and only if p is not an integer.

(46) $1 > \operatorname{den} p^{-1}$ if and only if p is not an integer.

(47) $\operatorname{num} p = \operatorname{den} p$ if and only if $p = 1$.

(48) $\operatorname{num} p = -\operatorname{den} p$ if and only if $p = -1$.

(49) $-\operatorname{num} p = \operatorname{den} p$ if and only if $p = -1$.

(50) Suppose $m \neq 0$. Then $p = \frac{\operatorname{num} p \cdot m}{\operatorname{den} p \cdot m}$ and $p = \frac{m \cdot \operatorname{num} p}{\operatorname{den} p \cdot m}$ and $p = \frac{m \cdot \operatorname{num} p}{m \cdot \operatorname{den} p}$ and $p = \frac{\operatorname{num} p \cdot m}{m \cdot \operatorname{den} p}$.

(51) Suppose $k \neq 0$. Then $p = \frac{\operatorname{num} p \cdot k}{\operatorname{den} p \cdot k}$ and $p = \frac{k \cdot \operatorname{num} p}{\operatorname{den} p \cdot k}$ and $p = \frac{k \cdot \operatorname{num} p}{k \cdot \operatorname{den} p}$ and $p = \frac{\operatorname{num} p \cdot k}{k \cdot \operatorname{den} p}$.

(52) Suppose $p = \frac{m}{n}$ and $n \neq 0$ and $m_1 \neq 0$. Then $p = \frac{m \cdot m_1}{n \cdot m_1}$ and $p = \frac{m_1 \cdot m}{n \cdot m_1}$ and $p = \frac{m_1 \cdot m}{m_1 \cdot n}$ and $p = \frac{m \cdot m_1}{m_1 \cdot n}$.

(53) Suppose $p = \frac{m}{l}$ and $l \neq 0$ and $m_1 \neq 0$. Then $p = \frac{m \cdot m_1}{l \cdot m_1}$ and $p = \frac{m_1 \cdot m}{l \cdot m_1}$ and $p = \frac{m_1 \cdot m}{m_1 \cdot l}$ and $p = \frac{m \cdot m_1}{m_1 \cdot l}$.

(54) Suppose $p = \frac{l}{n}$ and $n \neq 0$ and $m_1 \neq 0$. Then $p = \frac{l \cdot m_1}{n \cdot m_1}$ and $p = \frac{m_1 \cdot l}{n \cdot m_1}$ and $p = \frac{m_1 \cdot l}{m_1 \cdot n}$ and $p = \frac{l \cdot m_1}{m_1 \cdot n}$.

(55) Suppose $p = \frac{l}{l_1}$ and $l_1 \neq 0$ and $m_1 \neq 0$. Then $p = \frac{l \cdot m_1}{l_1 \cdot m_1}$ and $p = \frac{m_1 \cdot l}{l_1 \cdot m_1}$ and $p = \frac{m_1 \cdot l}{m_1 \cdot l_1}$ and $p = \frac{l \cdot m_1}{m_1 \cdot l_1}$.

(56) Suppose $p = \frac{m}{n}$ and $n \neq 0$ and $k \neq 0$. Then $p = \frac{m \cdot k}{n \cdot k}$ and $p = \frac{k \cdot m}{n \cdot k}$ and $p = \frac{k \cdot m}{k \cdot n}$ and $p = \frac{m \cdot k}{k \cdot n}$.

(57) Suppose $p = \frac{m}{l}$ and $l \neq 0$ and $k \neq 0$. Then $p = \frac{m \cdot k}{l \cdot k}$ and $p = \frac{k \cdot m}{l \cdot k}$ and $p = \frac{k \cdot m}{k \cdot l}$ and $p = \frac{m \cdot k}{k \cdot l}$.

(58) Suppose $p = \frac{l}{n}$ and $n \neq 0$ and $k \neq 0$. Then $p = \frac{l \cdot k}{n \cdot k}$ and $p = \frac{k \cdot l}{n \cdot k}$ and $p = \frac{k \cdot l}{k \cdot n}$ and $p = \frac{l \cdot k}{k \cdot n}$.

(59) Suppose $p = \frac{l}{l_1}$ and $l_1 \neq 0$ and $k \neq 0$. Then $p = \frac{l \cdot k}{l_1 \cdot k}$ and $p = \frac{k \cdot l}{l_1 \cdot k}$ and $p = \frac{k \cdot l}{k \cdot l_1}$ and $p = \frac{l \cdot k}{k \cdot l_1}$.

(60) If $k \neq 0$ and $p = \frac{m}{k}$, then there exists l such that $m = \operatorname{num} p \cdot l$ and $k = \operatorname{den} p \cdot l$.

(61) If $p = \frac{m}{n}$ and $n \neq 0$, then there exists m_1 such that $m = \operatorname{num} p \cdot m_1$ and $n = \operatorname{den} p \cdot m_1$.

(62) For no l holds $1 < l$ and there exist m, k such that $\operatorname{num} p = m \cdot l$ and $\operatorname{den} p = k \cdot l$.

(63) If $p = \frac{m}{k}$ and $k \neq 0$ and for no l holds $1 < l$ and there exist m_1, k_1 such that $m = m_1 \cdot l$ and $k = k_1 \cdot l$, then $k = \operatorname{den} p$ and $m = \operatorname{num} p$.

(64) $p < -1$ if and only if $\operatorname{num} p < -\operatorname{den} p$.

(65) $p \leq -1$ if and only if $\operatorname{num} p \leq -\operatorname{den} p$.

(66) $p < -1$ if and only if $\operatorname{den} p < -\operatorname{num} p$.

(67) $p \leq -1$ if and only if $\operatorname{den} p \leq -\operatorname{num} p$.

(68) $-1 < p$ if and only if $-\operatorname{den} p < \operatorname{num} p$.

(69) $p \geq -1$ if and only if $\operatorname{num} p \geq -\operatorname{den} p$.

(70) $-1 < p$ if and only if $-\operatorname{num} p < \operatorname{den} p$.

(71) $p \geq -1$ if and only if $\operatorname{den} p \geq -\operatorname{num} p$.

(72) $p < 1$ if and only if $\operatorname{num} p < \operatorname{den} p$.

(73) $p \leq 1$ if and only if $\operatorname{num} p \leq \operatorname{den} p$.

(74) $1 < p$ if and only if $\operatorname{den} p < \operatorname{num} p$.

(75) $p \geq 1$ if and only if $\operatorname{num} p \geq \operatorname{den} p$.

(76) $p < 0$ if and only if $\operatorname{num} p < 0$.

(77) $p \leq 0$ if and only if $\operatorname{num} p \leq 0$.

(78) $0 < p$ if and only if $0 < \operatorname{num} p$.

(79) $p \geq 0$ if and only if $\operatorname{num} p \geq 0$.

(80) $a < p$ if and only if $a \cdot \operatorname{den} p < \operatorname{num} p$.

(81) $a \leq p$ if and only if $a \cdot \operatorname{den} p \leq \operatorname{num} p$.

(82) $p < a$ if and only if $\operatorname{num} p < a \cdot \operatorname{den} p$.

(83) $a \geq p$ if and only if $a \cdot \operatorname{den} p \geq \operatorname{num} p$.

(84) $p = q$ if and only if $\operatorname{den} p = \operatorname{den} q$ and $\operatorname{num} p = \operatorname{num} q$.

(85) If $p = \frac{m}{n}$ and $n \neq 0$ and $q = \frac{m_1}{n_1}$ and $n_1 \neq 0$, then $p = q$ if and only if $m \cdot n_1 = m_1 \cdot n$.

(86) $p < q$ if and only if $\operatorname{num} p \cdot \operatorname{den} q < \operatorname{num} q \cdot \operatorname{den} p$.

(87) $\operatorname{den}(-p) = \operatorname{den} p$ and $\operatorname{num}(-p) = -\operatorname{num} p$.

(88) $0 < p$ and $q = \frac{1}{p}$ if and only if $\operatorname{num} q = \operatorname{den} p$ and $\operatorname{den} q = \operatorname{num} p$.

(89) $p < 0$ and $q = \frac{1}{p}$ if and only if $\operatorname{num} q = -\operatorname{den} p$ and $\operatorname{den} q = -\operatorname{num} p$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[3] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[4] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[5] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.

Received July 10, 1990

FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Basis of Real Linear Space

Wojciech A. Trybulec
Warsaw University

Summary. Notions of linear independence and dependence of set of vectors, the subspace generated by a set of vectors and basis of real linear space are introduced. Some theorems concerning those notion, are proved.

MML Identifier: `RLVECT_3`.

The papers [6], [2], [1], [3], [11], [4], [10], [9], [5], [8], and [7] provide the notation and terminology for this paper. For simplicity we follow a convention: x is arbitrary, a, b are real numbers, V is a real linear space, W, W_1, W_2, W_3 are subspaces of V, v, v_1, v_2 are vectors of V, A, B are subsets of the vectors of V, L, L_1, L_2 are linear combinations of V, l is a linear combination of A, F, G are finite sequences of elements of the vectors of V, f is a function from the vectors of V into $\mathbb{R}$, X, Y, Z are sets, M is a non-empty family of sets, and C_1 is a choice function of M. One can prove the following four propositions:

(1) $\sum(L_1 + L_2) = \sum L_1 + \sum L_2$.

(2) $\sum(a \cdot L) = a \cdot \sum L$.

(3) $\sum(-L) = -\sum L$.

(4) $\sum(L_1 - L_2) = \sum L_1 - \sum L_2$.

We now define two new predicates. Let us consider V, A. We say that A is linearly independent if and only if:

(Def.1) for every l such that $\sum l = 0_V$ holds $\operatorname{support} l = \emptyset$.

We say that A is linearly dependent if and only if A is not linearly independent.

One can prove the following propositions:

(5) A is linearly independent if and only if for every l such that $\sum l = 0_V$ holds $\operatorname{support} l = \emptyset$.

(6) If $A \subseteq B$ and B is linearly independent, then A is linearly independent.

(7) If A is linearly independent, then $0_V \notin A$.

(8) $\emptyset_{\text{the vectors of } V}$ is linearly independent.

ISSN 0777–4028

(9) $\{v\}$ is linearly independent if and only if $v \neq 0_V$.

(10) $\{0_V\}$ is linearly dependent.

(11) If $\{v_1, v_2\}$ is linearly independent, then $v_1 \neq 0_V$ and $v_2 \neq 0_V$.

(12) $\{v, 0_V\}$ is linearly dependent and $\{0_V, v\}$ is linearly dependent.

(13) $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent if and only if $v_2 \neq 0_V$ and for every a holds $v_1 \neq a \cdot v_2$.

(14) $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent if and only if for all a, b such that $a \cdot v_1 + b \cdot v_2 = 0_V$ holds $a = 0$ and $b = 0$.

Let us consider V, A. The functor $\mathrm{Lin}(A)$ yields a subspace of V and is defined by:

(Def.2) the vectors of $\mathrm{Lin}(A) = \{\sum l\}$.

We now state four propositions:

(15) If the vectors of $W = \{\sum l\}$, then $W = \mathrm{Lin}(A)$.

(16) The vectors of $\mathrm{Lin}(A) = \{\sum l\}$.

(17) $x \in \mathrm{Lin}(A)$ if and only if there exists l such that $x = \sum l$.

(18) If $x \in A$, then $x \in \mathrm{Lin}(A)$.

The following propositions are true:

(19) $\mathrm{Lin}(\emptyset_{\text{the vectors of } V}) = \mathbf{0}_V$.

(20) If $\mathrm{Lin}(A) = \mathbf{0}_V$, then $A = \emptyset$ or $A = \{0_V\}$.

(21) If $A =$ the vectors of W, then $\mathrm{Lin}(A) = W$.

(22) If $A =$ the vectors of V, then $\mathrm{Lin}(A) = V$.

(23) If $A \subseteq B$, then $\mathrm{Lin}(A)$ is a subspace of $\mathrm{Lin}(B)$.

(24) If $\mathrm{Lin}(A) = V$ and $A \subseteq B$, then $\mathrm{Lin}(B) = V$.

(25) $\mathrm{Lin}(A \cup B) = \mathrm{Lin}(A) + \mathrm{Lin}(B)$.

(26) $\mathrm{Lin}(A \cap B)$ is a subspace of $\mathrm{Lin}(A) \cap \mathrm{Lin}(B)$.

(27) If A is linearly independent, then there exists B such that $A \subseteq B$ and B is linearly independent and $\mathrm{Lin}(B) = V$.

(28) If $\mathrm{Lin}(A) = V$, then there exists B such that $B \subseteq A$ and B is linearly independent and $\mathrm{Lin}(B) = V$.

Let us consider V. A subset of the vectors of V is called a basis of V if:

(Def.3) it is linearly independent and $\mathrm{Lin}(\mathrm{it}) = V$.

The following proposition is true

(29) If A is linearly independent and $\mathrm{Lin}(A) = V$, then A is a basis of V.

In the sequel I is a basis of V. Next we state a number of propositions:

(30) I is linearly independent.

(31) $\mathrm{Lin}(I) = V$.

(32) If A is linearly independent, then there exists I such that $A \subseteq I$.

(33) If $\mathrm{Lin}(A) = V$, then there exists I such that $I \subseteq A$.

(34) If $Z \neq \emptyset$ and Z is finite and for all X, Y such that $X \in Z$ and $Y \in Z$ holds $X \subseteq Y$ or $Y \subseteq X$, then $\bigcup Z \in Z$.

(35) If $\emptyset \notin M$, then $\mathrm{dom}\, C_1 = M$ and $\mathrm{rng}\, C_1 \subseteq \bigcup M$.

(36) $x \in \mathbf{0}_V$ if and only if $x = 0_V$.

(37) If W_1 is a subspace of W_3, then $W_1 \cap W_2$ is a subspace of W_3.

(38) If W_1 is a subspace of W_2 and W_1 is a subspace of W_3, then W_1 is a subspace of $W_2 \cap W_3$.

(39) If W_1 is a subspace of W_3 and W_2 is a subspace of W_3, then $W_1 + W_2$ is a subspace of W_3.

(40) If W_1 is a subspace of W_2, then W_1 is a subspace of $W_2 + W_3$.

(41) $f \cdot (F ^\frown G) = (f \cdot F) ^\frown (f \cdot G)$.

References

[1] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[4] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[5] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[6] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[7] Wojciech A. Trybulec. Linear combinations in real linear space. *Formalized Mathematics*, 1(**3**):581–588, 1990.

[8] Wojciech A. Trybulec. Operations on subspaces in real linear space. *Formalized Mathematics*, 1(**2**):395–399, 1990.

[9] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.

[10] Wojciech A. Trybulec. Subspaces and cosets of subspaces in real linear space. *Formalized Mathematics*, 1(**2**):297–301, 1990.

[11] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

Received July 10, 1990

FORMALIZED MATHEMATICS
Vol.1, No.**5**, November–December 1990
Université Catholique de Louvain

Finite Sums of Vectors in Vector Space

Wojciech A. Trybulec
Warsaw University

Summary. We define the sum of finite sequences of vectors in vector space. Theorems concerning those sums are proved.

MML Identifier: `VECTSP_3`.

The terminology and notation used here have been introduced in the following papers: [7], [2], [3], [5], [6], [4], and [1]. Let F be a field. An element of F is an element of the carrier of F.

For simplicity we follow a convention: x will be arbitrary, G_1 will denote a field, a will denote an element of G_1, V will denote a vector space over G_1, and v, v_1, v_2, w, u will denote vectors of V. Let us consider G_1, V, x. The predicate $x \in V$ is defined by:

(Def.1) $x \in$ the carrier of the carrier of V.

Next we state two propositions:

(1) $x \in V$ if and only if $x \in$ the carrier of the carrier of V.

(2) $v \in V$.

We follow a convention: F, G, H will be finite sequences of elements of the carrier of the carrier of V, f will be a function from $\mathbb{N}$ into the carrier of the carrier of V, and i, j, k, n will be natural numbers. Let us consider G_1, V, f, j. Then $f(j)$ is a vector of V.

Let us consider G_1, V, F. The functor $\sum F$ yielding a vector of V is defined as follows:

(Def.2) there exists f such that $\sum F = f(\operatorname{len} F)$ and $f(0) = \Theta_V$ and for all j, v such that $j < \operatorname{len} F$ and $v = F(j+1)$ holds $f(j+1) = f(j) + v$.

We now state a number of propositions:

(3) If there exists f such that $u = f(\operatorname{len} F)$ and $f(0) = \Theta_V$ and for all j, v such that $j < \operatorname{len} F$ and $v = F(j+1)$ holds $f(j+1) = f(j) + v$, then $u = \sum F$.

ISSN 0777–4028

(4) There exists f such that $\sum F = f(\operatorname{len} F)$ and $f(0) = \Theta_V$ and for all j, v such that $j < \operatorname{len} F$ and $v = F(j+1)$ holds $f(j+1) = f(j) + v$.

(5) If $k \in \operatorname{Seg} n$ and $\operatorname{len} F = n$, then $F(k)$ is a vector of V.

(6) If $\operatorname{len} F = \operatorname{len} G + 1$ and $G = F \restriction \operatorname{Seg}(\operatorname{len} G)$ and $v = F(\operatorname{len} F)$, then $\sum F = \sum G + v$.

(7) $\sum (F \frown G) = \sum F + \sum G$.

(8) If $\operatorname{len} F = \operatorname{len} G$ and $\operatorname{len} F = \operatorname{len} H$ and for every k such that $k \in \operatorname{Seg}(\operatorname{len} F)$ holds $H(k) = \pi_k F + \pi_k G$, then $\sum H = \sum F + \sum G$.

(9) If $\operatorname{len} F = \operatorname{len} G$ and for all k, v such that $k \in \operatorname{Seg}(\operatorname{len} F)$ and $v = G(k)$ holds $F(k) = a \cdot v$, then $\sum F = a \cdot \sum G$.

(10) If $\operatorname{len} F = \operatorname{len} G$ and for every k such that $k \in \operatorname{Seg}(\operatorname{len} F)$ holds $G(k) = a \cdot \pi_k F$, then $\sum G = a \cdot \sum F$.

(11) If $\operatorname{len} F = \operatorname{len} G$ and for all k, v such that $k \in \operatorname{Seg}(\operatorname{len} F)$ and $v = G(k)$ holds $F(k) = -v$, then $\sum F = -\sum G$.

(12) If $\operatorname{len} F = \operatorname{len} G$ and for every k such that $k \in \operatorname{Seg}(\operatorname{len} F)$ holds $G(k) = -\pi_k F$, then $\sum G = -\sum F$.

(13) If $\operatorname{len} F = \operatorname{len} G$ and $\operatorname{len} F = \operatorname{len} H$ and for every k such that $k \in \operatorname{Seg}(\operatorname{len} F)$ holds $H(k) = \pi_k F - \pi_k G$, then $\sum H = \sum F - \sum G$.

(14) If $\operatorname{rng} F = \operatorname{rng} G$ and F is one-to-one and G is one-to-one, then $\sum F = \sum G$.

(15) For all F, G and for every permutation f of $\operatorname{dom} F$ such that $\operatorname{len} F = \operatorname{len} G$ and for every i such that $i \in \operatorname{dom} G$ holds $G(i) = F(f(i))$ holds $\sum F = \sum G$.

(16) For every permutation f of $\operatorname{dom} F$ such that $G = F \cdot f$ holds $\sum F = \sum G$.

(17) $\sum \varepsilon_{\text{the carrier of the carrier of } V} = \Theta_V$.

(18) $\sum \langle v \rangle = v$.

(19) $\sum \langle v, u \rangle = v + u$.

(20) $\sum \langle v, u, w \rangle = (v + u) + w$.

(21) $a \cdot \sum \varepsilon_{\text{the carrier of the carrier of } V} = \Theta_V$.

(22) $a \cdot \sum \langle v \rangle = a \cdot v$.

(23) $a \cdot \sum \langle v, u \rangle = a \cdot v + a \cdot u$.

(24) $a \cdot \sum \langle v, u, w \rangle = (a \cdot v + a \cdot u) + a \cdot w$.

(25) $-\sum \varepsilon_{\text{the carrier of the carrier of } V} = \Theta_V$.

(26) $-\sum \langle v \rangle = -v$.

(27) $-\sum \langle v, u \rangle = (-v) - u$.

(28) $-\sum \langle v, u, w \rangle = ((-v) - u) - w$.

(29) $\sum \langle v, w \rangle = \sum \langle w, v \rangle$.

(30) $\sum \langle v, w \rangle = \sum \langle v \rangle + \sum \langle w \rangle$.

(31) $\sum \langle \Theta_V, \Theta_V \rangle = \Theta_V$.

(32) $\sum \langle \Theta_V, v \rangle = v$ and $\sum \langle v, \Theta_V \rangle = v$.

(33) $\sum\langle v, -v\rangle = \Theta_V$ and $\sum\langle -v, v\rangle = \Theta_V$.
(34) $\sum\langle v, -w\rangle = v - w$ and $\sum\langle -w, v\rangle = v - w$.
(35) $\sum\langle -v, -w\rangle = -(v + w)$ and $\sum\langle -w, -v\rangle = -(v + w)$.
(36) $\sum\langle u, v, w\rangle = (\sum\langle u\rangle + \sum\langle v\rangle) + \sum\langle w\rangle$.
(37) $\sum\langle u, v, w\rangle = \sum\langle u, v\rangle + w$.
(38) $\sum\langle u, v, w\rangle = \sum\langle v, w\rangle + u$.
(39) $\sum\langle u, v, w\rangle = \sum\langle u, w\rangle + v$.
(40) $\sum\langle u, v, w\rangle = \sum\langle u, w, v\rangle$.
(41) $\sum\langle u, v, w\rangle = \sum\langle v, u, w\rangle$.
(42) $\sum\langle u, v, w\rangle = \sum\langle v, w, u\rangle$.
(43) $\sum\langle u, v, w\rangle = \sum\langle w, u, v\rangle$.
(44) $\sum\langle u, v, w\rangle = \sum\langle w, v, u\rangle$.
(45) $\sum\langle \Theta_V, \Theta_V, \Theta_V\rangle = \Theta_V$.
(46) $\sum\langle \Theta_V, \Theta_V, v\rangle = v$ and $\sum\langle \Theta_V, v, \Theta_V\rangle = v$ and $\sum\langle v, \Theta_V, \Theta_V\rangle = v$.
(47) $\sum\langle \Theta_V, u, v\rangle = u + v$ and $\sum\langle u, v, \Theta_V\rangle = u + v$ and $\sum\langle u, \Theta_V, v\rangle = u + v$.
(48) If $\operatorname{len} F = 0$, then $\sum F = \Theta_V$.
(49) If $\operatorname{len} F = 1$, then $\sum F = F(1)$.
(50) If $\operatorname{len} F = 2$ and $v_1 = F(1)$ and $v_2 = F(2)$, then $\sum F = v_1 + v_2$.
(51) If $\operatorname{len} F = 3$ and $v_1 = F(1)$ and $v_2 = F(2)$ and $v = F(3)$, then $\sum F = (v_1 + v_2) + v$.
(52) $v - v = \Theta_V$.
(53) $-(v + w) = (-v) + (-w)$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[4] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[5] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[6] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.

[7] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

Received July 12, 1990

FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Subgroup and Cosets of Subgroups

Wojciech A. Trybulec[1]
Warsaw University

Summary. We introduce notion of subgroup, coset of a subgroup, sets of left and right cosets of a subgroup. We define multiplication of two subset of a group, subset of reverse elemens of a group, intersection of two subgroups. We define the notion of an index of a subgroup and prove Lagrange theorem which states that in a finite group the order of the group equals the order of a subgroup multiplied by the index of the subgroup. Some theorems that belong rather to [1] are proved.

MML Identifier: `GROUP_2`.

The papers [9], [6], [3], [4], [1], [11], [10], [12], [5], [8], [7], and [2] provide the notation and terminology for this paper. Let D be a non-empty set. Then $\emptyset_D$ is a subset of D. Then Ω_D is a subset of D.

For simplicity we adopt the following convention: x is arbitrary, X, Y, Z are sets, k is a natural number, G, G_1, G_2, G_3 are groups, and a, b, g, g_1, g_2, h are elements of G. Let us consider G. A subset of G is a subset of the carrier of G.

In the sequel A, B, C denote subsets of G. The following propositions are true:

(1) If $x \in A$, then $x \in G$.

(2) If $x \in A$, then x is an element of G.

(3) If G is finite, then A is finite.

Let us consider G, A. The functor A^{-1} yielding a subset of G is defined by:

(Def.1) $A^{-1} = \{g^{-1} : g \in A\}$.

Next we state several propositions:

(4) $A^{-1} = \{g^{-1} : g \in A\}$.

(5) $x \in A^{-1}$ if and only if there exists g such that $x = g^{-1}$ and $g \in A$.

(6) $\{g\}^{-1} = \{g^{-1}\}$.

[1]Supported by RPBP.III-24.C1

ISSN 0777–4028

(7) $\{g, h\}^{-1} = \{g^{-1}, h^{-1}\}$.

(8) $(\emptyset_{\text{the carrier of } G})^{-1} = \emptyset$.

(9) $(\Omega_{\text{the carrier of } G})^{-1} =$ the carrier of G.

(10) $A \neq \emptyset$ if and only if $A^{-1} \neq \emptyset$.

Let us consider G, A, B. The functor $A \cdot B$ yielding a subset of G is defined as follows:

(Def.2) $A \cdot B = \{g \cdot h : g \in A \wedge h \in B\}$.

One can prove the following propositions:

(11) $A \cdot B = \{g \cdot h : g \in A \wedge h \in B\}$.

(12) $x \in A \cdot B$ if and only if there exist g, h such that $x = g \cdot h$ and $g \in A$ and $h \in B$.

(13) $A \neq \emptyset$ and $B \neq \emptyset$ if and only if $A \cdot B \neq \emptyset$.

(14) $(A \cdot B) \cdot C = A \cdot (B \cdot C)$.

(15) $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$.

(16) $A \cdot (B \cup C) = A \cdot B \cup A \cdot C$.

(17) $(A \cup B) \cdot C = A \cdot C \cup B \cdot C$.

(18) $A \cdot (B \cap C) \subseteq (A \cdot B) \cap (A \cdot C)$.

(19) $(A \cap B) \cdot C \subseteq (A \cdot C) \cap (B \cdot C)$.

(20) $\emptyset_{\text{the carrier of } G} \cdot A = \emptyset$ and $A \cdot \emptyset_{\text{the carrier of } G} = \emptyset$.

(21) If $A \neq \emptyset$, then $\Omega_{\text{the carrier of } G} \cdot A =$ the carrier of G and $A \cdot \Omega_{\text{the carrier of } G} =$ the carrier of G.

(22) $\{g\} \cdot \{h\} = \{g \cdot h\}$.

(23) $\{g\} \cdot \{g_1, g_2\} = \{g \cdot g_1, g \cdot g_2\}$.

(24) $\{g_1, g_2\} \cdot \{g\} = \{g_1 \cdot g, g_2 \cdot g\}$.

(25) $\{g, h\} \cdot \{g_1, g_2\} = \{g \cdot g_1, g \cdot g_2, h \cdot g_1, h \cdot g_2\}$.

(26) If for all g_1, g_2 such that $g_1 \in A$ and $g_2 \in A$ holds $g_1 \cdot g_2 \in A$ and for every g such that $g \in A$ holds $g^{-1} \in A$, then $A \cdot A = A$.

(27) If for every g such that $g \in A$ holds $g^{-1} \in A$, then $A^{-1} = A$.

(28) If for all a, b such that $a \in A$ and $b \in B$ holds $a \cdot b = b \cdot a$, then $A \cdot B = B \cdot A$.

(29) If G is an Abelian group, then $A \cdot B = B \cdot A$.

(30) If G is an Abelian group, then $(A \cdot B)^{-1} = A^{-1} \cdot B^{-1}$.

We now define two new functors. Let us consider G, g, A. The functor $g \cdot A$ yields a subset of G and is defined as follows:

(Def.3) $g \cdot A = \{g\} \cdot A$.

The functor $A \cdot g$ yielding a subset of G is defined as follows:

(Def.4) $A \cdot g = A \cdot \{g\}$.

Next we state a number of propositions:

(31) $g \cdot A = \{g\} \cdot A$.

(32) $A \cdot g = A \cdot \{g\}$.

(33) $x \in g \cdot A$ if and only if there exists h such that $x = g \cdot h$ and $h \in A$.

(34) $x \in A \cdot g$ if and only if there exists h such that $x = h \cdot g$ and $h \in A$.

(35) $(g \cdot A) \cdot B = g \cdot (A \cdot B)$.

(36) $(A \cdot g) \cdot B = A \cdot (g \cdot B)$.

(37) $(A \cdot B) \cdot g = A \cdot (B \cdot g)$.

(38) $(g \cdot h) \cdot A = g \cdot (h \cdot A)$.

(39) $(g \cdot A) \cdot h = g \cdot (A \cdot h)$.

(40) $(A \cdot g) \cdot h = A \cdot (g \cdot h)$.

(41) $\emptyset_{\text{the carrier of } G} \cdot a = \emptyset$ and $a \cdot \emptyset_{\text{the carrier of } G} = \emptyset$.

(42) $\Omega_{\text{the carrier of } G} \cdot a =$ the carrier of G and $a \cdot \Omega_{\text{the carrier of } G} =$ the carrier of G.

(43) $(1_G) \cdot A = A$ and $A \cdot (1_G) = A$.

(44) If G is an Abelian group, then $g \cdot A = A \cdot g$.

Let us consider G. A group is said to be a subgroup of G if:

(Def.5) the carrier of it $\subseteq$ the carrier of G and the operation of it $=$ (the operation of G) $\restriction [\!:$ the carrier of it, the carrier of it $:\!]$.

One can prove the following proposition

(45) If the carrier of $G_1 \subseteq$ the carrier of G_2 and the operation of $G_1 =$ (the operation of G_2) $\restriction [\!:$ the carrier of G_1, the carrier of $G_1 :\!]$, then G_1 is a subgroup of G_2.

We follow the rules: I, H, H_1, H_2, H_3 will be subgroups of G and h, h_1, h_2 will be elements of H. One can prove the following propositions:

(46) The carrier of $H \subseteq$ the carrier of G.

(47) The operation of $H =$ (the operation of G) $\restriction [\!:$ the carrier of H, the carrier of $H :\!]$.

(48) If G is finite, then H is finite.

(49) If $x \in H$, then $x \in G$.

(50) $h \in G$.

(51) h is an element of G.

(52) If $h_1 = g_1$ and $h_2 = g_2$, then $h_1 \cdot h_2 = g_1 \cdot g_2$.

(53) $1_H = 1_G$.

(54) $1_{H_1} = 1_{H_2}$.

(55) $1_G \in H$.

(56) $1_{H_1} \in H_2$.

(57) If $h = g$, then $h^{-1} = g^{-1}$.

(58) $\cdot_H^{-1} = \cdot_G^{-1} \restriction$ (the carrier of H).

(59) If $g_1 \in H$ and $g_2 \in H$, then $g_1 \cdot g_2 \in H$.

(60) If $g \in H$, then $g^{-1} \in H$.

(61) If $A \neq \emptyset$ and for all g_1, g_2 such that $g_1 \in A$ and $g_2 \in A$ holds $g_1 \cdot g_2 \in A$ and for every g such that $g \in A$ holds $g^{-1} \in A$, then there exists H such that the carrier of $H = A$.

(62) If G is an Abelian group, then H is an Abelian group.

Let G be an Abelian group. We see that the subgroup of G is an Abelian group.

We now state several propositions:

(63) G is a subgroup of G.

(64) If G_1 is a subgroup of G_2 and G_2 is a subgroup of G_1, then $G_1 = G_2$.

(65) If G_1 is a subgroup of G_2 and G_2 is a subgroup of G_3, then G_1 is a subgroup of G_3.

(66) If the carrier of $H_1 \subseteq$ the carrier of H_2, then H_1 is a subgroup of H_2.

(67) If for every g such that $g \in H_1$ holds $g \in H_2$, then H_1 is a subgroup of H_2.

(68) If the carrier of $H_1 =$ the carrier of H_2, then $H_1 = H_2$.

(69) If for every g holds $g \in H_1$ if and only if $g \in H_2$, then $H_1 = H_2$.

Let us consider G, H_1, H_2. Let us note that one can characterize the predicate $H_1 = H_2$ by the following (equivalent) condition:

(Def.6) for every g holds $g \in H_1$ if and only if $g \in H_2$.

The following two propositions are true:

(70) If the carrier of $H =$ the carrier of G, then $H = G$.

(71) If for every g holds $g \in H$, then $H = G$.

Let us consider G. The functor $\{\mathbf{1}\}_G$ yields a subgroup of G and is defined by:

(Def.7) the carrier of $\{\mathbf{1}\}_G = \{1_G\}$.

Let us consider G. The functor Ω_G yielding a subgroup of G is defined as follows:

(Def.8) $\Omega_G = G$.

The following propositions are true:

(72) If the carrier of $H = \{1_G\}$, then $H = \{\mathbf{1}\}_G$.

(73) The carrier of $\{\mathbf{1}\}_G = \{1_G\}$.

(74) $\Omega_G = G$.

(75) $\{\mathbf{1}\}_H = \{\mathbf{1}\}_G$.

(76) $\{\mathbf{1}\}_{H_1} = \{\mathbf{1}\}_{H_2}$.

(77) $\{\mathbf{1}\}_G$ is a subgroup of H.

(78) H is a subgroup of Ω_G.

(79) G is a subgroup of Ω_G.

(80) $\{\mathbf{1}\}_G$ is finite.

(81) $\operatorname{ord}(\{\mathbf{1}\}_G) = 1$.

(82) If H is finite and $\operatorname{ord}(H) = 1$, then $H = \{\mathbf{1}\}_G$.

(83) $\operatorname{Ord}(H) \leq \operatorname{Ord}(G)$.

(84) If G is finite, then $\operatorname{ord}(H) \leq \operatorname{ord}(G)$.

(85) If G is finite and $\operatorname{ord}(G) = \operatorname{ord}(H)$, then $H = G$.

Let us consider G, H. The functor $\overline{H}$ yields a subset of G and is defined by:

(Def.9) $\overline{H}$ = the carrier of H.

The following propositions are true:

(86) $\overline{H}$ = the carrier of H.

(87) $\overline{H} \neq \emptyset$.

(88) $x \in \overline{H}$ if and only if $x \in H$.

(89) If $g_1 \in \overline{H}$ and $g_2 \in \overline{H}$, then $g_1 \cdot g_2 \in \overline{H}$.

(90) If $g \in \overline{H}$, then $g^{-1} \in \overline{H}$.

(91) $\overline{H} \cdot \overline{H} = \overline{H}$.

(92) $\overline{H}^{-1} = \overline{H}$.

(93) $\overline{H_1} \cdot \overline{H_2} = \overline{H_2} \cdot \overline{H_1}$ if and only if there exists H such that the carrier of $H = \overline{H_1} \cdot \overline{H_2}$.

(94) If G is an Abelian group, then there exists H such that the carrier of $H = \overline{H_1} \cdot \overline{H_2}$.

Let us consider G, H_1, H_2. The functor $H_1 \cap H_2$ yields a subgroup of G and is defined as follows:

(Def.10) the carrier of $H_1 \cap H_2 = \overline{H_1} \cap \overline{H_2}$.

One can prove the following propositions:

(95) If the carrier of $H = \overline{H_1} \cap \overline{H_2}$, then $H = H_1 \cap H_2$.

(96) The carrier of $H_1 \cap H_2 = \overline{H_1} \cap \overline{H_2}$.

(97) $H = H_1 \cap H_2$ if and only if the carrier of H = (the carrier of H_1)$\cap$ (the carrier of H_2).

(98) $\overline{H_1 \cap H_2} = \overline{H_1} \cap \overline{H_2}$.

(99) $x \in H_1 \cap H_2$ if and only if $x \in H_1$ and $x \in H_2$.

(100) $H \cap H = H$.

(101) $H_1 \cap H_2 = H_2 \cap H_1$.

(102) $(H_1 \cap H_2) \cap H_3 = H_1 \cap (H_2 \cap H_3)$.

(103) $\{\mathbf{1}\}_G \cap H = \{\mathbf{1}\}_G$ and $H \cap \{\mathbf{1}\}_G = \{\mathbf{1}\}_G$.

(104) $H \cap \Omega_G = H$ and $\Omega_G \cap H = H$.

(105) $\Omega_G \cap \Omega_G = G$.

(106) $H_1 \cap H_2$ is a subgroup of H_1 and $H_1 \cap H_2$ is a subgroup of H_2.

(107) H_1 is a subgroup of H_2 if and only if $H_1 \cap H_2 = H_1$.

(108) If H_1 is a subgroup of H_2, then $H_1 \cap H_3$ is a subgroup of H_2.

(109) If H_1 is a subgroup of H_2 and H_1 is a subgroup of H_3, then H_1 is a subgroup of $H_2 \cap H_3$.

(110) If H_1 is a subgroup of H_2, then $H_1 \cap H_3$ is a subgroup of $H_2 \cap H_3$.

(111) If H_1 is finite or H_2 is finite, then $H_1 \cap H_2$ is finite.

We now define two new functors. Let us consider G, H, A. The functor $A \cdot H$ yielding a subset of G is defined as follows:

(Def.11) $A \cdot H = A \cdot \overline{H}$.

The functor $H \cdot A$ yields a subset of G and is defined as follows:

(Def.12) $H \cdot A = \overline{H} \cdot A$.

One can prove the following propositions:

(112) $A \cdot H = A \cdot \overline{H}$.

(113) $H \cdot A = \overline{H} \cdot A$.

(114) $x \in A \cdot H$ if and only if there exist g_1, g_2 such that $x = g_1 \cdot g_2$ and $g_1 \in A$ and $g_2 \in H$.

(115) $x \in H \cdot A$ if and only if there exist g_1, g_2 such that $x = g_1 \cdot g_2$ and $g_1 \in H$ and $g_2 \in A$.

(116) $(A \cdot B) \cdot H = A \cdot (B \cdot H)$.

(117) $(A \cdot H) \cdot B = A \cdot (H \cdot B)$.

(118) $(H \cdot A) \cdot B = H \cdot (A \cdot B)$.

(119) $(A \cdot H_1) \cdot H_2 = A \cdot (H_1 \cdot \overline{H_2})$.

(120) $(H_1 \cdot A) \cdot H_2 = H_1 \cdot (A \cdot H_2)$.

(121) $(H_1 \cdot \overline{H_2}) \cdot A = H_1 \cdot (H_2 \cdot A)$.

(122) If G is an Abelian group, then $A \cdot H = H \cdot A$.

We now define two new functors. Let us consider G, H, a. The functor $a \cdot H$ yielding a subset of G is defined as follows:

(Def.13) $a \cdot H = a \cdot \overline{H}$.

The functor $H \cdot a$ yielding a subset of G is defined by:

(Def.14) $H \cdot a = \overline{H} \cdot a$.

The following propositions are true:

(123) $a \cdot H = a \cdot \overline{H}$.

(124) $H \cdot a = \overline{H} \cdot a$.

(125) $x \in a \cdot H$ if and only if there exists g such that $x = a \cdot g$ and $g \in H$.

(126) $x \in H \cdot a$ if and only if there exists g such that $x = g \cdot a$ and $g \in H$.

(127) $(a \cdot b) \cdot H = a \cdot (b \cdot H)$.

(128) $(a \cdot H) \cdot b = a \cdot (H \cdot b)$.

(129) $(H \cdot a) \cdot b = H \cdot (a \cdot b)$.

(130) $a \in a \cdot H$ and $a \in H \cdot a$.

(131) $a \cdot H \neq \emptyset$ and $H \cdot a \neq \emptyset$.

(132) $(1_G) \cdot H = \overline{H}$ and $H \cdot (1_G) = \overline{H}$.

(133) $\{\mathbf{1}\}_G \cdot a = \{a\}$ and $a \cdot \{\mathbf{1}\}_G = \{a\}$.

(134) $a \cdot \Omega_G =$ the carrier of G and $\Omega_G \cdot a =$ the carrier of G.

(135) If G is an Abelian group, then $a \cdot H = H \cdot a$.

(136) $a \in H$ if and only if $a \cdot H = \overline{H}$.

(137) $a \cdot H = b \cdot H$ if and only if $b^{-1} \cdot a \in H$.

(138) $a \cdot H = b \cdot H$ if and only if $a \cdot H$ meets $b \cdot H$.

(139) $(a \cdot b) \cdot H \subseteq (a \cdot H) \cdot (b \cdot H)$.

(140) $\overline{H} \subseteq (a \cdot H) \cdot (a^{-1} \cdot H)$ and $\overline{H} \subseteq (a^{-1} \cdot H) \cdot (a \cdot H)$.

(141) $a^2 \cdot H \subseteq (a \cdot H) \cdot (a \cdot H)$.

(142) $a \in H$ if and only if $H \cdot a = \overline{H}$.

(143) $H \cdot a = H \cdot b$ if and only if $b \cdot a^{-1} \in H$.

(144) $H \cdot a = H \cdot b$ if and only if $H \cdot a$ meets $H \cdot b$.

(145) $(H \cdot a) \cdot b \subseteq (H \cdot a) \cdot (H \cdot b)$.

(146) $\overline{H} \subseteq (H \cdot a) \cdot (H \cdot a^{-1})$ and $\overline{H} \subseteq (H \cdot a^{-1}) \cdot (H \cdot a)$.

(147) $H \cdot a^2 \subseteq (H \cdot a) \cdot (H \cdot a)$.

(148) $a \cdot (H_1 \cap H_2) = (a \cdot H_1) \cap (a \cdot H_2)$.

(149) $(H_1 \cap H_2) \cdot a = (H_1 \cdot a) \cap (H_2 \cdot a)$.

(150) There exists H_1 such that the carrier of $H_1 = (a \cdot H_2) \cdot a^{-1}$.

(151) $a \cdot H \approx b \cdot H$.

(152) $a \cdot H \approx H \cdot b$.

(153) $H \cdot a \approx H \cdot b$.

(154) $\overline{H} \approx a \cdot H$ and $\overline{H} \approx H \cdot a$.

(155) $\mathrm{Ord}(H) = \overline{\overline{a \cdot H}}$ and $\mathrm{Ord}(H) = \overline{\overline{H \cdot a}}$.

(156) If H is finite, then $\mathrm{ord}(H) = \mathrm{card}(a \cdot H)$ and $\mathrm{ord}(H) = \mathrm{card}(H \cdot a)$.

The scheme *SubFamComp* deals with a set $\mathcal{A}$, a family $\mathcal{B}$ of subsets of $\mathcal{A}$, a family $\mathcal{C}$ of subsets of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{B} = \mathcal{C}$

provided the parameters meet the following requirements:

- for every subset X of $\mathcal{A}$ holds $X \in \mathcal{B}$ if and only if $\mathcal{P}[X]$,
- for every subset X of $\mathcal{A}$ holds $X \in \mathcal{C}$ if and only if $\mathcal{P}[X]$.

We now define two new functors. Let us consider G, H. The left cosets of H yielding a family of subsets of the carrier of G is defined as follows:

(Def.15) $A \in$ the left cosets of H if and only if there exists a such that $A = a \cdot H$.

The right cosets of H yielding a family of subsets of the carrier of G is defined by:

(Def.16) $A \in$ the right cosets of H if and only if there exists a such that $A = H \cdot a$.

In the sequel F denotes a family of subsets of the carrier of G. One can prove the following propositions:

(157) If for every A holds $A \in F$ if and only if there exists a such that $A = a \cdot H$, then $F =$ the left cosets of H.

(158) If for every A holds $A \in F$ if and only if there exists a such that $A = H \cdot a$, then $F =$ the right cosets of H.

(159) $A \in$ the left cosets of H if and only if there exists a such that $A = a \cdot H$.

(160) $A \in$ the right cosets of H if and only if there exists a such that $A = H \cdot a$.

(161) If $x \in$ the left cosets of H or $x \in$ the right cosets of H, then x is a subset of G.

(162) $x \in$ the left cosets of H if and only if there exists a such that $x = a \cdot H$.

(163) $x \in$ the right cosets of H if and only if there exists a such that $x = H \cdot a$.

(164) If G is finite, then the right cosets of H is finite and the left cosets of H is finite.

(165) $\overline{H} \in$ the left cosets of H and $\overline{H} \in$ the right cosets of H.

(166) The left cosets of $H \approx$ the right cosets of H.

(167) $\bigcup$(The left cosets of H) $=$ the carrier of G and $\bigcup$(the right cosets of H) $=$ the carrier of G.

(168) The left cosets of $\{\mathbf{1}\}_G = \{\{a\}\}$.

(169) The right cosets of $\{\mathbf{1}\}_G = \{\{a\}\}$.

(170) If the left cosets of $H = \{\{a\}\}$, then $H = \{\mathbf{1}\}_G$.

(171) If the right cosets of $H = \{\{a\}\}$, then $H = \{\mathbf{1}\}_G$.

(172) The left cosets of $\Omega_G = \{$ the carrier of $G\}$ and the right cosets of $\Omega_G = \{$ the carrier of $G\}$.

(173) If the left cosets of $H = \{$ the carrier of $G\}$, then $H = G$.

(174) If the right cosets of $H = \{$ the carrier of $G\}$, then $H = G$.

Let us consider G, H. The functor $|\bullet : H|$ yielding a cardinal number is defined by:

(Def.17) $|\bullet : H| = \overline{\overline{\text{the left cosets of } H}}$.

We now state the proposition

(175) $|\bullet : H| = \overline{\overline{\text{the left cosets of } H}}$ and $|\bullet : H| = \overline{\overline{\text{the right cosets of } H}}$.

Let us consider G, H. Let us assume that the left cosets of H is finite. The functor $|\bullet : H|_{\mathbb{N}}$ yielding a natural number is defined as follows:

(Def.18) $|\bullet : H|_{\mathbb{N}} = \mathrm{card}$(the left cosets of H).

Next we state the proposition

(176) If the left cosets of H is finite, then $|\bullet : H|_{\mathbb{N}} = \mathrm{card}$(the left cosets of H) and $|\bullet : H|_{\mathbb{N}} = \mathrm{card}$(the right cosets of H).

Let D be a non-empty set, and let d be an element of D. Then $\{d\}$ is an element of $\mathrm{Fin}\, D$.

The following two propositions are true:

(177) If G is finite, then $\mathrm{ord}(G) = \mathrm{ord}(H) \cdot |\bullet : H|_{\mathbb{N}}$.

(178) If G is finite, then $\mathrm{ord}(H) \mid \mathrm{ord}(G)$.

In the sequel J will denote a subgroup of H. One can prove the following propositions:

(179) If G is finite and $I = J$, then $|\bullet : I|_{\mathbb{N}} = |\bullet : J|_{\mathbb{N}} \cdot |\bullet : H|_{\mathbb{N}}$.

(180) $|\bullet : \Omega_G|_{\mathbb{N}} = 1$.

(181) If the left cosets of H is finite and $|\bullet : H|_{\mathbb{N}} = 1$, then $H = G$.

(182) $|\bullet : \{\mathbf{1}\}_G| = \mathrm{Ord}(G)$.

(183) If G is finite, then $|\bullet : \{\mathbf{1}\}_G|_{\mathbb{N}} = \mathrm{ord}(G)$.

(184) If G is finite and $|\bullet : H|_{\mathbb{N}} = \mathrm{ord}(G)$, then $H = \{\mathbf{1}\}_G$.

(185) If the left cosets of H is finite and $|\bullet : H| = \mathrm{Ord}(G)$, then G is finite and $H = \{\mathbf{1}\}_G$.

(186) If X is finite and for every Y such that $Y \in X$ holds Y is finite and $\mathrm{card}\, Y = k$ and for every Z such that $Z \in X$ and $Y \neq Z$ holds $Y \approx Z$ and Y misses Z, then $\mathrm{card}(\bigcup X) = k \cdot \mathrm{card}\, X$.

(187) If Y is finite and $X \subseteq Y$ and $\mathrm{card}\, X = \mathrm{card}\, Y$, then $X = Y$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[6] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[7] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[8] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.

[9] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[10] Andrzej Trybulec and Agata Darmochwał. Boolean domains. *Formalized Mathematics*, 1(**1**):187–190, 1990.

[11] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.

[12] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

Received July 23, 1990

FORMALIZED MATHEMATICS
Vol.1, No.**5**, November–December 1990
Université Catholique de Louvain

Subspaces and Cosets of Subspaces in Vector Space

Wojciech A. Trybulec[1]
Warsaw University

Summary. We introduce the notions of subspace of vector space and coset of a subspace. We prove a number of theorems concerning those notions. Some theorems that belong rather to [1] are proved.

MML Identifier: VECTSP_4.

The articles [3], [5], [2], [1], and [4] provide the terminology and notation for this paper. For simplicity we adopt the following rules: G_1 will denote a field, V, X, Y will denote vector spaces over G_1, u, v, v_1, v_2 will denote vectors of V, a, b, c will denote elements of G_1, and x will be arbitrary. Let us consider G_1, V. A subset of V is a subset of the carrier of the carrier of V.

In the sequel V_1, V_2, V_3 denote subsets of V. Let us consider G_1, V, V_1. We say that V_1 is linearly closed if and only if:

(Def.1) for all v, u such that $v \in V_1$ and $u \in V_1$ holds $v + u \in V_1$ and for all a, v such that $v \in V_1$ holds $a \cdot v \in V_1$.

The following propositions are true:

(1) If for all v, u such that $v \in V_1$ and $u \in V_1$ holds $v + u \in V_1$ and for all a, v such that $v \in V_1$ holds $a \cdot v \in V_1$, then V_1 is linearly closed.

(2) If V_1 is linearly closed, then for all v, u such that $v \in V_1$ and $u \in V_1$ holds $v + u \in V_1$.

(3) If V_1 is linearly closed, then for all a, v such that $v \in V_1$ holds $a \cdot v \in V_1$.

(4) If $V_1 \neq \emptyset$ and V_1 is linearly closed, then $\Theta_V \in V_1$.

(5) If V_1 is linearly closed, then for every v such that $v \in V_1$ holds $-v \in V_1$.

(6) If V_1 is linearly closed, then for all v, u such that $v \in V_1$ and $u \in V_1$ holds $v - u \in V_1$.

(7) $\{\Theta_V\}$ is linearly closed.

[1]Supported by RPBP.III-24.C1

ISSN 0777–4028

(8) If the carrier of the carrier of $V = V_1$, then V_1 is linearly closed.

(9) If V_1 is linearly closed and V_2 is linearly closed and $V_3 = \{v + u : v \in V_1 \wedge u \in V_2\}$, then V_3 is linearly closed.

(10) If V_1 is linearly closed and V_2 is linearly closed, then $V_1 \cap V_2$ is linearly closed.

Let us consider G_1, V. A vector space over G_1 is said to be a subspace of V if:

(Def.2) the carrier of the carrier of it $\subseteq$ the carrier of the carrier of V and the zero of the carrier of it $=$ the zero of the carrier of V and the addition of the carrier of it $=$ (the addition of the carrier of V) $\upharpoonright [\!:$ the carrier of the carrier of it, the carrier of the carrier of it $:\!]$ and the multiplication of it $=$ (the multiplication of V) $\upharpoonright [\!:$ the carrier of G_1, the carrier of the carrier of it $:\!]$.

Next we state the proposition

(11) If the carrier of the carrier of $X \subseteq$ the carrier of the carrier of V and the zero of the carrier of $X =$ the zero of the carrier of V and the addition of the carrier of $X =$ (the addition of the carrier of V) $\upharpoonright [\!:$ the carrier of the carrier of X, the carrier of the carrier of $X :\!]$ and the multiplication of $X =$ (the multiplication of V) $\upharpoonright [\!:$ the carrier of G_1, the carrier of the carrier of $X :\!]$, then X is a subspace of V.

We adopt the following convention: W, W_1, W_2 will be subspaces of V and w, w_1, w_2 will be vectors of W. Next we state a number of propositions:

(12) The carrier of the carrier of $W \subseteq$ the carrier of the carrier of V.

(13) The zero of the carrier of $W =$ the zero of the carrier of V.

(14) The addition of the carrier of $W =$ (the addition of the carrier of V) $\upharpoonright [\!:$ the carrier of the carrier of W, the carrier of the carrier of $W :\!]$.

(15) The multiplication of $W =$ (the multiplication of V) $\upharpoonright [\!:$ the carrier of G_1, the carrier of the carrier of $W :\!]$.

(16) If $x \in W_1$ and W_1 is a subspace of W_2, then $x \in W_2$.

(17) If $x \in W$, then $x \in V$.

(18) w is a vector of V.

(19) $\Theta_W = \Theta_V$.

(20) $\Theta_{W_1} = \Theta_{W_2}$.

(21) If $w_1 = v$ and $w_2 = u$, then $w_1 + w_2 = v + u$.

(22) If $w = v$, then $a \cdot w = a \cdot v$.

(23) If $w = v$, then $-v = -w$.

(24) If $w_1 = v$ and $w_2 = u$, then $w_1 - w_2 = v - u$.

(25) $\Theta_V \in W$.

(26) $\Theta_{W_1} \in W_2$.

(27) $\Theta_W \in V$.

(28) If $u \in W$ and $v \in W$, then $u + v \in W$.

(29) If $v \in W$, then $a \cdot v \in W$.
(30) If $v \in W$, then $-v \in W$.
(31) If $u \in W$ and $v \in W$, then $u - v \in W$.
(32) V is a subspace of V.
(33) If V is a subspace of X and X is a subspace of V, then $V = X$.
(34) If V is a subspace of X and X is a subspace of Y, then V is a subspace of Y.
(35) If the carrier of the carrier of $W_1 \subseteq$ the carrier of the carrier of W_2, then W_1 is a subspace of W_2.
(36) If for every v such that $v \in W_1$ holds $v \in W_2$, then W_1 is a subspace of W_2.
(37) If the carrier of the carrier of $W_1 =$ the carrier of the carrier of W_2, then $W_1 = W_2$.
(38) If for every v holds $v \in W_1$ if and only if $v \in W_2$, then $W_1 = W_2$.
(39) If the carrier of the carrier of $W =$ the carrier of the carrier of V, then $W = V$.
(40) If for every v holds $v \in W$, then $W = V$.
(41) If the carrier of the carrier of $W = V_1$, then V_1 is linearly closed.
(42) If $V_1 \neq \emptyset$ and V_1 is linearly closed, then there exists W such that $V_1 =$ the carrier of the carrier of W.

Let us consider G_1, V. The functor $\mathbf{0}_V$ yielding a subspace of V is defined by:

(Def.3) the carrier of the carrier of $\mathbf{0}_V = \{\Theta_V\}$.

Let us consider G_1, V. The functor Ω_V yields a subspace of V and is defined by:

(Def.4) $\Omega_V = V$.

The following propositions are true:

(43) The carrier of the carrier of $\mathbf{0}_V = \{\Theta_V\}$.
(44) If the carrier of the carrier of $W = \{\Theta_V\}$, then $W = \mathbf{0}_V$.
(45) $\Omega_V = V$.
(46) $x \in \mathbf{0}_V$ if and only if $x = \Theta_V$.
(47) $\mathbf{0}_W = \mathbf{0}_V$.
(48) $\mathbf{0}_{W_1} = \mathbf{0}_{W_2}$.
(49) $\mathbf{0}_W$ is a subspace of V.
(50) $\mathbf{0}_V$ is a subspace of W.
(51) $\mathbf{0}_{W_1}$ is a subspace of W_2.
(52) W is a subspace of Ω_V.
(53) V is a subspace of Ω_V.

Let us consider G_1, V, v, W. The functor $v + W$ yielding a subset of V is defined by:

(Def.5) $v + W = \{v + u : u \in W\}$.

Let us consider G_1, V, W. A subset of V is said to be a coset of W if:

(Def.6) there exists v such that it $= v + W$.

In the sequel B, C will denote cosets of W. The following propositions are true:

(54) $v + W = \{v + u : u \in W\}$.

(55) There exists v such that $C = v + W$.

(56) If $V_1 = v + W$, then V_1 is a coset of W.

(57) $x \in v + W$ if and only if there exists u such that $u \in W$ and $x = v + u$.

(58) $\Theta_V \in v + W$ if and only if $v \in W$.

(59) $v \in v + W$.

(60) $\Theta_V + W =$ the carrier of the carrier of W.

(61) $v + \mathbf{0}_V = \{v\}$.

(62) $v + \Omega_V =$ the carrier of the carrier of V.

(63) $\Theta_V \in v + W$ if and only if $v + W =$ the carrier of the carrier of W.

(64) $v \in W$ if and only if $v + W =$ the carrier of the carrier of W.

(65) If $v \in W$, then $a \cdot v + W =$ the carrier of the carrier of W.

(66) If $a \neq 0_{G_1}$ and $a \cdot v + W =$ the carrier of the carrier of W, then $v \in W$.

(67) $v \in W$ if and only if $(-v) + W =$ the carrier of the carrier of W.

(68) $u \in W$ if and only if $v + W = (v + u) + W$.

(69) $u \in W$ if and only if $v + W = (v - u) + W$.

(70) $v \in u + W$ if and only if $u + W = v + W$.

(71) If $u \in v_1 + W$ and $u \in v_2 + W$, then $v_1 + W = v_2 + W$.

(72) If $a \neq 1_{G_1}$ and $a \cdot v \in v + W$, then $v \in W$.

(73) If $v \in W$, then $a \cdot v \in v + W$.

(74) If $v \in W$, then $-v \in v + W$.

(75) $u + v \in v + W$ if and only if $u \in W$.

(76) $v - u \in v + W$ if and only if $u \in W$.

(77) $u \in v + W$ if and only if there exists v_1 such that $v_1 \in W$ and $u = v + v_1$.

(78) $u \in v + W$ if and only if there exists v_1 such that $v_1 \in W$ and $u = v - v_1$.

(79) There exists v such that $v_1 \in v + W$ and $v_2 \in v + W$ if and only if $v_1 - v_2 \in W$.

(80) If $v + W = u + W$, then there exists v_1 such that $v_1 \in W$ and $v + v_1 = u$.

(81) If $v + W = u + W$, then there exists v_1 such that $v_1 \in W$ and $v - v_1 = u$.

(82) $v + W_1 = v + W_2$ if and only if $W_1 = W_2$.

(83) If $v + W_1 = u + W_2$, then $W_1 = W_2$.

In the sequel C_1 denotes a coset of W_1 and C_2 denotes a coset of W_2. One can prove the following propositions:

(84) There exists C such that $v \in C$.

(85) C is linearly closed if and only if $C =$ the carrier of the carrier of W.
(86) If $C_1 = C_2$, then $W_1 = W_2$.
(87) $\{v\}$ is a coset of $\mathbf{0}_V$.
(88) If V_1 is a coset of $\mathbf{0}_V$, then there exists v such that $V_1 = \{v\}$.
(89) The carrier of the carrier of W is a coset of W.
(90) The carrier of the carrier of V is a coset of Ω_V.
(91) If V_1 is a coset of Ω_V, then $V_1 =$ the carrier of the carrier of V.
(92) $\Theta_V \in C$ if and only if $C =$ the carrier of the carrier of W.
(93) $u \in C$ if and only if $C = u + W$.
(94) If $u \in C$ and $v \in C$, then there exists v_1 such that $v_1 \in W$ and $u+v_1 = v$.
(95) If $u \in C$ and $v \in C$, then there exists v_1 such that $v_1 \in W$ and $u-v_1 = v$.
(96) There exists C such that $v_1 \in C$ and $v_2 \in C$ if and only if $v_1 - v_2 \in W$.
(97) If $u \in B$ and $u \in C$, then $B = C$.

In the sequel w will denote a vector of V. One can prove the following propositions:

(99)[2] $(u + v) - w = u + (v - w)$.
(100) $-(-v) = v$.
(101) $v - (u - w) = (v - u) + w$.
(102) If $v + u = v$ or $u + v = v$, then $u = \Theta_V$.
(103) $(a - b) \cdot v = a \cdot v - b \cdot v$.
(104) $a - 0_{G_1} = a$.
(105) $a - a = 0_{G_1}$.
(106) $a - (b - c) = (a - b) + c$.

References

[1] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[2] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[3] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[4] Wojciech A. Trybulec. Finite sums of vectors in vector space. *Formalized Mathematics*, 1(**5**):851–854, 1990.

[2]The proposition (98) became obvious.

[5] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

Received July 27, 1990

FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Operations on Subspaces in Vector Space

Wojciech A. Trybulec[1]
Warsaw University

Summary. Sum, direct sum and intersection of subspaces are introduced. We prove some theorems concerning those notions and the decomposition of vector onto two subspaces. Linear complement of a subspace is also defined. We prove theorem that belong rather to [3].

MML Identifier: `VECTSP_5`.

The papers [2], [8], [9], [5], [3], [4], [6], [1], and [7] provide the terminology and notation for this paper. For simplicity we adopt the following rules: G_1 will denote a field, V will denote a vector space over G_1, W, W_1, W_2, W_3 will denote subspaces of V, u, u_1, u_2, v, v_1, v_2 will denote vectors of V, and x will be arbitrary. Let us consider G_1, V, W_1, W_2. The functor $W_1 + W_2$ yields a subspace of V and is defined by:

(Def.1) the carrier of the carrier of $W_1 + W_2 = \{v + u : v \in W_1 \wedge u \in W_2\}$.

Let us consider G_1, V, W_1, W_2. The functor $W_1 \cap W_2$ yields a subspace of V and is defined by:

(Def.2) the carrier of the carrier of $W_1 \cap W_2 =$ (the carrier of the carrier of W_1)$\cap$ (the carrier of the carrier of W_2).

We now state a number of propositions:

(1) The carrier of the carrier of $W_1 + W_2 = \{v + u : v \in W_1 \wedge u \in W_2\}$.

(2) If the carrier of the carrier of $W = \{v + u : v \in W_1 \wedge u \in W_2\}$, then $W = W_1 + W_2$.

(3) The carrier of the carrier of $W_1 \cap W_2 =$ (the carrier of the carrier of W_1)$\cap$ (the carrier of the carrier of W_2).

(4) If the carrier of the carrier of $W =$ (the carrier of the carrier of W_1)$\cap$ (the carrier of the carrier of W_2), then $W = W_1 \cap W_2$.

(5) $x \in W_1 + W_2$ if and only if there exist v_1, v_2 such that $v_1 \in W_1$ and $v_2 \in W_2$ and $x = v_1 + v_2$.

[1]Supported by RPBP.III-24.C1

ISSN 0777–4028

(6) If $v \in W_1$ or $v \in W_2$, then $v \in W_1 + W_2$.

(7) $x \in W_1 \cap W_2$ if and only if $x \in W_1$ and $x \in W_2$.

(8) $W + W = W$.

(9) $W_1 + W_2 = W_2 + W_1$.

(10) $W_1 + (W_2 + W_3) = (W_1 + W_2) + W_3$.

(11) W_1 is a subspace of $W_1 + W_2$ and W_2 is a subspace of $W_1 + W_2$.

(12) W_1 is a subspace of W_2 if and only if $W_1 + W_2 = W_2$.

(13) $\mathbf{0}_V + W = W$ and $W + \mathbf{0}_V = W$.

(14) $\mathbf{0}_V + \Omega_V = V$ and $\Omega_V + \mathbf{0}_V = V$.

(15) $\Omega_V + W = V$ and $W + \Omega_V = V$.

(16) $\Omega_V + \Omega_V = V$.

(17) $W \cap W = W$.

(18) $W_1 \cap W_2 = W_2 \cap W_1$.

(19) $W_1 \cap (W_2 \cap W_3) = (W_1 \cap W_2) \cap W_3$.

(20) $W_1 \cap W_2$ is a subspace of W_1 and $W_1 \cap W_2$ is a subspace of W_2.

(21) W_1 is a subspace of W_2 if and only if $W_1 \cap W_2 = W_1$.

(22) If W_1 is a subspace of W_2, then $W_1 \cap W_3$ is a subspace of $W_2 \cap W_3$.

(23) If W_1 is a subspace of W_3, then $W_1 \cap W_2$ is a subspace of W_3.

(24) If W_1 is a subspace of W_2 and W_1 is a subspace of W_3, then W_1 is a subspace of $W_2 \cap W_3$.

(25) $\mathbf{0}_V \cap W = \mathbf{0}_V$ and $W \cap \mathbf{0}_V = \mathbf{0}_V$.

(26) $\mathbf{0}_V \cap \Omega_V = \mathbf{0}_V$ and $\Omega_V \cap \mathbf{0}_V = \mathbf{0}_V$.

(27) $\Omega_V \cap W = W$ and $W \cap \Omega_V = W$.

(28) $\Omega_V \cap \Omega_V = V$.

(29) $W_1 \cap W_2$ is a subspace of $W_1 + W_2$.

(30) $W_1 \cap W_2 + W_2 = W_2$.

(31) $W_1 \cap (W_1 + W_2) = W_1$.

(32) $W_1 \cap W_2 + W_2 \cap W_3$ is a subspace of $W_2 \cap (W_1 + W_3)$.

(33) If W_1 is a subspace of W_2, then $W_2 \cap (W_1 + W_3) = W_1 \cap W_2 + W_2 \cap W_3$.

(34) $W_2 + W_1 \cap W_3$ is a subspace of $(W_1 + W_2) \cap (W_2 + W_3)$.

(35) If W_1 is a subspace of W_2, then $W_2 + W_1 \cap W_3 = (W_1 + W_2) \cap (W_2 + W_3)$.

(36) If W_1 is a subspace of W_3, then $W_1 + W_2 \cap W_3 = (W_1 + W_2) \cap W_3$.

(37) $W_1 + W_2 = W_2$ if and only if $W_1 \cap W_2 = W_1$.

(38) If W_1 is a subspace of W_2, then $W_1 + W_3$ is a subspace of $W_2 + W_3$.

(39) If W_1 is a subspace of W_2, then W_1 is a subspace of $W_2 + W_3$.

(40) If W_1 is a subspace of W_3 and W_2 is a subspace of W_3, then $W_1 + W_2$ is a subspace of W_3.

(41) There exists W such that the carrier of the carrier of $W =$ (the carrier of the carrier of W_1)$\cup$ (the carrier of the carrier of W_2) if and only if W_1 is a subspace of W_2 or W_2 is a subspace of W_1.

Let us consider G_1, V. The functor Subspaces V yielding a non-empty set is defined as follows:

(Def.3) for every x holds $x \in$ Subspaces V if and only if x is a subspace of V.

In the sequel D denotes a non-empty set. The following three propositions are true:

(42) If for every x holds $x \in D$ if and only if x is a subspace of V, then $D =$ Subspaces V.

(43) $x \in$ Subspaces V if and only if x is a subspace of V.

(44) $V \in$ Subspaces V.

Let us consider G_1, V, W_1, W_2. We say that V is the direct sum of W_1 and W_2 if and only if:

(Def.4) $V = W_1 + W_2$ and $W_1 \cap W_2 = \mathbf{0}_V$.

Let us consider G_1, V, W. A subspace of V is said to be a linear complement of W if:

(Def.5) V is the direct sum of it and W.

We now state three propositions:

(45) V is the direct sum of W_1 and W_2 if and only if $V = W_1 + W_2$ and $W_1 \cap W_2 = \mathbf{0}_V$.

(46) If V is the direct sum of W_1 and W_2, then W_1 is a linear complement of W_2.

(47) If V is the direct sum of W_1 and W_2, then W_2 is a linear complement of W_1.

In the sequel L denotes a linear complement of W. The following propositions are true:

(48) V is the direct sum of L and W and V is the direct sum of W and L.

(49) $W + L = V$ and $L + W = V$.

(50) $W \cap L = \mathbf{0}_V$ and $L \cap W = \mathbf{0}_V$.

(51) If V is the direct sum of W_1 and W_2, then V is the direct sum of W_2 and W_1.

(52) V is the direct sum of $\mathbf{0}_V$ and Ω_V and V is the direct sum of Ω_V and $\mathbf{0}_V$.

(53) W is a linear complement of L.

(54) $\mathbf{0}_V$ is a linear complement of Ω_V and Ω_V is a linear complement of $\mathbf{0}_V$.

In the sequel C_1 is a coset of W_1 and C_2 is a coset of W_2. We now state several propositions:

(55) If $C_1 \cap C_2 \neq \emptyset$, then $C_1 \cap C_2$ is a coset of $W_1 \cap W_2$.

(56) V is the direct sum of W_1 and W_2 if and only if for every C_1, C_2 there exists v such that $C_1 \cap C_2 = \{v\}$.

(57) $W_1 + W_2 = V$ if and only if for every v there exist v_1, v_2 such that $v_1 \in W_1$ and $v_2 \in W_2$ and $v = v_1 + v_2$.

(58) If V is the direct sum of W_1 and W_2 and $v = v_1 + v_2$ and $v = u_1 + u_2$ and $v_1 \in W_1$ and $u_1 \in W_1$ and $v_2 \in W_2$ and $u_2 \in W_2$, then $v_1 = u_1$ and $v_2 = u_2$.

(59) Suppose $V = W_1 + W_2$ and there exists v such that for all v_1, v_2, u_1, u_2 such that $v = v_1 + v_2$ and $v = u_1 + u_2$ and $v_1 \in W_1$ and $u_1 \in W_1$ and $v_2 \in W_2$ and $u_2 \in W_2$ holds $v_1 = u_1$ and $v_2 = u_2$. Then V is the direct sum of W_1 and W_2.

In the sequel t will denote an element of $[\!:$ the carrier of the carrier of V, the carrier of the carrier of $V :\!]$. Let us consider G_1, V, t. Then $t_{\mathbf{1}}$ is a vector of V. Then $t_{\mathbf{2}}$ is a vector of V.

Let us consider G_1, V, v, W_1, W_2. Let us assume that V is the direct sum of W_1 and W_2. The functor $v \lhd (W_1, W_2)$ yielding an element of $[\!:$ the carrier of the carrier of V, the carrier of the carrier of $V :\!]$ is defined by:

(Def.6) $v = (v \lhd (W_1, W_2))_{\mathbf{1}} + (v \lhd (W_1, W_2))_{\mathbf{2}}$ and $(v \lhd (W_1, W_2))_{\mathbf{1}} \in W_1$ and $(v \lhd (W_1, W_2))_{\mathbf{2}} \in W_2$.

Next we state a number of propositions:

(60) If V is the direct sum of W_1 and W_2 and $t_{\mathbf{1}} + t_{\mathbf{2}} = v$ and $t_{\mathbf{1}} \in W_1$ and $t_{\mathbf{2}} \in W_2$, then $t = v \lhd (W_1, W_2)$.

(61) If V is the direct sum of W_1 and W_2, then $(v \lhd (W_1, W_2))_{\mathbf{1}} + (v \lhd (W_1, W_2))_{\mathbf{2}} = v$.

(62) If V is the direct sum of W_1 and W_2, then $(v \lhd (W_1, W_2))_{\mathbf{1}} \in W_1$.

(63) If V is the direct sum of W_1 and W_2, then $(v \lhd (W_1, W_2))_{\mathbf{2}} \in W_2$.

(64) If V is the direct sum of W_1 and W_2, then $(v \lhd (W_1, W_2))_{\mathbf{1}} = (v \lhd (W_2, W_1))_{\mathbf{2}}$.

(65) If V is the direct sum of W_1 and W_2, then $(v \lhd (W_1, W_2))_{\mathbf{2}} = (v \lhd (W_2, W_1))_{\mathbf{1}}$.

(66) If $t_{\mathbf{1}} + t_{\mathbf{2}} = v$ and $t_{\mathbf{1}} \in W$ and $t_{\mathbf{2}} \in L$, then $t = v \lhd (W, L)$.

(67) $(v \lhd (W, L))_{\mathbf{1}} + (v \lhd (W, L))_{\mathbf{2}} = v$.

(68) $(v \lhd (W, L))_{\mathbf{1}} \in W$ and $(v \lhd (W, L))_{\mathbf{2}} \in L$.

(69) $(v \lhd (W, L))_{\mathbf{1}} = (v \lhd (L, W))_{\mathbf{2}}$.

(70) $(v \lhd (W, L))_{\mathbf{2}} = (v \lhd (L, W))_{\mathbf{1}}$.

In the sequel A_1, A_2 will be elements of Subspaces V. Let us consider G_1, V. The functor SubJoin V yields a binary operation on Subspaces V and is defined by:

(Def.7) for all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubJoin}\, V)(A_1, A_2) = W_1 + W_2$.

Let us consider G_1, V. The functor SubMeet V yielding a binary operation on Subspaces V is defined by:

(Def.8) for all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubMeet}\,V)(A_1, A_2) = W_1 \cap W_2$.

In the sequel o denotes a binary operation on Subspaces V. One can prove the following propositions:

(71) If $A_1 = W_1$ and $A_2 = W_2$, then $\mathrm{SubJoin}\,V(A_1, A_2) = W_1 + W_2$.

(72) If for all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $o(A_1, A_2) = W_1 + W_2$, then $o = \mathrm{SubJoin}\,V$.

(73) If $A_1 = W_1$ and $A_2 = W_2$, then $\mathrm{SubMeet}\,V(A_1, A_2) = W_1 \cap W_2$.

(74) If for all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $o(A_1, A_2) = W_1 \cap W_2$, then $o = \mathrm{SubMeet}\,V$.

(75) $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is a lattice.

(76) $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is a lower bound lattice.

(77) $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is an upper bound lattice.

(78) $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is a bound lattice.

(79) $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is a modular lattice.

(80) $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is a complemented lattice.

(81) $v = v_1 + v_2$ if and only if $v_1 = v - v_2$.

References

[1] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[2] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[3] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[4] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[5] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

[6] Wojciech A. Trybulec. Finite sums of vectors in vector space. *Formalized Mathematics*, 1(**5**):851–854, 1990.

[7] Wojciech A. Trybulec. Subspaces and cosets of subspaces in vector space. *Formalized Mathematics*, 1(**5**):865–870, 1990.

[8] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

[9] Stanisław Żukowski. Introduction to lattice theory. *Formalized Mathematics*, 1(**1**):215–222, 1990.

Received July 27, 1990

FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Linear Combinations in Vector Space

Wojciech A. Trybulec[1]
Warsaw University

Summary. The notion of linear combination of vectors is introduced as a function from the carrier of a vector space to the carrier of the field. Definition of linear combination of set of vectors is also presented. We define addition and substraction of combinations and multiplication of combination by element of the field. Sum of finite set of vectors and sum of linear combination is defined. We prove theorems that belong rather to [5].

MML Identifier: `VECTSP_6`.

The articles [12], [4], [2], [1], [3], [11], [7], [6], [9], [5], [8], and [10] provide the terminology and notation for this paper. Let D be a non-empty set. Then $\emptyset_D$ is a subset of D.

For simplicity we adopt the following rules: x will be arbitrary, i will be a natural number, G_1 will be a field, V will be a vector space over G_1, u, v, v_1, v_2, v_3 will be vectors of V, a, b, c will be elements of G_1, F, G will be finite sequences of elements of the carrier of the carrier of V, A, B will be subsets of V, and f will be a function from the carrier of the carrier of V into the carrier of G_1. Let us consider G_1, V. A subset of V is called a finite subset of V if:

(Def.1) it is finite.

We now state the proposition

(1) A is a finite subset of V if and only if A is finite.

In the sequel S, T are finite subsets of V. Let us consider G_1, V, S, T. Then $S \cup T$ is a finite subset of V. Then $S \cap T$ is a finite subset of V. Then $S \setminus T$ is a finite subset of V. Then $S \dot{-} T$ is a finite subset of V.

Let us consider G_1, V. The functor 0_V yields a finite subset of V and is defined as follows:

(Def.2) $0_V = \emptyset$.

One can prove the following proposition

[1]Supported by RPBP.III-24.C1

ISSN 0777–4028

(2) $0_V = \emptyset$.

Let us consider G_1, V, T. The functor $\sum T$ yields a vector of V and is defined as follows:

(Def.3) there exists F such that $\operatorname{rng} F = T$ and F is one-to-one and $\sum T = \sum F$.

We now state two propositions:

(3) There exists F such that $\operatorname{rng} F = T$ and F is one-to-one and $\sum T = \sum F$.

(4) If $\operatorname{rng} F = T$ and F is one-to-one and $v = \sum F$, then $v = \sum T$.

Let us consider G_1, V, v. Then $\{v\}$ is a finite subset of V.

Let us consider G_1, V, v_1, v_2. Then $\{v_1, v_2\}$ is a finite subset of V.

Let us consider G_1, V, v_1, v_2, v_3. Then $\{v_1, v_2, v_3\}$ is a finite subset of V.

One can prove the following propositions:

(5) $\sum(0_V) = \Theta_V$.

(6) $\sum\{v\} = v$.

(7) If $v_1 \neq v_2$, then $\sum\{v_1, v_2\} = v_1 + v_2$.

(8) If $v_1 \neq v_2$ and $v_2 \neq v_3$ and $v_1 \neq v_3$, then $\sum\{v_1, v_2, v_3\} = (v_1 + v_2) + v_3$.

(9) If T misses S, then $\sum(T \cup S) = \sum T + \sum S$.

(10) $\sum(T \cup S) = (\sum T + \sum S) - \sum(T \cap S)$.

(11) $\sum(T \cap S) = (\sum T + \sum S) - \sum(T \cup S)$.

(12) $\sum(T \setminus S) = \sum(T \cup S) - \sum S$.

(13) $\sum(T \setminus S) = \sum T - \sum(T \cap S)$.

(14) $\sum(T \dot{-} S) = \sum(T \cup S) - \sum(T \cap S)$.

(15) $\sum(T \dot{-} S) = \sum(T \setminus S) + \sum(S \setminus T)$.

Let us consider G_1, V. An element of (the carrier of $G_1)^{\text{the carrier of the carrier of } V}$ is called a linear combination of V if:

(Def.4) there exists T such that for every v such that $v \notin T$ holds $\mathrm{it}(v) = 0_{G_1}$.

In the sequel K, L, L_1, L_2, L_3 are linear combinations of V. Next we state the proposition

(16) There exists T such that for every v such that $v \notin T$ holds $L(v) = 0_{G_1}$.

In the sequel E is an element of (the carrier of $G_1)^{\text{the carrier of the carrier of } V}$. We now state the proposition

(17) If there exists T such that for every v such that $v \notin T$ holds $E(v) = 0_{G_1}$, then E is a linear combination of V.

Let us consider G_1, V, L. The functor $\operatorname{support} L$ yields a finite subset of V and is defined as follows:

(Def.5) $\operatorname{support} L = \{v : L(v) \neq 0_{G_1}\}$.

The following propositions are true:

(18) $\operatorname{support} L = \{v : L(v) \neq 0_{G_1}\}$.

(19) $x \in \operatorname{support} L$ if and only if there exists v such that $x = v$ and $L(v) \neq 0_{G_1}$.

(20) $L(v) = 0_{G_1}$ if and only if $v \notin \operatorname{support} L$.

Let us consider G_1, V. The functor $\mathbf{0}_{\mathrm{LC}_V}$ yielding a linear combination of V is defined as follows:

(Def.6) $\operatorname{support} \mathbf{0}_{\mathrm{LC}_V} = \emptyset$.

Next we state two propositions:

(21) $L = \mathbf{0}_{\mathrm{LC}_V}$ if and only if $\operatorname{support} L = \emptyset$.

(22) $\mathbf{0}_{\mathrm{LC}_V}(v) = 0_{G_1}$.

Let us consider G_1, V, A. A linear combination of V is said to be a linear combination of A if:

(Def.7) $\operatorname{support} \mathrm{it} \subseteq A$.

One can prove the following proposition

(23) If $\operatorname{support} L \subseteq A$, then L is a linear combination of A.

In the sequel l denotes a linear combination of A. Next we state several propositions:

(24) $\operatorname{support} l \subseteq A$.

(25) If $A \subseteq B$, then l is a linear combination of B.

(26) $\mathbf{0}_{\mathrm{LC}_V}$ is a linear combination of A.

(27) For every linear combination l of $\emptyset_{\text{the carrier of the carrier of } V}$ holds $l = \mathbf{0}_{\mathrm{LC}_V}$.

(28) L is a linear combination of $\operatorname{support} L$.

Let us consider G_1, V, F, f. The functor $f \cdot F$ yields a finite sequence of elements of the carrier of the carrier of V and is defined by:

(Def.8) $\operatorname{len}(f \cdot F) = \operatorname{len} F$ and for every i such that $i \in \operatorname{dom}(f \cdot F)$ holds $(f \cdot F)(i) = f(\pi_i F) \cdot \pi_i F$.

Next we state several propositions:

(29) $\operatorname{len}(f \cdot F) = \operatorname{len} F$.

(30) For every i such that $i \in \operatorname{dom}(f \cdot F)$ holds $(f \cdot F)(i) = f(\pi_i F) \cdot \pi_i F$.

(31) If $\operatorname{len} G = \operatorname{len} F$ and for every i such that $i \in \operatorname{dom} G$ holds $G(i) = f(\pi_i F) \cdot \pi_i F$, then $G = f \cdot F$.

(32) If $i \in \operatorname{dom} F$ and $v = F(i)$, then $(f \cdot F)(i) = f(v) \cdot v$.

(33) $f \cdot \varepsilon_{\text{the carrier of the carrier of } V} = \varepsilon_{\text{the carrier of the carrier of } V}$.

(34) $f \cdot \langle v \rangle = \langle f(v) \cdot v \rangle$.

(35) $f \cdot \langle v_1, v_2 \rangle = \langle f(v_1) \cdot v_1, f(v_2) \cdot v_2 \rangle$.

(36) $f \cdot \langle v_1, v_2, v_3 \rangle = \langle f(v_1) \cdot v_1, f(v_2) \cdot v_2, f(v_3) \cdot v_3 \rangle$.

(37) $f \cdot (F \frown G) = (f \cdot F) \frown (f \cdot G)$.

Let us consider G_1, V, L. The functor $\sum L$ yielding a vector of V is defined as follows:

(Def.9) there exists F such that F is one-to-one and $\operatorname{rng} F = \operatorname{support} L$ and $\sum L = \sum(L \cdot F)$.

The following propositions are true:

(38) There exists F such that F is one-to-one and $\operatorname{rng} F = \operatorname{support} L$ and $\sum L = \sum(L \cdot F)$.

(39) If F is one-to-one and $\operatorname{rng} F = \operatorname{support} L$ and $u = \sum(L \cdot F)$, then $u = \sum L$.

(40) $A \neq \emptyset$ and A is linearly closed if and only if for every l holds $\sum l \in A$.

(41) $\sum \mathbf{0}_{\mathrm{LC}_V} = \Theta_V$.

(42) For every linear combination l of $\emptyset_{\text{the carrier of the carrier of } V}$ holds $\sum l = \Theta_V$.

(43) For every linear combination l of $\{v\}$ holds $\sum l = l(v) \cdot v$.

(44) If $v_1 \neq v_2$, then for every linear combination l of $\{v_1, v_2\}$ holds $\sum l = l(v_1) \cdot v_1 + l(v_2) \cdot v_2$.

(45) If $\operatorname{support} L = \emptyset$, then $\sum L = \Theta_V$.

(46) If $\operatorname{support} L = \{v\}$, then $\sum L = L(v) \cdot v$.

(47) If $\operatorname{support} L = \{v_1, v_2\}$ and $v_1 \neq v_2$, then $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2$.

Let us consider G_1, V, L_1, L_2. Let us note that one can characterize the predicate $L_1 = L_2$ by the following (equivalent) condition:

(Def.10) for every v holds $L_1(v) = L_2(v)$.

One can prove the following proposition

(48) If for every v holds $L_1(v) = L_2(v)$, then $L_1 = L_2$.

Let us consider G_1, V, L_1, L_2. The functor $L_1 + L_2$ yields a linear combination of V and is defined as follows:

(Def.11) for every v holds $(L_1 + L_2)(v) = L_1(v) + L_2(v)$.

Next we state several propositions:

(49) If for every v holds $L(v) = L_1(v) + L_2(v)$, then $L = L_1 + L_2$.

(50) $(L_1 + L_2)(v) = L_1(v) + L_2(v)$.

(51) $\operatorname{support}(L_1 + L_2) \subseteq \operatorname{support} L_1 \cup \operatorname{support} L_2$.

(52) If L_1 is a linear combination of A and L_2 is a linear combination of A, then $L_1 + L_2$ is a linear combination of A.

(53) $L_1 + L_2 = L_2 + L_1$.

(54) $L_1 + (L_2 + L_3) = (L_1 + L_2) + L_3$.

(55) $L + \mathbf{0}_{\mathrm{LC}_V} = L$ and $\mathbf{0}_{\mathrm{LC}_V} + L = L$.

Let us consider G_1, V, a, L. The functor $a \cdot L$ yielding a linear combination of V is defined by:

(Def.12) for every v holds $(a \cdot L)(v) = a \cdot L(v)$.

The following propositions are true:

(56) If for every v holds $K(v) = a \cdot L(v)$, then $K = a \cdot L$.

(57) $(a \cdot L)(v) = a \cdot L(v)$.

(58) If $a \neq 0_{G_1}$, then $\mathrm{support}(a \cdot L) = \mathrm{support}\, L$.

(59) $0_{G_1} \cdot L = \mathbf{0}_{\mathrm{LC}_V}$.

(60) If L is a linear combination of A, then $a \cdot L$ is a linear combination of A.

(61) $(a + b) \cdot L = a \cdot L + b \cdot L$.

(62) $a \cdot (L_1 + L_2) = a \cdot L_1 + a \cdot L_2$.

(63) $a \cdot (b \cdot L) = (a \cdot b) \cdot L$.

(64) $(1_{G_1}) \cdot L = L$.

Let us consider G_1, V, L. The functor $-L$ yields a linear combination of V and is defined by:

(Def.13) $-L = (-1_{G_1}) \cdot L$.

The following propositions are true:

(65) $-L = (-1_{G_1}) \cdot L$.

(66) $(-L)(v) = -L(v)$.

(67) If $L_1 + L_2 = \mathbf{0}_{\mathrm{LC}_V}$, then $L_2 = -L_1$.

(68) $\mathrm{support}(-L) = \mathrm{support}\, L$.

(69) If L is a linear combination of A, then $-L$ is a linear combination of A.

(70) $-(-L) = L$.

Let us consider G_1, V, L_1, L_2. The functor $L_1 - L_2$ yielding a linear combination of V is defined by:

(Def.14) $L_1 - L_2 = L_1 + (-L_2)$.

Next we state a number of propositions:

(71) $L_1 - L_2 = L_1 + (-L_2)$.

(72) $(L_1 - L_2)(v) = L_1(v) - L_2(v)$.

(73) $\mathrm{support}(L_1 - L_2) \subseteq \mathrm{support}\, L_1 \cup \mathrm{support}\, L_2$.

(74) If L_1 is a linear combination of A and L_2 is a linear combination of A, then $L_1 - L_2$ is a linear combination of A.

(75) $L - L = \mathbf{0}_{\mathrm{LC}_V}$.

(76) $\sum(L_1 + L_2) = \sum L_1 + \sum L_2$.

(77) $\sum(a \cdot L) = a \cdot \sum L$.

(78) $\sum(-L) = -\sum L$.

(79) $\sum(L_1 - L_2) = \sum L_1 - \sum L_2$.

(80) $(-1_{G_1}) \cdot a = -a$.

(81) $-1_{G_1} \neq 0_{G_1}$.

(82) $-a = 0_{G_1} - a$.

(83) $-a = -(1_{G_1}) \cdot a$.

(84) $(a - b) \cdot c = a \cdot c - b \cdot c$.

(85) If $a + b = 0_{G_1}$, then $b = -a$.

References

[1] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[4] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[5] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[6] Andrzej Trybulec. Function domains and Frænkel operator. *Formalized Mathematics*, 1(**3**):495–500, 1990.

[7] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[8] Wojciech A. Trybulec. Finite sums of vectors in vector space. *Formalized Mathematics*, 1(**5**):851–854, 1990.

[9] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.

[10] Wojciech A. Trybulec. Subspaces and cosets of subspaces in vector space. *Formalized Mathematics*, 1(**5**):865–870, 1990.

[11] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[12] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

Received July 27, 1990

FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Basis of Vector Space

Wojciech A. Trybulec[1]
Warsaw University

Summary. We prove the existence of a basis of a vector space, i.e., a set of vectors that generates the vector space and is linearly independent. We also introduce the notion of a subspace generated by a set of vectors and linear independence of set of vectors.

MML Identifier: `VECTSP_7`.

The terminology and notation used in this paper are introduced in the following papers: [5], [2], [9], [4], [3], [6], [1], [10], [8], and [7]. For simplicity we follow the rules: x will be arbitrary, G_1 will denote a field, a, b will denote elements of G_1, V will denote a vector space over G_1, W will denote a subspace of V, v, v_1, v_2 will denote vectors of V, A, B will denote subsets of V, and l will denote a linear combination of A. We now define two new predicates. Let us consider G_1, V, A. We say that A is linearly independent if and only if:

(Def.1) for every l such that $\sum l = \Theta_V$ holds $\mathrm{support}\, l = \emptyset$.

We say that A is linearly dependent if A is not linearly independent.

One can prove the following propositions:

(1) A is linearly independent if and only if for every l such that $\sum l = \Theta_V$ holds $\mathrm{support}\, l = \emptyset$.

(2) If $A \subseteq B$ and B is linearly independent, then A is linearly independent.

(3) If A is linearly independent, then $\Theta_V \notin A$.

(4) $\emptyset_{\text{the carrier of the carrier of } V}$ is linearly independent.

(5) $\{v\}$ is linearly independent if and only if $v \neq \Theta_V$.

(6) If $\{v_1, v_2\}$ is linearly independent, then $v_1 \neq \Theta_V$ and $v_2 \neq \Theta_V$.

(7) $\{v, \Theta_V\}$ is linearly dependent and $\{\Theta_V, v\}$ is linearly dependent.

(8) $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent if and only if $v_2 \neq \Theta_V$ and for every a holds $v_1 \neq a \cdot v_2$.

[1]Supported by RPBP.III-24.C1

ISSN 0777–4028

(9) $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent if and only if for all a, b such that $a \cdot v_1 + b \cdot v_2 = \Theta_V$ holds $a = 0_{G_1}$ and $b = 0_{G_1}$.

Let us consider G_1, V, A. The functor $\mathrm{Lin}(A)$ yields a subspace of V and is defined by:

(Def.2) the carrier of the carrier of $\mathrm{Lin}(A) = \{\sum l\}$.

The following propositions are true:

(10) If the carrier of the carrier of $W = \{\sum l\}$, then $W = \mathrm{Lin}(A)$.

(11) The carrier of the carrier of $\mathrm{Lin}(A) = \{\sum l\}$.

(12) $x \in \mathrm{Lin}(A)$ if and only if there exists l such that $x = \sum l$.

(13) If $x \in A$, then $x \in \mathrm{Lin}(A)$.

The following propositions are true:

(14) $\mathrm{Lin}(\emptyset_{\text{the carrier of the carrier of } V}) = \mathbf{0}_V$.

(15) If $\mathrm{Lin}(A) = \mathbf{0}_V$, then $A = \emptyset$ or $A = \{\Theta_V\}$.

(16) If $A =$ the carrier of the carrier of W, then $\mathrm{Lin}(A) = W$.

(17) If $A =$ the carrier of the carrier of V, then $\mathrm{Lin}(A) = V$.

(18) If $A \subseteq B$, then $\mathrm{Lin}(A)$ is a subspace of $\mathrm{Lin}(B)$.

(19) If $\mathrm{Lin}(A) = V$ and $A \subseteq B$, then $\mathrm{Lin}(B) = V$.

(20) $\mathrm{Lin}(A \cup B) = \mathrm{Lin}(A) + \mathrm{Lin}(B)$.

(21) $\mathrm{Lin}(A \cap B)$ is a subspace of $\mathrm{Lin}(A) \cap \mathrm{Lin}(B)$.

(22) If A is linearly independent, then there exists B such that $A \subseteq B$ and B is linearly independent and $\mathrm{Lin}(B) = V$.

(23) If $\mathrm{Lin}(A) = V$, then there exists B such that $B \subseteq A$ and B is linearly independent and $\mathrm{Lin}(B) = V$.

Let us consider G_1, V. A subset of V is called a basis of V if:

(Def.3) it is linearly independent and $\mathrm{Lin}(\mathrm{it}) = V$.

We now state the proposition

(24) If A is linearly independent and $\mathrm{Lin}(A) = V$, then A is a basis of V.

In the sequel I will denote a basis of V. We now state four propositions:

(25) I is linearly independent.

(26) $\mathrm{Lin}(I) = V$.

(27) If A is linearly independent, then there exists I such that $A \subseteq I$.

(28) If $\mathrm{Lin}(A) = V$, then there exists I such that $I \subseteq A$.

References

[1] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[2] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[3] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[4] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[5] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[6] Wojciech A. Trybulec. Finite sums of vectors in vector space. *Formalized Mathematics*, 1(**5**):851–854, 1990.

[7] Wojciech A. Trybulec. Linear combinations in vector space. *Formalized Mathematics*, 1(**5**):877–882, 1990.

[8] Wojciech A. Trybulec. Operations on subspaces in vector space. *Formalized Mathematics*, 1(**5**):871–876, 1990.

[9] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.

[10] Wojciech A. Trybulec. Subspaces and cosets of subspaces in vector space. *Formalized Mathematics*, 1(**5**):865–870, 1990.

Received July 27, 1990

FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Factorial and Newton coeffitients

Rafał Kwiatek[1]
Nicolaus Copernicus University
Toruń

Summary. We define the following functions: exponential function (for natural exponent), factorial function and Newton coefficients. We prove some basic properties of notions introduced. There is also a proof of binominal formula. We prove also that $\sum_{k=0}^{n}\binom{n}{k} = 2^n$.

MML Identifier: `NEWTON`.

The notation and terminology used in this paper have been introduced in the following articles: [4], [7], [6], [2], [3], [1], and [5]. We adopt the following rules: i, k, n, m, l denote natural numbers, a, b, x, y, z denote real numbers, and F, G denote finite sequences of elements of $\mathbb{R}$. One can prove the following propositions:

(1) For all x, y, z such that $y \neq 0$ and $z \neq 0$ holds $\frac{z \cdot x}{z \cdot y} = \frac{x}{y}$.

(2) If $k \geq l$, then $k - l$ is a natural number.

(3) For all F, G such that $\operatorname{len} F = \operatorname{len} G$ and for every i such that $i \in \operatorname{dom} F$ holds $F(i) = G(i)$ holds $F = G$.

(4) For every n such that $n \geq 1$ holds $1 \in \operatorname{Seg} n$.

(5) For every n such that $n \geq 1$ holds $\operatorname{Seg} n = (\{1\} \cup \{k : 1 < k \wedge k < n\}) \cup \{n\}$.

(6) For every F holds $\operatorname{len}(a \cdot F) = \operatorname{len} F$.

(7) $n \in \operatorname{dom} G$ if and only if $n \in \operatorname{dom}(a \cdot G)$.

Let us consider i, x. Then $i \longmapsto x$ is a finite sequence of elements of $\mathbb{R}$.

Let us consider x, n. The functor x^n yielding a real number is defined as follows:

(Def.1) $x^n = \prod(n \longmapsto x)$.

One can prove the following propositions:

[1] Supported by RPBP.III-24.B5.

ISSN 0777–4028

(8) $x^n = \prod(n \longmapsto x)$.

(9) For every x holds $x^0 = 1$.

(10) For every x holds $x^1 = x$.

(11) For every n holds $x^{n+1} = x^n \cdot x$ and $x^{n+1} = x \cdot x^n$.

(12) $(x \cdot y)^n = x^n \cdot y^n$.

(13) $x^{n+m} = x^n \cdot x^m$.

(14) $(x^n)^m = x^{n \cdot m}$.

(15) For every n holds $1^n = 1$.

(16) For every n such that $n \geq 1$ holds $0^n = 0$.

Let us consider n. Then id_n is a finite sequence of elements of $\mathbb{R}$.

Let us consider x. Then $\langle x \rangle$ is a finite sequence of elements of $\mathbb{R}$. Let us consider y. Then $\langle x, y \rangle$ is a finite sequence of elements of $\mathbb{R}$.

Let us consider n. The functor $n!$ yielding a real number is defined by:

(Def.2) $n! = \prod(\mathrm{id}_n)$.

We now state several propositions:

(17) $n! = \prod(\mathrm{id}_n)$.

(18) $0! = 1$.

(19) $1! = 1$.

(20) $2! = 2$.

(21) For every n holds $(n+1)! = (n+1) \cdot (n!)$ and $(n+1)! = (n!) \cdot (n+1)$.

(22) For every n holds $n!$ is a natural number.

(23) For every n holds $n! > 0$.

(24) For every n holds $n! \neq 0$.

(25) For all n, k holds $(n!) \cdot (k!) \neq 0$.

Let us consider k, n. The functor $\binom{n}{k}$ yielding a real number is defined as follows:

(Def.3) for every l such that $l = n - k$ holds $\binom{n}{k} = \frac{n!}{(k!) \cdot (l!)}$ if $n \geq k$, $\binom{n}{k} = 0$, otherwise.

We now state a number of propositions:

(26) For every l such that $l = n - k$ holds $\binom{n}{k} = \frac{n!}{(k!) \cdot (l!)}$ if and only if $n \geq k$ or if $\binom{n}{k} = 1$, then $n < k$.

(27) $\binom{0}{0} = 1$.

(28) For every k such that $k > 0$ holds $\binom{0}{k} = 0$.

(29) For every n holds $\binom{n}{0} = 1$.

(30) For all n, k such that $n \geq k$ for every l such that $l = n - k$ holds $\binom{n}{k} = \binom{n}{l}$.

(31) For every n holds $\binom{n}{n} = 1$.

(32) For all k, n such that $k < n$ holds $\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$ and $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$.

(33) For every n such that $n \geq 1$ holds $\binom{n}{1} = n$.

(34) For all n, l such that $n \geq 1$ and $l = n - 1$ holds $\binom{n}{l} = n$.

(35) For every n and for every k holds $\binom{n}{k}$ is a natural number.

(36) For all m, F such that $m \neq 0$ and $\operatorname{len} F = m$ and for all i, l such that $i \in \operatorname{dom} F$ and $l = (n + i) - 1$ holds $F(i) = \binom{l}{n}$ holds $\sum F = \binom{n+m}{n+1}$.

Let a, b be real numbers, and let n be a natural number. The functor $\langle\binom{n}{0}a^0b^n, \ldots, \binom{n}{n}a^nb^0\rangle$ yields a finite sequence of elements of $\mathbb{R}$ and is defined as follows:

(Def.4) $\operatorname{len}\langle\binom{n}{0}a^0b^n, \ldots, \binom{n}{n}a^nb^0\rangle = n + 1$ and for all i, l, m such that $i \in \operatorname{dom}\langle\binom{n}{0}a^0b^n, \ldots, \binom{n}{n}a^nb^0\rangle$ and $m = i - 1$ and $l = n - m$ holds $\langle\binom{n}{0}a^0b^n, \ldots, \binom{n}{n}a^nb^0\rangle(i) = (\binom{n}{m} \cdot a^l) \cdot b^m$.

Next we state several propositions:

(37) Given F. Then the following conditions are equivalent:

(i) $\operatorname{len} F = n + 1$ and for all i, l, m such that $i \in \operatorname{dom} F$ and $m = i - 1$ and $l = n - m$ holds $F(i) = (\binom{n}{m} \cdot a^l) \cdot b^m$,

(ii) $F = \langle\binom{n}{0}a^0b^n, \ldots, \binom{n}{n}a^nb^0\rangle$.

(38) $\langle\binom{0}{0}a^0b^0, \ldots, \binom{0}{0}a^0b^0\rangle = \langle 1\rangle$.

(39) $\langle\binom{n}{0}a^0b^n, \ldots, \binom{n}{n}a^nb^0\rangle(1) = a^n$.

(40) $\langle\binom{n}{0}a^0b^n, \ldots, \binom{n}{n}a^nb^0\rangle(n + 1) = b^n$.

(41) For every n holds $(a + b)^n = \sum\langle\binom{n}{0}a^0b^n, \ldots, \binom{n}{n}a^nb^0\rangle$.

Let us consider n. The functor $\langle\binom{n}{0}, \ldots, \binom{n}{n}\rangle$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by:

(Def.5) $\operatorname{len}\langle\binom{n}{0}, \ldots, \binom{n}{n}\rangle = n + 1$ and for all i, k such that $i \in \operatorname{dom}\langle\binom{n}{0}, \ldots, \binom{n}{n}\rangle$ and $k = i - 1$ holds $\langle\binom{n}{0}, \ldots, \binom{n}{n}\rangle(i) = \binom{n}{k}$.

We now state three propositions:

(42) For every F holds $\operatorname{len} F = n + 1$ and for all i, m such that $i \in \operatorname{dom} F$ and $m = i - 1$ holds $F(i) = \binom{n}{m}$ if and only if $F = \langle\binom{n}{0}, \ldots, \binom{n}{n}\rangle$.

(43) For every n holds $\langle\binom{n}{0}, \ldots, \binom{n}{n}\rangle = \langle\binom{n}{0}1^01^n, \ldots, \binom{n}{n}1^n1^0\rangle$.

(44) For every n holds $2^n = \sum\langle\binom{n}{0}, \ldots, \binom{n}{n}\rangle$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.

[6] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[7] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

Received July 27, 1990

FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Analytical Metric Affine Spaces and Planes [1]

Henryk Oryszczyszyn
Warsaw University
Białystok

Krzysztof Prażmowski
Warsaw University
Białystok

Summary. We introduce relations of orthogonality of vectors and of orthogonality of segments (considered as pairs of vectors) in real linear space of dimension two. This enables us to show an example of (in fact anisotropic and satisfying theorem on three perpendiculars) metric affine space (and plane as well). These two types of objects are defined formally as "Mizar" modes. They are to be understood as structures consisting of a point universe and two binary relations on segments - a parallelity relation and orthogonality relation, satisfying appropriate axioms. With every such structure we correlate a structure obtained as a reduct of the given one to the parallelity relation only. Some relationships between metric affine spaces and their affine parts are proved; they enable us to use "affine" facts and constructions in investigating metric affine geometry. We define the notions of line, parallelity of lines and two derived relations of orthogonality: between segments and lines, and between lines. Some basic properties of the introduced notions are proved.

MML Identifier: ANALMETR.

The articles [5], [1], [7], [6], [2], [3], and [4] provide the notation and terminology for this paper. For simplicity we follow a convention: V denotes a real linear space, u, u_1, u_2, v, v_1, v_2, w, y denote vectors of V, a, a_1, a_2, b, b_1, b_2 denote real numbers, and x, z are arbitrary. Let us consider V, w, y. We say that w, y span the space if and only if:

(Def.1) for every u there exist a_1, a_2 such that $u = a_1 \cdot w + a_2 \cdot y$ and for all a_1, a_2 such that $a_1 \cdot w + a_2 \cdot y = 0_V$ holds $a_1 = 0$ and $a_2 = 0$.

One can prove the following propositions:

(1) For all w, y holds w, y span the space if and only if for every u there exist a_1, a_2 such that $u = a_1 \cdot w + a_2 \cdot y$ and for all a_1, a_2 such that $a_1 \cdot w + a_2 \cdot y = 0_V$ holds $a_1 = 0$ and $a_2 = 0$.

[1]Supported by RPBP.III-24.C2.

ISSN 0777–4028

(2) If w, y span the space, then there exist a_1, a_2 such that $u = a_1 \cdot w + a_2 \cdot y$.

(3) If w, y span the space and $a_1 \cdot w + a_2 \cdot y = 0_V$, then $a_1 = 0$ and $a_2 = 0$.

Let us consider V, u, v, w, y. We say that u, v are orthogonal w.r.t. w, y if and only if:

(Def.2) there exist a_1, a_2, b_1, b_2 such that $u = a_1 \cdot w + a_2 \cdot y$ and $v = b_1 \cdot w + b_2 \cdot y$ and $a_1 \cdot b_1 + a_2 \cdot b_2 = 0$.

The following propositions are true:

(4) For all u, v, w, y holds u, v are orthogonal w.r.t. w, y if and only if there exist a_1, a_2, b_1, b_2 such that $u = a_1 \cdot w + a_2 \cdot y$ and $v = b_1 \cdot w + b_2 \cdot y$ and $a_1 \cdot b_1 + a_2 \cdot b_2 = 0$.

(5) For all w, y such that w, y span the space holds u, v are orthogonal w.r.t. w, y if and only if for all a_1, a_2, b_1, b_2 such that $u = a_1 \cdot w + a_2 \cdot y$ and $v = b_1 \cdot w + b_2 \cdot y$ holds $a_1 \cdot b_1 + a_2 \cdot b_2 = 0$.

(6) w, y are orthogonal w.r.t. w, y.

(7) There exists V and there exist w, y such that w, y span the space.

(8) If u, v are orthogonal w.r.t. w, y, then v, u are orthogonal w.r.t. w, y.

(9) If w, y span the space, then for all u, v holds u, 0_V are orthogonal w.r.t. w, y and 0_V, v are orthogonal w.r.t. w, y.

(10) If u, v are orthogonal w.r.t. w, y, then $a \cdot u$, $b \cdot v$ are orthogonal w.r.t. w, y.

(11) If u, v are orthogonal w.r.t. w, y, then $a \cdot u$, v are orthogonal w.r.t. w, y and u, $b \cdot v$ are orthogonal w.r.t. w, y.

(12) If w, y span the space, then for every u there exists v such that u, v are orthogonal w.r.t. w, y and $v \neq 0_V$.

(13) If w, y span the space and v, u_1 are orthogonal w.r.t. w, y and v, u_2 are orthogonal w.r.t. w, y and $v \neq 0_V$, then there exist a, b such that $a \cdot u_1 = b \cdot u_2$ but $a \neq 0$ or $b \neq 0$.

(14) If w, y span the space and u, v_1 are orthogonal w.r.t. w, y and u, v_2 are orthogonal w.r.t. w, y, then u, $v_1 + v_2$ are orthogonal w.r.t. w, y and u, $v_1 - v_2$ are orthogonal w.r.t. w, y.

(15) If w, y span the space and u, u are orthogonal w.r.t. w, y, then $u = 0_V$.

(16) If w, y span the space and u, $u_1 - u_2$ are orthogonal w.r.t. w, y and u_1, $u_2 - u$ are orthogonal w.r.t. w, y, then u_2, $u - u_1$ are orthogonal w.r.t. w, y.

(17) If w, y span the space and $u \neq 0_V$, then there exists a such that $v - a \cdot u$, u are orthogonal w.r.t. w, y.

(18) $u, v \upuparrows u_1, v_1$ or $u, v \upuparrows v_1, u_1$ if and only if there exist a, b such that $a \cdot (v - u) = b \cdot (v_1 - u_1)$ but $a \neq 0$ or $b \neq 0$.

(19) $\langle\langle u, v\rangle, \langle u_1, v_1\rangle\rangle \in \lambda(\upuparrows_V)$ if and only if there exist a, b such that $a \cdot (v - u) = b \cdot (v_1 - u_1)$ but $a \neq 0$ or $b \neq 0$.

Let us consider V, u, u_1, v, v_1, w, y. We say that u, u_1, v and v_1 are orthogonal w.r.t. w, y if and only if:

(Def.3) $u_1 - u$, $v_1 - v$ are orthogonal w.r.t. w, y.

One can prove the following proposition

(20) For all u, u_1, v, v_1, w, y holds u, u_1, v and v_1 are orthogonal w.r.t. w, y if and only if $u_1 - u$, $v_1 - v$ are orthogonal w.r.t. w, y.

Let us consider V, w, y. The ortogonality determined by w, y in V yielding a binary relation on $[\!:$ the vectors of V, the vectors of $V :\!]$ is defined as follows:

(Def.4) $\langle x, z\rangle \in$ the ortogonality determined by w, y in V if and only if there exist u, u_1, v, v_1 such that $x = \langle u, u_1\rangle$ and $z = \langle v, v_1\rangle$ and u, u_1, v and v_1 are orthogonal w.r.t. w, y.

We now state the proposition

(21) For every binary relation R on $[\!:$ the vectors of V, the vectors of $V :\!]$ holds $R =$ the ortogonality determined by w, y in V if and only if for all x, z holds $\langle x, z\rangle \in R$ if and only if there exist u, u_1, v, v_1 such that $x = \langle u, u_1\rangle$ and $z = \langle v, v_1\rangle$ and u, u_1, v and v_1 are orthogonal w.r.t. w, y.

In the sequel p, p_1, q, q_1 will denote elements of the points of $\Lambda(\text{OASpace}\, V)$. We now state three propositions:

(22) The points of $\Lambda(\text{OASpace}\, V) =$ the vectors of V.

(23) The congruence of $\Lambda(\text{OASpace}\, V) = \lambda(\upuparrows_V)$.

(24) If $p = u$ and $q = v$ and $p_1 = u_1$ and $q_1 = v_1$, then $p, q \parallel p_1, q_1$ if and only if there exist a, b such that $a \cdot (v - u) = b \cdot (v_1 - u_1)$ but $a \neq 0$ or $b \neq 0$.

We consider metric affine structures which are systems

$\langle$points, a parallelity, an orthogonality$\rangle$,

where the points constitute a non-empty set, the parallelity is a binary relation on $[\!:$ the points, the points $:\!]$, and the orthogonality is a binary relation on $[\!:$ the points, the points $:\!]$. In the sequel P_1 will denote a metric-affine structure. We now define two new predicates. Let us consider P_1, and let a, b, c, d be elements of the points of P_1. The predicate $a, b \parallel c, d$ is defined as follows:

(Def.5) $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in$ the parallelity of P_1.

The predicate $a, b \perp c, d$ is defined as follows:

(Def.6) $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in$ the orthogonality of P_1.

One can prove the following propositions:

(25) For all elements a, b, c, d of the points of P_1 holds $a, b \parallel c, d$ if and only if $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in$ the parallelity of P_1.

(26) For all elements a, b, c, d of the points of P_1 holds $a, b \perp c, d$ if and only if $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in$ the orthogonality of P_1.

Let us consider V, w, y. Let us assume that w, y span the space. The functor **AMSp**(V, w, y) yielding a metric-affine structure is defined by:

(Def.7) $\mathbf{AMSp}(V, w, y) = \langle$ the vectors of $V, \lambda(\Uparrow_V)$, the ortogonality determined by w, y in $V\rangle$.

Next we state two propositions:

(27) If w, y span the space, then $P_1 = \mathbf{AMSp}(V, w, y)$ if and only if $P_1 = \langle$ the vectors of $V, \lambda(\Uparrow_V)$, the ortogonality determined by w, y in $V\rangle$.

(28) If w, y span the space, then the points of $\mathbf{AMSp}(V, w, y) =$ the vectors of V and the parallelity of $\mathbf{AMSp}(V, w, y) = \lambda(\Uparrow_V)$ and the orthogonality of $\mathbf{AMSp}(V, w, y) =$ the ortogonality determined by w, y in V.

Let us consider P_1. The affine reduct of P_1 yielding an affine structure is defined by:

(Def.8) the affine reduct of $P_1 = \langle$ the points of P_1, the parallelity of $P_1\rangle$.

We now state two propositions:

(29) For every P_1 and for every A_1 being an affine structure holds $A_1 =$ the affine reduct of P_1 if and only if $A_1 = \langle$ the points of P_1, the parallelity of $P_1\rangle$.

(30) If w, y span the space, then the affine reduct of $\mathbf{AMSp}(V, w, y) = \Lambda(\text{OASpace}\, V)$.

In the sequel p, p_1, p_2, q, q_1, r, r_1, r_2 denote elements of the points of $\mathbf{AMSp}(V, w, y)$. One can prove the following propositions:

(31) If w, y span the space and $p = u$ and $p_1 = u_1$ and $q = v$ and $q_1 = v_1$, then $p, q \perp p_1, q_1$ if and only if u, v, u_1 and v_1 are orthogonal w.r.t. w, y.

(32) If w, y span the space and $p = u$ and $q = v$ and $p_1 = u_1$ and $q_1 = v_1$, then $p, q \parallel p_1, q_1$ if and only if there exist a, b such that $a \cdot (v - u) = b \cdot (v_1 - u_1)$ but $a \neq 0$ or $b \neq 0$.

(33) If w, y span the space and $p, q \perp p_1, q_1$, then $p_1, q_1 \perp p, q$.

(34) If w, y span the space and $p, q \perp p_1, q_1$, then $p, q \perp q_1, p_1$.

(35) If w, y span the space, then for all p, q, r holds $p, q \perp r, r$.

(36) If w, y span the space and $p, p_1 \perp q, q_1$ and $p, p_1 \parallel r, r_1$, then $p = p_1$ or $q, q_1 \perp r, r_1$.

(37) If w, y span the space, then for every p, q, r there exists r_1 such that $p, q \perp r, r_1$ and $r \neq r_1$.

(38) If w, y span the space and $p, p_1 \perp q, q_1$ and $p, p_1 \perp r, r_1$, then $p = p_1$ or $q, q_1 \parallel r, r_1$.

(39) If w, y span the space and $p, q \perp r, r_1$ and $p, q \perp r, r_2$, then $p, q \perp r_1, r_2$.

(40) If w, y span the space and $p, q \perp p, q$, then $p = q$.

(41) If w, y span the space and $p, q \perp p_1, p_2$ and $p_1, q \perp p_2, p$, then $p_2, q \perp p, p_1$.

(42) If w, y span the space and $p \neq p_1$, then for every q there exists q_1 such that $p, p_1 \parallel p, q_1$ and $p, p_1 \perp q_1, q$.

A metric-affine structure is called a metric affine space if:

(Def.9) (i) $\langle$ the points of it, the parallelity of it$\rangle$ is an affine space,

(ii) for all elements a, b, c, d, p, q, r, s of the points of it holds if $a,b \perp a,b$, then $a=b$ but $a,b \perp c,c$ but if $a,b \perp c,d$, then $a,b \perp d,c$ and $c,d \perp a,b$ but if $a,b \perp p,q$ and $a,b \parallel r,s$, then $p,q \perp r,s$ or $a=b$ but if $a,b \perp p,q$ and $a,b \perp p,s$, then $a,b \perp q,s$,

(iii) for all elements a, b, c of the points of it such that $a \neq b$ there exists an element x of the points of it such that $a,b \parallel a,x$ and $a,b \perp x,c$,

(iv) for every elements a, b, c of the points of it there exists an element x of the points of it such that $a,b \perp c,x$ and $c \neq x$.

We now state two propositions:

(43) Given P_1. Then P_1 is a metric affine space if and only if the following conditions are satisfied:

(i) $\langle$ the points of P_1, the parallelity of $P_1\rangle$ is an affine space,

(ii) for all elements a, b, c, d, p, q, r, s of the points of P_1 holds if $a,b \perp a,b$, then $a=b$ but $a,b \perp c,c$ but if $a,b \perp c,d$, then $a,b \perp d,c$ and $c,d \perp a,b$ but if $a,b \perp p,q$ and $a,b \parallel r,s$, then $p,q \perp r,s$ or $a=b$ but if $a,b \perp p,q$ and $a,b \perp p,s$, then $a,b \perp q,s$,

(iii) for all elements a, b, c of the points of P_1 such that $a \neq b$ there exists an element x of the points of P_1 such that $a,b \parallel a,x$ and $a,b \perp x,c$,

(iv) for every elements a, b, c of the points of P_1 there exists an element x of the points of P_1 such that $a,b \perp c,x$ and $c \neq x$.

(44) If w, y span the space, then $\mathbf{AMSp}(V,w,y)$ is a metric affine space.

A metric-affine structure is said to be a metric affine plane if:

(Def.10) (i) $\langle$ the points of it, the parallelity of it$\rangle$ is an affine plane,

(ii) for all elements a, b, c, d, p, q, r, s of the points of it holds if $a,b \perp a,b$, then $a=b$ but $a,b \perp c,c$ but if $a,b \perp c,d$, then $a,b \perp d,c$ and $c,d \perp a,b$ but if $a,b \perp p,q$ and $a,b \parallel r,s$, then $p,q \perp r,s$ or $a=b$ but if $a,b \perp p,q$ and $a,b \perp r,s$, then $p,q \parallel r,s$ or $a=b$,

(iii) for every elements a, b, c of the points of it there exists an element x of the points of it such that $a,b \perp c,x$ and $c \neq x$.

Next we state four propositions:

(45) Given P_1. Then P_1 is a metric affine plane if and only if the following conditions are satisfied:

(i) $\langle$ the points of P_1, the parallelity of $P_1\rangle$ is an affine plane,

(ii) for all elements a, b, c, d, p, q, r, s of the points of P_1 holds if $a,b \perp a,b$, then $a=b$ but $a,b \perp c,c$ but if $a,b \perp c,d$, then $a,b \perp d,c$ and $c,d \perp a,b$ but if $a,b \perp p,q$ and $a,b \parallel r,s$, then $p,q \perp r,s$ or $a=b$ but if $a,b \perp p,q$ and $a,b \perp r,s$, then $p,q \parallel r,s$ or $a=b$,

(iii) for every elements a, b, c of the points of P_1 there exists an element x of the points of P_1 such that $a,b \perp c,x$ and $c \neq x$.

(46) If w, y span the space, then $\mathbf{AMSp}(V,w,y)$ is a metric affine plane.

(47) For an arbitrary x holds x is an element of the points of P_1 if and only if x is an element of the points of the affine reduct of P_1.

(48) For all elements a, b, c, d of the points of P_1 and for all elements a', b', c', d' of the points of the affine reduct of P_1 such that $a = a'$ and $b = b'$ and $c = c'$ and $d = d'$ holds $a, b \parallel c, d$ if and only if $a', b' \parallel c', d'$.

Let P_1 be a metric affine space. Then the affine reduct of P_1 is an affine space.

Let P_1 be a metric affine plane. Then the affine reduct of P_1 is an affine plane.

The following proposition is true

(49) For every metric affine plane P_1 holds P_1 is a metric affine space.

We see that the metric affine plane is a metric affine space.

The following two propositions are true:

(50) For every metric affine space P_1 such that the affine reduct of P_1 is an affine plane holds P_1 is a metric affine plane.

(51) Let P_1 be a metric-affine structure. Then P_1 is a metric affine plane if and only if the following conditions are satisfied:

(i) there exist elements a, b of the points of P_1 such that $a \neq b$,

(ii) for all elements a, b, c, d, p, q, r, s of the points of P_1 holds $a, b \parallel b, a$ and $a, b \parallel c, c$ but if $a, b \parallel p, q$ and $a, b \parallel r, s$, then $p, q \parallel r, s$ or $a = b$ but if $a, b \parallel a, c$, then $b, a \parallel b, c$ and there exists an element x of the points of P_1 such that $a, b \parallel c, x$ and $a, c \parallel b, x$ and there exist elements x, y, z of the points of P_1 such that $x, y \nparallel x, z$ and there exists an element x of the points of P_1 such that $a, b \parallel c, x$ and $c \neq x$ but if $a, b \parallel b, d$ and $b \neq a$, then there exists an element x of the points of P_1 such that $c, b \parallel b, x$ and $c, a \parallel d, x$ but if $a, b \perp a, b$, then $a = b$ and $a, b \perp c, c$ but if $a, b \perp c, d$, then $a, b \perp d, c$ and $c, d \perp a, b$ but if $a, b \perp p, q$ and $a, b \parallel r, s$, then $p, q \perp r, s$ or $a = b$ but if $a, b \perp p, q$ and $a, b \perp r, s$, then $p, q \parallel r, s$ or $a = b$ and there exists an element x of the points of P_1 such that $a, b \perp c, x$ and $c \neq x$ but if $a, b \nparallel c, d$, then there exists an element x of the points of P_1 such that $a, b \parallel a, x$ and $c, d \parallel c, x$.

In the sequel x, a, b, c, d, p, q will denote elements of the points of P_1. Let us consider P_1, a, b, c. The predicate $\mathbf{L}(a, b, c)$ is defined as follows:

(Def.11) $a, b \parallel a, c$.

We now state the proposition

(52) For every P_1 and for all a, b, c holds $\mathbf{L}(a, b, c)$ if and only if $a, b \parallel a, c$.

Let us consider P_1, a, b. The functor $\mathrm{Line}(a, b)$ yielding a subset of the points of P_1 is defined by:

(Def.12) for every element x of the points of P_1 holds $x \in \mathrm{Line}(a, b)$ if and only if $\mathbf{L}(a, b, x)$.

In the sequel A, K, M denote subsets of the points of P_1. The following proposition is true

(53) $A = \mathrm{Line}(a, b)$ if and only if for every x holds $x \in A$ if and only if $\mathbf{L}(a, b, x)$.

Let us consider P_1, A. We say that A is a line if and only if:

(Def.13) there exist a, b such that $a \neq b$ and $A = \mathrm{Line}(a, b)$.

Next we state several propositions:

(54) A is a line if and only if there exist a, b such that $a \neq b$ and $A = \mathrm{Line}(a, b)$.

(55) For every metric affine space P_1 and for all elements a, b, c of the points of P_1 and for all elements a', b', c' of the points of the affine reduct of P_1 such that $a = a'$ and $b = b'$ and $c = c'$ holds $\mathbf{L}(a, b, c)$ if and only if $\mathbf{L}(a', b', c')$.

(56) For every metric affine space P_1 and for all elements a, b of the points of P_1 and for all elements a', b' of the points of the affine reduct of P_1 such that $a = a'$ and $b = b'$ holds $\mathrm{Line}(a, b) = \mathrm{Line}(a', b')$.

(57) For an arbitrary X holds X is a subset of the points of P_1 if and only if X is a subset of the points of the affine reduct of P_1.

(58) For every metric affine space P_1 and for every subset X of the points of P_1 and for every subset Y of the points of the affine reduct of P_1 such that $X = Y$ holds X is a line if and only if Y is a line.

Let us consider P_1, a, b, K. The predicate $a, b \perp K$ is defined as follows:

(Def.14) there exist p, q such that $p \neq q$ and $K = \mathrm{Line}(p, q)$ and $a, b \perp p, q$.

Let us consider P_1, K, M. The predicate $K \perp M$ is defined by:

(Def.15) there exist p, q such that $p \neq q$ and $K = \mathrm{Line}(p, q)$ and $p, q \perp M$.

Let us consider P_1, K, M. The predicate $K \parallel M$ is defined by:

(Def.16) there exist a, b, c, d such that $a \neq b$ and $c \neq d$ and $K = \mathrm{Line}(a, b)$ and $M = \mathrm{Line}(c, d)$ and $a, b \parallel c, d$.

One can prove the following propositions:

(59) For all a, b, K holds $a, b \perp K$ if and only if there exist p, q such that $p \neq q$ and $K = \mathrm{Line}(p, q)$ and $a, b \perp p, q$.

(60) For all K, M holds $K \perp M$ if and only if there exist p, q such that $p \neq q$ and $K = \mathrm{Line}(p, q)$ and $p, q \perp M$.

(61) For all K, M holds $K \parallel M$ if and only if there exist a, b, c, d such that $a \neq b$ and $c \neq d$ and $K = \mathrm{Line}(a, b)$ and $M = \mathrm{Line}(c, d)$ and $a, b \parallel c, d$.

(62) If $a, b \perp K$, then K is a line but if $K \perp M$, then K is a line and M is a line.

(63) $K \perp M$ if and only if there exist a, b, c, d such that $a \neq b$ and $c \neq d$ and $K = \mathrm{Line}(a, b)$ and $M = \mathrm{Line}(c, d)$ and $a, b \perp c, d$.

(64) For every metric affine space P_1 and for all subsets M, N of the points of P_1 and for all subsets M', N' of the points of the affine reduct of P_1 such that $M = M'$ and $N = N'$ holds $M \parallel N$ if and only if $M' \parallel N'$.

We adopt the following rules: P_1 denotes a metric affine space, A, K, M, N denote subsets of the points of P_1, and a, b, c, d, p, q, r, s denote elements of the points of P_1. The following propositions are true:

(65) If K is a line, then $a, a \perp K$.

(66) If $a, b \perp K$ but $a, b \parallel c, d$ or $c, d \parallel a, b$ and $a \neq b$, then $c, d \perp K$.

(67) If $a, b \perp K$, then $b, a \perp K$.

(68) If $K \parallel M$, then $M \parallel K$.

(69) If $r, s \perp K$ but $K \parallel M$ or $M \parallel K$, then $r, s \perp M$.

(70) If $K \perp M$, then $M \perp K$.

(71) If $a \in K$ and $b \in K$ and $a, b \perp K$, then $a = b$.

(72) If K is a line, then $K \not\perp K$.

(73) If $K \perp M$ or $M \perp K$ but $K \parallel N$ or $N \parallel K$, then $M \perp N$ and $N \perp M$.

(74) If $K \parallel N$, then $K \not\perp N$.

(75) If $a \in K$ and $b \in K$ and $c, d \perp K$, then $c, d \perp a, b$ and $a, b \perp c, d$.

(76) If $a \in K$ and $b \in K$ and $a \neq b$ and K is a line, then $K = \mathrm{Line}(a, b)$.

(77) If $a \in K$ and $b \in K$ and $a \neq b$ and K is a line but $a, b \perp c, d$ or $c, d \perp a, b$, then $c, d \perp K$.

(78) If $a \in M$ and $b \in M$ and $c \in N$ and $d \in N$ and $M \perp N$, then $a, b \perp c, d$.

(79) If $p \in M$ and $p \in N$ and $a \in M$ and $b \in N$ and $a \neq b$ and $a \in K$ and $b \in K$ and $A \perp M$ and $A \perp N$ and K is a line, then $A \perp K$.

(80) $b, c \perp a, a$ and $a, a \perp b, c$ and $b, c \parallel a, a$ and $a, a \parallel b, c$.

(81) If $a, b \parallel c, d$, then $a, b \parallel d, c$ and $b, a \parallel c, d$ and $b, a \parallel d, c$ and $c, d \parallel a, b$ and $c, d \parallel b, a$ and $d, c \parallel a, b$ and $d, c \parallel b, a$.

(82) Suppose that

(i) $p \neq q$,

(ii) $p, q \parallel a, b$ and $p, q \parallel c, d$ or $p, q \parallel a, b$ and $c, d \parallel p, q$ or $a, b \parallel p, q$ and $c, d \parallel p, q$ or $a, b \parallel p, q$ and $p, q \parallel c, d$.

Then $a, b \parallel c, d$.

(83) If $a, b \perp c, d$, then $a, b \perp d, c$ and $b, a \perp c, d$ and $b, a \perp d, c$ and $c, d \perp a, b$ and $c, d \perp b, a$ and $d, c \perp a, b$ and $d, c \perp b, a$.

(84) Suppose that

(i) $p \neq q$,

(ii) $p, q \parallel a, b$ and $p, q \perp c, d$ or $p, q \parallel c, d$ and $p, q \perp a, b$ or $p, q \parallel a, b$ and $c, d \perp p, q$ or $p, q \parallel c, d$ and $a, b \perp p, q$ or $a, b \parallel p, q$ and $c, d \perp p, q$ or $c, d \parallel p, q$ and $a, b \perp p, q$ or $a, b \parallel p, q$ and $p, q \perp c, d$ or $c, d \parallel p, q$ and $p, q \perp a, b$.

Then $a, b \perp c, d$.

We follow the rules: P_1 is a metric affine plane, K, M, N are subsets of the points of P_1, and x, a, b, c, d, p, q are elements of the points of P_1. The following propositions are true:

(85) Suppose that

(i) $p \neq q$,

(ii) $p, q \perp a, b$ and $p, q \perp c, d$ or $p, q \perp a, b$ and $c, d \perp p, q$ or $a, b \perp p, q$ and $c, d \perp p, q$ or $a, b \perp p, q$ and $p, q \perp c, d$.

Then $a, b \parallel c, d$.

(86) If $a \in M$ and $b \in M$ and $a \neq b$ and M is a line and $c \in N$ and $d \in N$ and $c \neq d$ and N is a line and $a, b \parallel c, d$, then $M \parallel N$.

(87) If $K \perp M$ or $M \perp K$ but $K \perp N$ or $N \perp K$, then $M \parallel N$ and $N \parallel M$.

(88) If $M \perp N$, then there exists p such that $p \in M$ and $p \in N$.

(89) If $a, b \perp c, d$, then there exists p such that $\mathbf{L}(a, b, p)$ and $\mathbf{L}(c, d, p)$.

(90) If $a, b \perp K$, then there exists p such that $\mathbf{L}(a, b, p)$ and $p \in K$.

(91) There exists x such that $a, x \perp p, q$ and $\mathbf{L}(p, q, x)$.

(92) If K is a line, then there exists x such that $a, x \perp K$ and $x \in K$.

References

[1] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[2] Henryk Oryszczyszyn and Krzysztof Prażmowski. Analytical ordered affine spaces. *Formalized Mathematics*, 1(**3**):601–605, 1990.

[3] Henryk Oryszczyszyn and Krzysztof Prażmowski. Ordered affine spaces defined in terms of directed parallelity - part I. *Formalized Mathematics*, 1(**3**):611–615, 1990.

[4] Henryk Oryszczyszyn and Krzysztof Prażmowski. Parallelity and lines in affine spaces. *Formalized Mathematics*, 1(**3**):617–621, 1990.

[5] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[6] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[7] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received August 10, 1990

FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Projective Spaces - part II

Wojciech Leończuk[1]
Warsaw University
Białystok

Krzysztof Prażmowski[2]
Warsaw University
Białystok

Summary. Distinction is made among several types of many dimensional projective spaces - at least three dimensional and exactly three-dimensional projective structures. We prove that analytical projective spaces defined over appropiate real linear spaces may serve as examples of the introduced classes of projective spaces. Corresponding subclasses of Fano projective structures are distinguished. Note that in projective geometry the axiom which assures that the dimension is not greater than three can be formulated as the statement: there exists a plane which intersects every line.

MML Identifier: ANPROJ_3.

The terminology and notation used in this paper have been introduced in the following articles: [1], [4], [2], and [3]. We follow a convention: V will be a real linear space, p, q, r, s, u, v, w, y, u_1, v_1 will be vectors of V, and a, b, c, d, a_1, b_1, c_1 will be real numbers. The following two propositions are true:

(1) Suppose that

(i) for every w there exist a, b, c, d such that $w = ((a \cdot p + b \cdot q) + c \cdot r) + d \cdot s$,

(ii) for all a, b, c, d such that $((a \cdot p + b \cdot q) + c \cdot r) + d \cdot s = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$.

Then for all u, v such that u is a proper vector and v is a proper vector there exist y, w such that u, v and w are lineary dependent and q, r and y are lineary dependent and p, w and y are lineary dependent and y is a proper vector and w is a proper vector.

(2) Suppose for all a, b, a_1, b_1 such that $((a \cdot u + b \cdot v) + a_1 \cdot u_1) + b_1 \cdot v_1 = 0_V$ holds $a = 0$ and $b = 0$ and $a_1 = 0$ and $b_1 = 0$. Then for no y holds y is a proper vector and u, v and y are lineary dependent and u_1, v_1 and y are lineary dependent.

[1]Supported by RPBP.III-24.C6.
[2]Supported by RPBP.III-24.C2.

ISSN 0777–4028

We adopt the following rules: V will be a non-trivial real linear space, u, v, w, y, w_1 will be vectors of V, and p, p_1, q, q_1, q_2, q_3, r, r_1, r_2, r_3 will be elements of the points of the projective space over V. We now state two propositions:

(3) If there exist p, q, r such that p, q and r are not collinear, then for all p, q such that $p \neq q$ there exists r such that p, q and r are not collinear.

(4) Suppose that

(i) there exist y, u, v, w such that for every w_1 there exist a, b, a_1, b_1 such that $w_1 = ((a \cdot y + b \cdot u) + a_1 \cdot v) + b_1 \cdot w$ and for all a, b, a_1, b_1 such that $((a \cdot y + b \cdot u) + a_1 \cdot v) + b_1 \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $a_1 = 0$ and $b_1 = 0$.

Then there exist p, q_1, q_2 such that p, q_1 and q_2 are not collinear and for every r_1, r_2 there exist q_3, r_3 such that r_1, r_2 and r_3 are collinear and q_1, q_2 and q_3 are collinear and p, r_3 and q_3 are collinear.

Next we state the proposition

(5) Suppose that

(i) there exist p, q, r such that p, q and r are not collinear,

(ii) for every p, q there exists r such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iii) there exist p, q_1, q_2 such that p, q_1 and q_2 are not collinear and for every r_1, r_2 there exist q_3, r_3 such that r_1, r_2 and r_3 are collinear and q_1, q_2 and q_3 are collinear and p, r_3 and q_3 are collinear.

Then for every p, p_1, q, q_1, r_2 there exist r, r_1 such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear.

In the sequel u, v, w, y, u_1, v_1, w_1 will be vectors of V. Next we state three propositions:

(6) Suppose that

(i) there exist y, u, v, w such that for every w_1 there exist a, b, c, c_1 such that $w_1 = ((a \cdot y + b \cdot u) + c \cdot v) + c_1 \cdot w$ and for all a, b, a_1, b_1 such that $((a \cdot y + b \cdot u) + a_1 \cdot v) + b_1 \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $a_1 = 0$ and $b_1 = 0$.

Then for every p, p_1, q, q_1, r_2 there exist r, r_1 such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear.

(7) Suppose there exist u, v, u_1, v_1 such that for all a, b, a_1, b_1 such that $((a \cdot u + b \cdot v) + a_1 \cdot u_1) + b_1 \cdot v_1 = 0_V$ holds $a = 0$ and $b = 0$ and $a_1 = 0$ and $b_1 = 0$. Then there exist p, p_1, q, q_1 such that for no r holds p, p_1 and r are collinear and q, q_1 and r are collinear.

(8) Suppose that

(i) there exist u, v, u_1, v_1 such that for every w there exist a, b, a_1, b_1 such that $w = ((a \cdot u + b \cdot v) + a_1 \cdot u_1) + b_1 \cdot v_1$ and for all a, b, a_1, b_1 such that $((a \cdot u + b \cdot v) + a_1 \cdot u_1) + b_1 \cdot v_1 = 0_V$ holds $a = 0$ and $b = 0$ and $a_1 = 0$ and $b_1 = 0$.

Then

(ii) for every p, p_1, q, q_1, r_2 there exist r, r_1 such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear,

(iii) for every p, q there exists r such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) there exist p, q, r such that p, q and r are not collinear,

(v) there exist p, p_1, q, q_1 such that for no r holds p, p_1 and r are collinear and q, q_1 and r are collinear.

A projective space defined in terms of collinearity is called an at least 3 dimensional projective space defined in terms of collinearity if:

(Def.1) there exist elements p, p_1, q, q_1 of the points of it such that for no element r of the points of it holds p, p_1 and r are collinear and q, q_1 and r are collinear.

We now state three propositions:

(9) For every projective space C_1 defined in terms of collinearity holds C_1 is an at least 3 dimensional projective space defined in terms of collinearity if and only if there exist elements p, p_1, q, q_1 of the points of C_1 such that for no element r of the points of C_1 holds p, p_1 and r are collinear and q, q_1 and r are collinear.

(10) If there exist u, v, u_1, v_1 such that for all a, b, a_1, b_1 such that $((a \cdot u + b \cdot v) + a_1 \cdot u_1) + b_1 \cdot v_1 = 0_V$ holds $a = 0$ and $b = 0$ and $a_1 = 0$ and $b_1 = 0$, then the projective space over V is an at least 3 dimensional projective space defined in terms of collinearity.

(11) Let C_1 be a collinearity structure. Then C_1 is an at least 3 dimensional projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(ii) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(iii) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) for all elements p, p_1, p_2, r, r_1 of the points of C_1 such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of C_1 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(v) there exist elements p, p_1, q, q_1 of the points of C_1 such that for no element r of the points of C_1 holds p, p_1 and r are collinear and q, q_1 and r are collinear.

An at least 3 dimensional projective space defined in terms of collinearity is said to be a Fanoian at least 3 dimensional projective space defined in terms of collinearity if:

(Def.2) Let p_1, r_2, q, r_1, q_1, p, r be elements of the points of it . Suppose p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear. Then p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

One can prove the following propositions:

(12) Let C_1 be an at least 3 dimensional projective space defined in terms of collinearity. Then C_1 is a Fanoian at least 3 dimensional projective space defined in terms of collinearity if and only if for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

(13) If there exist u, v, u_1, v_1 such that for all a, b, a_1, b_1 such that $((a \cdot u + b \cdot v) + a_1 \cdot u_1) + b_1 \cdot v_1 = 0_V$ holds $a = 0$ and $b = 0$ and $a_1 = 0$ and $b_1 = 0$, then the projective space over V is a Fanoian at least 3 dimensional projective space defined in terms of collinearity.

(14) Let C_1 be a collinearity structure. Then C_1 is a Fanoian at least 3 dimensional projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(ii) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(iii) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) for all elements p, p_1, p_2, r, r_1 of the points of C_1 such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of C_1 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(v) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear,

(vi) there exist elements p, p_1, q, q_1 of the points of C_1 such that for no element r of the points of C_1 holds p, p_1 and r are collinear and q, q_1 and r are collinear.

(15) For every C_1 being a collinearity structure holds C_1 is a Fanoian at least 3 dimensional projective space defined in terms of collinearity if and

only if C_1 is a Fanoian projective space defined in terms of collinearity and there exist elements p, p_1, q, q_1 of the points of C_1 such that for no element r of the points of C_1 holds p, p_1 and r are collinear and q, q_1 and r are collinear.

An at least 3 dimensional projective space defined in terms of collinearity is called a 3 dimensional projective space defined in terms of collinearity if:

(Def.3) for every elements p, p_1, q, q_1, r_2 of the points of it there exist elements r, r_1 of the points of it such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear.

The following propositions are true:

(16) For every at least 3 dimensional projective space C_1 defined in terms of collinearity holds C_1 is a 3 dimensional projective space defined in terms of collinearity if and only if for every elements p, p_1, q, q_1, r_2 of the points of C_1 there exist elements r, r_1 of the points of C_1 such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear.

(17) Suppose that

(i) there exist u, v, w, u_1 such that for all a, b, c, d such that $((a \cdot u + b \cdot v) + c \cdot w) + d \cdot u_1 = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$ and for every y there exist a, b, c, d such that $y = ((a \cdot u + b \cdot v) + c \cdot w) + d \cdot u_1$.

Then the projective space over V is a 3 dimensional projective space defined in terms of collinearity.

(18) Let C_1 be a collinearity structure. Then C_1 is a 3 dimensional projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(ii) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(iii) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) for all elements p, p_1, p_2, r, r_1 of the points of C_1 such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of C_1 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(v) there exist elements p, p_1, q, q_1 of the points of C_1 such that for no element r of the points of C_1 holds p, p_1 and r are collinear and q, q_1 and r are collinear,

(vi) for every elements p, p_1, q, q_1, r_2 of the points of C_1 there exist elements r, r_1 of the points of C_1 such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear.

A 3 dimensional projective space defined in terms of collinearity is called a Fanoian 3 dimensional projective space defined in terms of collinearity if:

(Def.4) Let p_1, r_2, q, r_1, q_1, p, r be elements of the points of it . Suppose p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear. Then p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

We now state four propositions:

(19) Let C_1 be a 3 dimensional projective space defined in terms of collinearity. Then C_1 is a Fanoian 3 dimensional projective space defined in terms of collinearity if and only if for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

(20) Suppose that

(i) there exist u, v, w, u_1 such that for all a, b, c, d such that $((a \cdot u + b \cdot v) + c \cdot w) + d \cdot u_1 = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$ and for every y there exist a, b, c, d such that $y = ((a \cdot u + b \cdot v) + c \cdot w) + d \cdot u_1$.

Then the projective space over V is a Fanoian 3 dimensional projective space defined in terms of collinearity.

(21) Let C_1 be a collinearity structure. Then C_1 is a Fanoian 3 dimensional projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(ii) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(iii) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) for all elements p, p_1, p_2, r, r_1 of the points of C_1 such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of C_1 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(v) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear,

(vi) there exist elements p, p_1, q, q_1 of the points of C_1 such that for no element r of the points of C_1 holds p, p_1 and r are collinear and q, q_1 and r are collinear,

(vii) for every elements p, p_1, q, q_1, r_2 of the points of C_1 there exist elements r, r_1 of the points of C_1 such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear.

(22) For every C_1 being a collinearity structure holds C_1 is a Fanoian 3 dimensional projective space defined in terms of collinearity if and only if C_1 is a Fanoian at least 3 dimensional projective space defined in terms of collinearity and for every elements p, p_1, q, q_1, r_2 of the points of C_1 there exist elements r, r_1 of the points of C_1 such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear.

References

[1] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[2] Wojciech Leończuk and Krzysztof Prażmowski. A construction of analytical projective space. *Formalized Mathematics*, 1(**4**):761–766, 1990.

[3] Wojciech Leończuk and Krzysztof Prażmowski. Projective spaces - part I. *Formalized Mathematics*, 1(**4**):767–776, 1990.

[4] Wojciech Skaba. The collinearity structure. *Formalized Mathematics*, 1(**4**):657–659, 1990.

Received August 10, 1990

FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Projective Spaces - part III

Wojciech Leończuk[1]
Warsaw University
Białystok

Krzysztof Prażmowski[2]
Warsaw University
Białystok

Summary. In the classes of projective spaces, defined in terms of collinearity, introduced in the article [3], we distinguish the subclasses of Desarguesian projective structures. As examples of these types of objects we consider analytical projective spaces defined over suitable real linear spaces; analytical counterpart of the Desargues Axiom is proved without any assumption on the dimension of the underlying linear space.

MML Identifier: `ANPROJ_4`.

The articles [1], [4], [2], and [3] provide the notation and terminology for this paper. We adopt the following rules: V will denote a real linear space, o, p, p_1, p_2, p_3, q, q_1, q_2, q_3, r, r_1, r_2, r_3 will denote vectors of V, and a, b, c, a_1, b_1, a_2, c_2 will denote real numbers. Let us consider V, p_1, p_2, p_3. We say that p_1, p_2 and p_3 are proper vectors if and only if:

(Def.1) p_1 is a proper vector and p_2 is a proper vector and p_3 is a proper vector.

Next we state the proposition

(1) p_1, p_2 and p_3 are proper vectors if and only if p_1 is a proper vector and p_2 is a proper vector and p_3 is a proper vector.

Let us consider V, p_1, p_2, p_3, r_1, r_2, r_3. We say that p_1, p_2, p_3, r_1, r_2, and r_3 lie on a triangle if and only if:

(Def.2) p_1, p_2 and r_3 are lineary dependent and p_1, p_3 and r_2 are lineary dependent and p_2, p_3 and r_1 are lineary dependent.

Next we state the proposition

(2) p_1, p_2, p_3, r_1, r_2, and r_3 lie on a triangle if and only if p_1, p_2 and r_3 are lineary dependent and p_1, p_3 and r_2 are lineary dependent and p_2, p_3 and r_1 are lineary dependent.

[1]Supported by RPBP.III-24.C6.
[2]Supported by RPBP.III-24.C2.

ISSN 0777–4028

Let us consider V, o, p_1, p_2, p_3, q_1, q_2, q_3. We say that o, p_1, p_2, p_3, q_1, q_2, and q_3 are perspective if and only if:

(Def.3) o, p_1 and q_1 are lineary dependent and o, p_2 and q_2 are lineary dependent and o, p_3 and q_3 are lineary dependent.

The following propositions are true:

(3) o, p_1, p_2, p_3, q_1, q_2, and q_3 are perspective if and only if o, p_1 and q_1 are lineary dependent and o, p_2 and q_2 are lineary dependent and o, p_3 and q_3 are lineary dependent.

(4) Suppose o, p_1 and q_1 are lineary dependent and o and p_1 are not proportional and o and q_1 are not proportional and p_1 and q_1 are not proportional and o, p_1 and q_1 are proper vectors. Then there exist a_1, b_1 such that $b_1 \cdot q_1 = o + a_1 \cdot p_1$ and $a_1 \neq 0$ and $b_1 \neq 0$ and there exist a_2, c_2 such that $q_1 = c_2 \cdot o + a_2 \cdot p_1$ and $c_2 \neq 0$ and $a_2 \neq 0$.

(5) If p, q and r are lineary dependent and p and q are not proportional and p, q and r are proper vectors, then there exist a, b such that $r = a \cdot p + b \cdot q$.

(6) Suppose that

(i) o is a proper vector,
(ii) p_1, p_2 and p_3 are proper vectors,
(iii) q_1, q_2 and q_3 are proper vectors,
(iv) r_1, r_2 and r_3 are proper vectors,
(v) o, p_1, p_2, p_3, q_1, q_2, and q_3 are perspective,
(vi) o and q_1 are not proportional,
(vii) o and q_2 are not proportional,
(viii) o and q_3 are not proportional,
(ix) p_1 and q_1 are not proportional,
(x) p_2 and q_2 are not proportional,
(xi) p_3 and q_3 are not proportional,
(xii) o, p_1 and p_2 are not lineary dependent,
(xiii) o, p_1 and p_3 are not lineary dependent,
(xiv) o, p_2 and p_3 are not lineary dependent,
(xv) p_1, p_2, p_3, r_1, r_2, and r_3 lie on a triangle,
(xvi) q_1, q_2, q_3, r_1, r_2, and r_3 lie on a triangle.

Then r_1, r_2 and r_3 are lineary dependent.

We adopt the following rules: V will be a non-trivial real linear space and o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 will be elements of the points of the projective space over V. The following proposition is true

(7) Suppose that

(i) $o \neq q_1$,
(ii) $p_1 \neq q_1$,
(iii) $o \neq q_2$,
(iv) $p_2 \neq q_2$,
(v) $o \neq q_3$,
(vi) $p_3 \neq q_3$,

(vii) o, p_1 and p_2 are not collinear,
(viii) o, p_1 and p_3 are not collinear,
(ix) o, p_2 and p_3 are not collinear,
(x) p_1, p_2 and r_3 are collinear,
(xi) q_1, q_2 and r_3 are collinear,
(xii) p_2, p_3 and r_1 are collinear,
(xiii) q_2, q_3 and r_1 are collinear,
(xiv) p_1, p_3 and r_2 are collinear,
(xv) q_1, q_3 and r_2 are collinear,
(xvi) o, p_1 and q_1 are collinear,
(xvii) o, p_2 and q_2 are collinear,
(xviii) o, p_3 and q_3 are collinear.

Then r_1, r_2 and r_3 are collinear.

In the sequel u, v, w, y will denote vectors of V. A projective space defined in terms of collinearity is said to be a Desarguesian projective space defined in terms of collinearity if:

(Def.4) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of the points of it . Suppose that

(i) $o \neq q_1$,
(ii) $p_1 \neq q_1$,
(iii) $o \neq q_2$,
(iv) $p_2 \neq q_2$,
(v) $o \neq q_3$,
(vi) $p_3 \neq q_3$,
(vii) o, p_1 and p_2 are not collinear,
(viii) o, p_1 and p_3 are not collinear,
(ix) o, p_2 and p_3 are not collinear,
(x) p_1, p_2 and r_3 are collinear,
(xi) q_1, q_2 and r_3 are collinear,
(xii) p_2, p_3 and r_1 are collinear,
(xiii) q_2, q_3 and r_1 are collinear,
(xiv) p_1, p_3 and r_2 are collinear,
(xv) q_1, q_3 and r_2 are collinear,
(xvi) o, p_1 and q_1 are collinear,
(xvii) o, p_2 and q_2 are collinear,
(xviii) o, p_3 and q_3 are collinear.

Then r_1, r_2 and r_3 are collinear.

We now state three propositions:

(8) Let C_1 be a projective space defined in terms of collinearity. Then C_1 is a Desarguesian projective space defined in terms of collinearity if and only if for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear

and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear.

(9) If there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$, then the projective space over V is a Desarguesian projective space defined in terms of collinearity.

(10) Let C_1 be a collinearity structure. Then C_1 is a Desarguesian projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(ii) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(iii) for all elements p, p_1, p_2, r, r_1 of the points of C_1 such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of C_1 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(iv) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(v) there exist elements p, q, r of the points of C_1 such that p, q and r are not collinear,

(vi) for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear.

A Fanoian projective space defined in terms of collinearity is called a Fano-Desarguesian projective space defined in terms of collinearity if:

(Def.5) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of the points of it . Suppose that

(i) $o \neq q_1$,
(ii) $p_1 \neq q_1$,
(iii) $o \neq q_2$,
(iv) $p_2 \neq q_2$,
(v) $o \neq q_3$,
(vi) $p_3 \neq q_3$,
(vii) o, p_1 and p_2 are not collinear,
(viii) o, p_1 and p_3 are not collinear,
(ix) o, p_2 and p_3 are not collinear,

(x) p_1, p_2 and r_3 are collinear,
(xi) q_1, q_2 and r_3 are collinear,
(xii) p_2, p_3 and r_1 are collinear,
(xiii) q_2, q_3 and r_1 are collinear,
(xiv) p_1, p_3 and r_2 are collinear,
(xv) q_1, q_3 and r_2 are collinear,
(xvi) o, p_1 and q_1 are collinear,
(xvii) o, p_2 and q_2 are collinear,
(xviii) o, p_3 and q_3 are collinear.

Then r_1, r_2 and r_3 are collinear.

One can prove the following propositions:

(11) Let C_1 be a Fanoian projective space defined in terms of collinearity. Then C_1 is a Fano-Desarguesian projective space defined in terms of collinearity if and only if for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear.

(12) If there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$, then the projective space over V is a Fano-Desarguesian projective space defined in terms of collinearity.

(13) Let C_1 be a collinearity structure. Then C_1 is a Fano-Desarguesian projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(ii) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(iii) for all elements p, p_1, p_2, r, r_1 of the points of C_1 such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of C_1 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(iv) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(v) there exist elements p, q, r of the points of C_1 such that p, q and r are not collinear,

(vi) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1

are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear,

(vii) for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear.

(14) Let C_1 be a collinearity structure. Then C_1 is a Fano-Desarguesian projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) C_1 is a Desarguesian projective space defined in terms of collinearity,

(ii) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

A projective plane defined in terms of collinearity is called a Desarguesian projective plane defined in terms of collinearity if:

(Def.6) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of the points of it . Suppose that

(i) $o \neq q_1$,

(ii) $p_1 \neq q_1$,

(iii) $o \neq q_2$,

(iv) $p_2 \neq q_2$,

(v) $o \neq q_3$,

(vi) $p_3 \neq q_3$,

(vii) o, p_1 and p_2 are not collinear,

(viii) o, p_1 and p_3 are not collinear,

(ix) o, p_2 and p_3 are not collinear,

(x) p_1, p_2 and r_3 are collinear,

(xi) q_1, q_2 and r_3 are collinear,

(xii) p_2, p_3 and r_1 are collinear,

(xiii) q_2, q_3 and r_1 are collinear,

(xiv) p_1, p_3 and r_2 are collinear,

(xv) q_1, q_3 and r_2 are collinear,

(xvi) o, p_1 and q_1 are collinear,

(xvii) o, p_2 and q_2 are collinear,

(xviii) o, p_3 and q_3 are collinear.

Then r_1, r_2 and r_3 are collinear.

We now state four propositions:

(15) Let C_1 be a projective plane defined in terms of collinearity. Then C_1 is a Desarguesian projective plane defined in terms of collinearity if and only if for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear.

(16) Suppose that

(i) there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and for every y there exist a, b, c such that $y = (a \cdot u + b \cdot v) + c \cdot w$.

Then the projective space over V is a Desarguesian projective plane defined in terms of collinearity.

(17) Let C_1 be a collinearity structure. Then C_1 is a Desarguesian projective plane defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(ii) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(iii) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) there exist elements p, q, r of the points of C_1 such that p, q and r are not collinear,

(v) for every elements p, p_1, q, q_1 of the points of C_1 there exists an element r of the points of C_1 such that p, p_1 and r are collinear and q, q_1 and r are collinear,

(vi) for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear.

(18) For every C_1 being a collinearity structure holds C_1 is a Desarguesian projective plane defined in terms of collinearity if and only if C_1 is a Desarguesian projective space defined in terms of collinearity and for every elements p, p_1, q, q_1 of the points of C_1 there exists an element r of the points of C_1 such that p, p_1 and r are collinear and q, q_1 and r are collinear.

A Fanoian projective plane defined in terms of collinearity is called a Fano-Desarguesian projective plane defined in terms of collinearity if:

(Def.7) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of the points of it . Suppose that

(i) $o \neq q_1$,
(ii) $p_1 \neq q_1$,
(iii) $o \neq q_2$,
(iv) $p_2 \neq q_2$,
(v) $o \neq q_3$,
(vi) $p_3 \neq q_3$,
(vii) o, p_1 and p_2 are not collinear,
(viii) o, p_1 and p_3 are not collinear,
(ix) o, p_2 and p_3 are not collinear,
(x) p_1, p_2 and r_3 are collinear,
(xi) q_1, q_2 and r_3 are collinear,
(xii) p_2, p_3 and r_1 are collinear,
(xiii) q_2, q_3 and r_1 are collinear,
(xiv) p_1, p_3 and r_2 are collinear,
(xv) q_1, q_3 and r_2 are collinear,
(xvi) o, p_1 and q_1 are collinear,
(xvii) o, p_2 and q_2 are collinear,
(xviii) o, p_3 and q_3 are collinear.

Then r_1, r_2 and r_3 are collinear.

One can prove the following propositions:

(19) Let C_1 be a Fanoian projective plane defined in terms of collinearity. Then C_1 is a Fano-Desarguesian projective plane defined in terms of collinearity if and only if for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear.

(20) Suppose that

(i) there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and for every y there exist a, b, c such that $y = (a \cdot u + b \cdot v) + c \cdot w$.

Then the projective space over V is a Fano-Desarguesian projective plane defined in terms of collinearity.

(21) Let C_1 be a collinearity structure. Then C_1 is a Fano-Desarguesian projective plane defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(ii) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(iii) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) there exist elements p, q, r of the points of C_1 such that p, q and r are not collinear,

(v) for every elements p, p_1, q, q_1 of the points of C_1 there exists an element r of the points of C_1 such that p, p_1 and r are collinear and q, q_1 and r are collinear,

(vi) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear,

(vii) for all elements $o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3$ of the points of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear.

(22) Let C_1 be a collinearity structure. Then C_1 is a Fano-Desarguesian projective plane defined in terms of collinearity if and only if the following conditions are satisfied:

(i) C_1 is a Desarguesian projective plane defined in terms of collinearity,

(ii) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

(23) For every C_1 being a collinearity structure holds
C_1
is a Fano-Desarguesian projective plane defined in terms of collinearity if and only if C_1 is a Fano-Desarguesian projective space defined in terms of collinearity and for every elements p, p_1, q, q_1 of the points of C_1 there exists an element r of the points of C_1 such that p, p_1 and r are collinear and q, q_1 and r are collinear.

References

[1] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[2] Wojciech Leończuk and Krzysztof Prażmowski. A construction of analytical projective space. *Formalized Mathematics*, 1(**4**):761–766, 1990.

[3] Wojciech Leończuk and Krzysztof Prażmowski. Projective spaces - part I. *Formalized Mathematics*, 1(**4**):767–776, 1990.

[4] Wojciech Skaba. The collinearity structure. *Formalized Mathematics*, 1(**4**):657–659, 1990.

Received August 10, 1990

FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Projective Spaces - part IV

Wojciech Leończuk[1]
Warsaw University
Białystok

Krzysztof Prażmowski[2]
Warsaw University
Białystok

Summary. A continuation of [4]. In the classes of projective spaces, defined in terms of collinearity, introduced in the article [3], we distinguish the subclasses of Desarguesian projective structures. As examples of these objects we consider analytical projective spaces defined over suitable real linear spaces.

MML Identifier: ANPROJ_5.

The notation and terminology used here have been introduced in the following papers: [1], [5], [2], [3], and [4]. We adopt the following convention: a, b, c, d denote real numbers, V denotes a non-trivial real linear space, and u, v, w, y, u_1 denote vectors of V. An at least 3 dimensional projective space defined in terms of collinearity is said to be a Desarguesian at least 3 dimensional projective space defined in terms of collinearity if:

(Def.1) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of the points of it . Suppose that

(i) $o \neq q_1$,
(ii) $p_1 \neq q_1$,
(iii) $o \neq q_2$,
(iv) $p_2 \neq q_2$,
(v) $o \neq q_3$,
(vi) $p_3 \neq q_3$,
(vii) o, p_1 and p_2 are not collinear,
(viii) o, p_1 and p_3 are not collinear,
(ix) o, p_2 and p_3 are not collinear,
(x) p_1, p_2 and r_3 are collinear,
(xi) q_1, q_2 and r_3 are collinear,
(xii) p_2, p_3 and r_1 are collinear,

[1]Supported by RPBP.III-24.C6.
[2]Supported by RPBP.III-24.C2.

ISSN 0777–4028

(xiii) q_2, q_3 and r_1 are collinear,
(xiv) p_1, p_3 and r_2 are collinear,
(xv) q_1, q_3 and r_2 are collinear,
(xvi) o, p_1 and q_1 are collinear,
(xvii) o, p_2 and q_2 are collinear,
(xviii) o, p_3 and q_3 are collinear.

Then r_1, r_2 and r_3 are collinear.

The following propositions are true:

(1) Let C_1 be an at least 3 dimensional projective space defined in terms of collinearity. Then C_1 is a Desarguesian at least 3 dimensional projective space defined in terms of collinearity if and only if for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear.

(2) If there exist u, v, w, u_1 such that for all a, b, c, d such that $((a \cdot u + b \cdot v) + c \cdot w) + d \cdot u_1 = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$, then the projective space over V is a Desarguesian at least 3 dimensional projective space defined in terms of collinearity.

(3) Let C_1 be a collinearity structure. Then C_1 is a Desarguesian at least 3 dimensional projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(ii) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(iii) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) for all elements p, p_1, p_2, r, r_1 of the points of C_1 such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of C_1 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(v) there exist elements p, p_1, q, q_1 of the points of C_1 such that for no element r of the points of C_1 holds p, p_1 and r are collinear and q, q_1 and r are collinear,

(vi) for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear and q_1, q_2

and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear.

(4) For every C_1 being a collinearity structure holds C_1 is a Desarguesian at least 3 dimensional projective space defined in terms of collinearity if and only if C_1 is a Desarguesian projective space defined in terms of collinearity and there exist elements p, p_1, q, q_1 of the points of C_1 such that for no element r of the points of C_1 holds p, p_1 and r are collinear and q, q_1 and r are collinear.

A Fanoian at least 3 dimensional projective space defined in terms of collinearity is called a Fano-Desarguesian at least 3 dimensional projective space defined in terms of collinearity if:

(Def.2) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of the points of it . Suppose that

(i) $o \neq q_1$,
(ii) $p_1 \neq q_1$,
(iii) $o \neq q_2$,
(iv) $p_2 \neq q_2$,
(v) $o \neq q_3$,
(vi) $p_3 \neq q_3$,
(vii) o, p_1 and p_2 are not collinear,
(viii) o, p_1 and p_3 are not collinear,
(ix) o, p_2 and p_3 are not collinear,
(x) p_1, p_2 and r_3 are collinear,
(xi) q_1, q_2 and r_3 are collinear,
(xii) p_2, p_3 and r_1 are collinear,
(xiii) q_2, q_3 and r_1 are collinear,
(xiv) p_1, p_3 and r_2 are collinear,
(xv) q_1, q_3 and r_2 are collinear,
(xvi) o, p_1 and q_1 are collinear,
(xvii) o, p_2 and q_2 are collinear,
(xviii) o, p_3 and q_3 are collinear.

Then r_1, r_2 and r_3 are collinear.

We now state several propositions:

(5) Let C_1 be a Fanoian at least 3 dimensional projective space defined in terms of collinearity. Then C_1 is a Fano-Desarguesian at least 3 dimensional projective space defined in terms of collinearity if and only if for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1

and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear.

(6) If there exist u, v, w, u_1 such that for all a, b, c, d such that $((a \cdot u + b \cdot v) + c \cdot w) + d \cdot u_1 = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$, then the projective space over V is a Fano-Desarguesian at least 3 dimensional projective space defined in terms of collinearity.

(7) Let C_1 be a collinearity structure. Then C_1 is a Fano-Desarguesian at least 3 dimensional projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(ii) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(iii) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) for all elements p, p_1, p_2, r, r_1 of the points of C_1 such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of C_1 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(v) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear,

(vi) there exist elements p, p_1, q, q_1 of the points of C_1 such that for no element r of the points of C_1 holds p, p_1 and r are collinear and q, q_1 and r are collinear,

(vii) for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear.

(8) Let C_1 be a collinearity structure. Then C_1 is a Fano-Desarguesian at least 3 dimensional projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) C_1 is a Desarguesian at least 3 dimensional projective space defined in terms of collinearity,

(ii) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

(9) For every C_1 being a collinearity structure holds
C_1
is a Fano-Desarguesian at least 3 dimensional projective space defined in terms of collinearity if and only if C_1 is a Fano-Desarguesian projective space defined in terms of collinearity and there exist elements p, p_1, q, q_1 of the points of C_1 such that for no element r of the points of C_1 holds p, p_1 and r are collinear and q, q_1 and r are collinear.

A 3 dimensional projective space defined in terms of collinearity is called a Desarguesian 3 dimensional projective space defined in terms of collinearity if:

(Def.3) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of the points of it . Suppose that

- (i) $o \neq q_1$,
- (ii) $p_1 \neq q_1$,
- (iii) $o \neq q_2$,
- (iv) $p_2 \neq q_2$,
- (v) $o \neq q_3$,
- (vi) $p_3 \neq q_3$,
- (vii) o, p_1 and p_2 are not collinear,
- (viii) o, p_1 and p_3 are not collinear,
- (ix) o, p_2 and p_3 are not collinear,
- (x) p_1, p_2 and r_3 are collinear,
- (xi) q_1, q_2 and r_3 are collinear,
- (xii) p_2, p_3 and r_1 are collinear,
- (xiii) q_2, q_3 and r_1 are collinear,
- (xiv) p_1, p_3 and r_2 are collinear,
- (xv) q_1, q_3 and r_2 are collinear,
- (xvi) o, p_1 and q_1 are collinear,
- (xvii) o, p_2 and q_2 are collinear,
- (xviii) o, p_3 and q_3 are collinear.

Then r_1, r_2 and r_3 are collinear.

We now state four propositions:

(10) Let C_1 be a 3 dimensional projective space defined in terms of collinearity. Then C_1 is a Desarguesian 3 dimensional projective space defined in terms of collinearity if and only if for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2

and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear.

(11) Suppose that

(i) there exist u, v, w, u_1 such that for all a, b, c, d such that $((a \cdot u + b \cdot v) + c \cdot w) + d \cdot u_1 = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$ and for every y there exist a, b, c, d such that $y = ((a \cdot u + b \cdot v) + c \cdot w) + d \cdot u_1$. Then the projective space over V is a Desarguesian 3 dimensional projective space defined in terms of collinearity.

(12) Let C_1 be a collinearity structure. Then C_1 is a Desarguesian 3 dimensional projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(ii) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(iii) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) for all elements p, p_1, p_2, r, r_1 of the points of C_1 such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of C_1 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(v) there exist elements p, p_1, q, q_1 of the points of C_1 such that for no element r of the points of C_1 holds p, p_1 and r are collinear and q, q_1 and r are collinear,

(vi) for every elements p, p_1, q, q_1, r_2 of the points of C_1 there exist elements r, r_1 of the points of C_1 such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear,

(vii) for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear.

(13) For every C_1 being a collinearity structure holds C_1 is a Desarguesian 3 dimensional projective space defined in terms of collinearity if and only if C_1 is a Desarguesian at least 3 dimensional projective space defined in terms of collinearity and for every elements p, p_1, q, q_1, r_2 of the points of C_1 there exist elements r, r_1 of the points of C_1 such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear.

A Fanoian 3 dimensional projective space defined in terms of collinearity is called a Fano-Desarguesian 3 dimensional projective space defined in terms of collinearity if:

(Def.4) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of the points of it . Suppose that

(i) $o \neq q_1$,
(ii) $p_1 \neq q_1$,
(iii) $o \neq q_2$,
(iv) $p_2 \neq q_2$,
(v) $o \neq q_3$,
(vi) $p_3 \neq q_3$,
(vii) o, p_1 and p_2 are not collinear,
(viii) o, p_1 and p_3 are not collinear,
(ix) o, p_2 and p_3 are not collinear,
(x) p_1, p_2 and r_3 are collinear,
(xi) q_1, q_2 and r_3 are collinear,
(xii) p_2, p_3 and r_1 are collinear,
(xiii) q_2, q_3 and r_1 are collinear,
(xiv) p_1, p_3 and r_2 are collinear,
(xv) q_1, q_3 and r_2 are collinear,
(xvi) o, p_1 and q_1 are collinear,
(xvii) o, p_2 and q_2 are collinear,
(xviii) o, p_3 and q_3 are collinear.

Then r_1, r_2 and r_3 are collinear.

We now state several propositions:

(14) Let C_1 be a Fanoian 3 dimensional projective space defined in terms of collinearity. Then C_1 is a Fano-Desarguesian 3 dimensional projective space defined in terms of collinearity if and only if for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear.

(15) Suppose that

(i) there exist u, v, w, u_1 such that for all a, b, c, d such that $((a \cdot u + b \cdot v) + c \cdot w) + d \cdot u_1 = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$ and for every y there exist a, b, c, d such that $y = ((a \cdot u + b \cdot v) + c \cdot w) + d \cdot u_1$.

Then the projective space over V is a Fano-Desarguesian 3 dimensional projective space defined in terms of collinearity.

(16) Let C_1 be a collinearity structure. Then C_1 is a Fano-Desarguesian 3 dimensional projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(ii) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(iii) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) for all elements p, p_1, p_2, r, r_1 of the points of C_1 such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of C_1 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(v) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear,

(vi) there exist elements p, p_1, q, q_1 of the points of C_1 such that for no element r of the points of C_1 holds p, p_1 and r are collinear and q, q_1 and r are collinear,

(vii) for every elements p, p_1, q, q_1, r_2 of the points of C_1 there exist elements r, r_1 of the points of C_1 such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear,

(viii) for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear.

(17) Let C_1 be a collinearity structure. Then C_1 is a Fano-Desarguesian 3 dimensional projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) C_1 is a Desarguesian 3 dimensional projective space defined in terms of collinearity,

(ii) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

(18) For every C_1 being a collinearity structure holds
C_1

is a Fano-Desarguesian 3 dimensional projective space defined in terms of collinearity if and only if C_1 is a Fano-Desarguesian at least 3 dimensional projective space defined in terms of collinearity and for every elements p, p_1, q, q_1, r_2 of the points of C_1 there exist elements r, r_1 of the points of C_1 such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear.

References

[1] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[2] Wojciech Leończuk and Krzysztof Prażmowski. A construction of analytical projective space. *Formalized Mathematics*, 1(**4**):761–766, 1990.

[3] Wojciech Leończuk and Krzysztof Prażmowski. Projective spaces - part II. *Formalized Mathematics*, 1(**5**):901–907, 1990.

[4] Wojciech Leończuk and Krzysztof Prażmowski. Projective spaces - part III. *Formalized Mathematics*, 1(**5**):909–918, 1990.

[5] Wojciech Skaba. The collinearity structure. *Formalized Mathematics*, 1(**4**):657–659, 1990.

Received August 10, 1990

FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Projective Spaces - part V

Wojciech Leończuk[1]
Warsaw University
Białystok

Krzysztof Prażmowski[2]
Warsaw University
Białystok

Summary. In the classes of projective spaces, defined in terms of collinearity, introduced in the article [3], we distinguish the subclasses of Pappian projective structures. As examples of these objects we consider analytical projective spaces defined over suitable real linear spaces; analytical counterpart of the Pappus Axiom is proved without any assumption on the dimension of the underlying linear space.

MML Identifier: `ANPROJ_6`.

The terminology and notation used in this paper are introduced in the following papers: [1], [5], [2], [3], and [4]. We follow a convention: V will denote a real linear space, o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 will denote vectors of V, and a, b, c will denote real numbers. Let us consider V, o, p_1, p_2, p_3, q_1, q_2, q_3. We say that o, p_1, p_2, p_3, q_1, q_2, and q_3 lie on an angle if and only if:

(Def.1) o, p_1 and q_1 are not lineary dependent and o, p_1 and p_2 are lineary dependent and o, p_1 and p_3 are lineary dependent and o, q_1 and q_2 are lineary dependent and o, q_1 and q_3 are lineary dependent.

One can prove the following proposition

(1) o, p_1, p_2, p_3, q_1, q_2, and q_3 lie on an angle if and only if o, p_1 and q_1 are not lineary dependent and o, p_1 and p_2 are lineary dependent and o, p_1 and p_3 are lineary dependent and o, q_1 and q_2 are lineary dependent and o, q_1 and q_3 are lineary dependent.

Let us consider V, o, p_1, p_2, p_3, q_1, q_2, q_3. We say that o, p_1, p_2, p_3, q_1, q_2, q_3 are half-mutually not proportional if and only if:

(Def.2) o and p_2 are not proportional and o and p_3 are not proportional and o and q_2 are not proportional and o and q_3 are not proportional and p_1 and p_2 are not proportional and p_1 and p_3 are not proportional and q_1 and

[1]Supported by RPBP.III-24.C6.
[2]Supported by RPBP.III-24.C2.

ISSN 0777–4028

q_2 are not proportional and q_1 and q_3 are not proportional and p_2 and p_3 are not proportional and q_2 and q_3 are not proportional.

Next we state two propositions:

(2) o, p_1, p_2, p_3, q_1, q_2, q_3 are half-mutually not proportional if and only if the following conditions are satisfied:

(i) o and p_2 are not proportional,
(ii) o and p_3 are not proportional,
(iii) o and q_2 are not proportional,
(iv) o and q_3 are not proportional,
(v) p_1 and p_2 are not proportional,
(vi) p_1 and p_3 are not proportional,
(vii) q_1 and q_2 are not proportional,
(viii) q_1 and q_3 are not proportional,
(ix) p_2 and p_3 are not proportional,
(x) q_2 and q_3 are not proportional.

(3) Suppose that

(i) o is a proper vector,
(ii) p_1, p_2 and p_3 are proper vectors,
(iii) q_1, q_2 and q_3 are proper vectors,
(iv) r_1, r_2 and r_3 are proper vectors,
(v) o, p_1, p_2, p_3, q_1, q_2, and q_3 lie on an angle,
(vi) o, p_1, p_2, p_3, q_1, q_2, q_3 are half-mutually not proportional,
(vii) p_1, q_2 and r_3 are lineary dependent,
(viii) q_1, p_2 and r_3 are lineary dependent,
(ix) p_1, q_3 and r_2 are lineary dependent,
(x) p_3, q_1 and r_2 are lineary dependent,
(xi) p_2, q_3 and r_1 are lineary dependent,
(xii) p_3, q_2 and r_1 are lineary dependent.

Then r_1, r_2 and r_3 are lineary dependent.

We adopt the following convention: V will denote a non-trivial real linear space and o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 will denote elements of the points of the projective space over V. The following proposition is true

(4) Suppose that

(i) $o \neq p_2$,
(ii) $o \neq p_3$,
(iii) $p_2 \neq p_3$,
(iv) $p_1 \neq p_2$,
(v) $p_1 \neq p_3$,
(vi) $o \neq q_2$,
(vii) $o \neq q_3$,
(viii) $q_2 \neq q_3$,
(ix) $q_1 \neq q_2$,
(x) $q_1 \neq q_3$,
(xi) o, p_1 and q_1 are not collinear,

(xii) o, p_1 and p_2 are collinear,
(xiii) o, p_1 and p_3 are collinear,
(xiv) o, q_1 and q_2 are collinear,
(xv) o, q_1 and q_3 are collinear,
(xvi) p_1, q_2 and r_3 are collinear,
(xvii) q_1, p_2 and r_3 are collinear,
(xviii) p_1, q_3 and r_2 are collinear,
(xix) p_3, q_1 and r_2 are collinear,
(xx) p_2, q_3 and r_1 are collinear,
(xxi) p_3, q_2 and r_1 are collinear.

Then r_1, r_2 and r_3 are collinear.

In the sequel u, v, w, y are vectors of V. A projective space defined in terms of collinearity is said to be a Pappian projective space defined in terms of collinearity if:

(Def.3) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of the points of it . Suppose that

(i) $o \neq p_2$,
(ii) $o \neq p_3$,
(iii) $p_2 \neq p_3$,
(iv) $p_1 \neq p_2$,
(v) $p_1 \neq p_3$,
(vi) $o \neq q_2$,
(vii) $o \neq q_3$,
(viii) $q_2 \neq q_3$,
(ix) $q_1 \neq q_2$,
(x) $q_1 \neq q_3$,
(xi) o, p_1 and q_1 are not collinear,
(xii) o, p_1 and p_2 are collinear,
(xiii) o, p_1 and p_3 are collinear,
(xiv) o, q_1 and q_2 are collinear,
(xv) o, q_1 and q_3 are collinear,
(xvi) p_1, q_2 and r_3 are collinear,
(xvii) q_1, p_2 and r_3 are collinear,
(xviii) p_1, q_3 and r_2 are collinear,
(xix) p_3, q_1 and r_2 are collinear,
(xx) p_2, q_3 and r_1 are collinear,
(xxi) p_3, q_2 and r_1 are collinear.

Then r_1, r_2 and r_3 are collinear.

We now state three propositions:

(5) Let C_1 be a projective space defined in terms of collinearity. Then C_1 is a Pappian projective space defined in terms of collinearity if and only if for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not

collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear.

(6) If there exist u, v, w such that for all a, b, c such that $(a\cdot u+b\cdot v)+c\cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$, then the projective space over V is a Pappian projective space defined in terms of collinearity.

(7) Let C_1 be a collinearity structure. Then C_1 is a Pappian projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(ii) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(iii) for all elements p, p_1, p_2, r, r_1 of the points of C_1 such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of C_1 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(iv) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(v) there exist elements p, q, r of the points of C_1 such that p, q and r are not collinear,

(vi) for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear.

A Fanoian projective space defined in terms of collinearity is said to be a Fano-Pappian projective space defined in terms of collinearity if:

(Def.4) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of the points of it . Suppose that

(i) $o \neq p_2$,

(ii) $o \neq p_3$,

(iii) $p_2 \neq p_3$,

(iv) $p_1 \neq p_2$,

(v) $p_1 \neq p_3$,

(vi) $o \neq q_2$,

(vii) $o \neq q_3$,

(viii) $q_2 \neq q_3$,

(ix) $q_1 \neq q_2$,
(x) $q_1 \neq q_3$,
(xi) o, p_1 and q_1 are not collinear,
(xii) o, p_1 and p_2 are collinear,
(xiii) o, p_1 and p_3 are collinear,
(xiv) o, q_1 and q_2 are collinear,
(xv) o, q_1 and q_3 are collinear,
(xvi) p_1, q_2 and r_3 are collinear,
(xvii) q_1, p_2 and r_3 are collinear,
(xviii) p_1, q_3 and r_2 are collinear,
(xix) p_3, q_1 and r_2 are collinear,
(xx) p_2, q_3 and r_1 are collinear,
(xxi) p_3, q_2 and r_1 are collinear.

Then r_1, r_2 and r_3 are collinear.

We now state four propositions:

(8) Let C_1 be a Fanoian projective space defined in terms of collinearity. Then C_1 is a Fano-Pappian projective space defined in terms of collinearity if and only if for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear.

(9) If there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$, then the projective space over V is a Fano-Pappian projective space defined in terms of collinearity.

(10) Let C_1 be a collinearity structure. Then C_1 is a Fano-Pappian projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(ii) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(iii) for all elements p, p_1, p_2, r, r_1 of the points of C_1 such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of C_1 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(iv) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(v) there exist elements p, q, r of the points of C_1 such that p, q and r are not collinear,

(vi) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear,

(vii) for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear.

(11) Let C_1 be a collinearity structure. Then C_1 is a Fano-Pappian projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) C_1 is a Pappian projective space defined in terms of collinearity,

(ii) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

A projective plane defined in terms of collinearity is called a Pappian projective plane defined in terms of collinearity if:

(Def.5) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of the points of it . Suppose that

(i) $o \neq p_2$,

(ii) $o \neq p_3$,

(iii) $p_2 \neq p_3$,

(iv) $p_1 \neq p_2$,

(v) $p_1 \neq p_3$,

(vi) $o \neq q_2$,

(vii) $o \neq q_3$,

(viii) $q_2 \neq q_3$,

(ix) $q_1 \neq q_2$,

(x) $q_1 \neq q_3$,

(xi) o, p_1 and q_1 are not collinear,

(xii) o, p_1 and p_2 are collinear,

(xiii) o, p_1 and p_3 are collinear,

(xiv) o, q_1 and q_2 are collinear,

(xv) o, q_1 and q_3 are collinear,

(xvi) p_1, q_2 and r_3 are collinear,

(xvii) q_1, p_2 and r_3 are collinear,
(xviii) p_1, q_3 and r_2 are collinear,
(xix) p_3, q_1 and r_2 are collinear,
(xx) p_2, q_3 and r_1 are collinear,
(xxi) p_3, q_2 and r_1 are collinear.
Then r_1, r_2 and r_3 are collinear.

We now state four propositions:

(12) Let C_1 be a projective plane defined in terms of collinearity. Then C_1 is a Pappian projective plane defined in terms of collinearity if and only if for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear.

(13) Suppose that

(i) there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and for every y there exist a, b, c such that $y = (a \cdot u + b \cdot v) + c \cdot w$.

Then the projective space over V is a Pappian projective plane defined in terms of collinearity.

(14) Let C_1 be a collinearity structure. Then C_1 is a Pappian projective plane defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(ii) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(iii) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) there exist elements p, q, r of the points of C_1 such that p, q and r are not collinear,

(v) for every elements p, p_1, q, q_1 of the points of C_1 there exists an element r of the points of C_1 such that p, p_1 and r are collinear and q, q_1 and r are collinear,

(vi) for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2

are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear.

(15) For every C_1 being a collinearity structure holds C_1 is a Pappian projective plane defined in terms of collinearity if and only if C_1 is a Pappian projective space defined in terms of collinearity and for every elements p, p_1, q, q_1 of the points of C_1 there exists an element r of the points of C_1 such that p, p_1 and r are collinear and q, q_1 and r are collinear.

A Fanoian projective plane defined in terms of collinearity is called a Fano-Pappian projective plane defined in terms of collinearity if:

(Def.6) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of the points of it . Suppose that

(i) $o \neq p_2$,
(ii) $o \neq p_3$,
(iii) $p_2 \neq p_3$,
(iv) $p_1 \neq p_2$,
(v) $p_1 \neq p_3$,
(vi) $o \neq q_2$,
(vii) $o \neq q_3$,
(viii) $q_2 \neq q_3$,
(ix) $q_1 \neq q_2$,
(x) $q_1 \neq q_3$,
(xi) o, p_1 and q_1 are not collinear,
(xii) o, p_1 and p_2 are collinear,
(xiii) o, p_1 and p_3 are collinear,
(xiv) o, q_1 and q_2 are collinear,
(xv) o, q_1 and q_3 are collinear,
(xvi) p_1, q_2 and r_3 are collinear,
(xvii) q_1, p_2 and r_3 are collinear,
(xviii) p_1, q_3 and r_2 are collinear,
(xix) p_3, q_1 and r_2 are collinear,
(xx) p_2, q_3 and r_1 are collinear,
(xxi) p_3, q_2 and r_1 are collinear.

Then r_1, r_2 and r_3 are collinear.

We now state several propositions:

(16) Let C_1 be a Fanoian projective plane defined in terms of collinearity. Then C_1 is a Fano-Pappian projective plane defined in terms of collinearity if and only if for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear.

(17) Suppose that

(i) there exist u, v, w such that for all a, b, c such that $(a \cdot u+b \cdot v)+c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and for every y there exist a, b, c such that $y = (a \cdot u + b \cdot v) + c \cdot w$.

Then the projective space over V is a Fano-Pappian projective plane defined in terms of collinearity.

(18) Let C_1 be a collinearity structure. Then C_1 is a Fano-Pappian projective plane defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(ii) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(iii) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) there exist elements p, q, r of the points of C_1 such that p, q and r are not collinear,

(v) for every elements p, p_1, q, q_1 of the points of C_1 there exists an element r of the points of C_1 such that p, p_1 and r are collinear and q, q_1 and r are collinear,

(vi) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear,

(vii) for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear.

(19) Let C_1 be a collinearity structure. Then C_1 is a Fano-Pappian projective plane defined in terms of collinearity if and only if the following conditions are satisfied:

(i) C_1 is a Pappian projective plane defined in terms of collinearity,

(ii) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or

r_2, r_1 and q_1 are collinear.

(20) For every C_1 being a collinearity structure holds C_1 is a Fano-Pappian projective plane defined in terms of collinearity if and only if C_1 is a Fano-Pappian projective space defined in terms of collinearity and for every elements p, p_1, q, q_1 of the points of C_1 there exists an element r of the points of C_1 such that p, p_1 and r are collinear and q, q_1 and r are collinear.

References

[1] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[2] Wojciech Leończuk and Krzysztof Prażmowski. A construction of analytical projective space. *Formalized Mathematics*, 1(**4**):761–766, 1990.

[3] Wojciech Leończuk and Krzysztof Prażmowski. Projective spaces - part I. *Formalized Mathematics*, 1(**4**):767–776, 1990.

[4] Wojciech Leończuk and Krzysztof Prażmowski. Projective spaces - part III. *Formalized Mathematics*, 1(**5**):909–918, 1990.

[5] Wojciech Skaba. The collinearity structure. *Formalized Mathematics*, 1(**4**):657–659, 1990.

Received August 10, 1990

FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Projective Spaces - part VI

Wojciech Leończuk[1]
Warsaw University
Białystok

Krzysztof Prażmowski[2]
Warsaw University
Białystok

Summary. The article is a continuation of [4]. In the classes of projective spaces, defined in terms of collinearity, introduced in the article [3], we distinguish the subclasses of Pappian projective structures. As examples of these types of objects we consider analytical projective spaces defined over suitable real linear spaces.

MML Identifier: ANPROJ_7.

The terminology and notation used in this paper have been introduced in the following articles: [1], [5], [2], [3], and [4]. We adopt the following rules: a, b, c, d will be real numbers, V will be a non-trivial real linear space, and u, v, w, y, u_1 will be vectors of V. An at least 3 dimensional projective space defined in terms of collinearity is said to be a Pappian at least 3 dimensional projective space defined in terms of collinearity if:

(Def.1) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of the points of it . Suppose that

(i) $o \neq p_2$,
(ii) $o \neq p_3$,
(iii) $p_2 \neq p_3$,
(iv) $p_1 \neq p_2$,
(v) $p_1 \neq p_3$,
(vi) $o \neq q_2$,
(vii) $o \neq q_3$,
(viii) $q_2 \neq q_3$,
(ix) $q_1 \neq q_2$,
(x) $q_1 \neq q_3$,
(xi) o, p_1 and q_1 are not collinear,
(xii) o, p_1 and p_2 are collinear,

[1]Supported by RPBP.III-24.C6.
[2]Supported by RPBP.III-24.C2.

ISSN 0777–4028

(xiii) o, p_1 and p_3 are collinear,
(xiv) o, q_1 and q_2 are collinear,
(xv) o, q_1 and q_3 are collinear,
(xvi) p_1, q_2 and r_3 are collinear,
(xvii) q_1, p_2 and r_3 are collinear,
(xviii) p_1, q_3 and r_2 are collinear,
(xix) p_3, q_1 and r_2 are collinear,
(xx) p_2, q_3 and r_1 are collinear,
(xxi) p_3, q_2 and r_1 are collinear.

Then r_1, r_2 and r_3 are collinear.

We now state four propositions:

(1) Let C_1 be an at least 3 dimensional projective space defined in terms of collinearity. Then C_1 is a Pappian at least 3 dimensional projective space defined in terms of collinearity if and only if for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear.

(2) If there exist u, v, w, u_1 such that for all a, b, c, d such that $((a \cdot u + b \cdot v) + c \cdot w) + d \cdot u_1 = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$, then the projective space over V is a Pappian at least 3 dimensional projective space defined in terms of collinearity.

(3) Let C_1 be a collinearity structure. Then C_1 is a Pappian at least 3 dimensional projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(ii) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(iii) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) for all elements p, p_1, p_2, r, r_1 of the points of C_1 such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of C_1 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(v) there exist elements p, p_1, q, q_1 of the points of C_1 such that for no element r of the points of C_1 holds p, p_1 and r are collinear and q, q_1 and r are collinear,

(vi) for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear.

(4) For every C_1 being a collinearity structure holds C_1 is a Pappian at least 3 dimensional projective space defined in terms of collinearity if and only if C_1 is a Pappian projective space defined in terms of collinearity and there exist elements p, p_1, q, q_1 of the points of C_1 such that for no element r of the points of C_1 holds p, p_1 and r are collinear and q, q_1 and r are collinear.

A Fanoian at least 3 dimensional projective space defined in terms of collinearity is called a Fano-Pappian at least 3 dimensional projective space defined in terms of collinearity if:

(Def.2) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of the points of it . Suppose that

(i) $o \neq p_2$,
(ii) $o \neq p_3$,
(iii) $p_2 \neq p_3$,
(iv) $p_1 \neq p_2$,
(v) $p_1 \neq p_3$,
(vi) $o \neq q_2$,
(vii) $o \neq q_3$,
(viii) $q_2 \neq q_3$,
(ix) $q_1 \neq q_2$,
(x) $q_1 \neq q_3$,
(xi) o, p_1 and q_1 are not collinear,
(xii) o, p_1 and p_2 are collinear,
(xiii) o, p_1 and p_3 are collinear,
(xiv) o, q_1 and q_2 are collinear,
(xv) o, q_1 and q_3 are collinear,
(xvi) p_1, q_2 and r_3 are collinear,
(xvii) q_1, p_2 and r_3 are collinear,
(xviii) p_1, q_3 and r_2 are collinear,
(xix) p_3, q_1 and r_2 are collinear,
(xx) p_2, q_3 and r_1 are collinear,
(xxi) p_3, q_2 and r_1 are collinear.

Then r_1, r_2 and r_3 are collinear.

One can prove the following propositions:

(5) Let C_1 be a Fanoian at least 3 dimensional projective space defined in terms of collinearity. Then C_1 is a Fano-Pappian at least 3 dimensional

projective space defined in terms of collinearity if and only if for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear.

(6) If there exist u, v, w, u_1 such that for all a, b, c, d such that $((a \cdot u + b \cdot v) + c \cdot w) + d \cdot u_1 = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$, then the projective space over V is a Fano-Pappian at least 3 dimensional projective space defined in terms of collinearity.

(7) Let C_1 be a collinearity structure. Then C_1 is a Fano-Pappian at least 3 dimensional projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(ii) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(iii) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) for all elements p, p_1, p_2, r, r_1 of the points of C_1 such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of C_1 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(v) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear,

(vi) there exist elements p, p_1, q, q_1 of the points of C_1 such that for no element r of the points of C_1 holds p, p_1 and r are collinear and q, q_1 and r are collinear,

(vii) for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear.

(8) Let C_1 be a collinearity structure. Then C_1 is a Fano-Pappian at least 3 dimensional projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) C_1 is a Pappian at least 3 dimensional projective space defined in terms of collinearity,

(ii) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

(9) For every C_1 being a collinearity structure holds C_1 is a Fano-Pappian at least 3 dimensional projective space defined in terms of collinearity if and only if C_1 is a Fano-Pappian projective space defined in terms of collinearity and there exist elements p, p_1, q, q_1 of the points of C_1 such that for no element r of the points of C_1 holds p, p_1 and r are collinear and q, q_1 and r are collinear.

A 3 dimensional projective space defined in terms of collinearity is called a Pappian 3 dimensional projective space defined in terms of collinearity if:

(Def.3) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of the points of it . Suppose that

(i) $o \neq p_2$,
(ii) $o \neq p_3$,
(iii) $p_2 \neq p_3$,
(iv) $p_1 \neq p_2$,
(v) $p_1 \neq p_3$,
(vi) $o \neq q_2$,
(vii) $o \neq q_3$,
(viii) $q_2 \neq q_3$,
(ix) $q_1 \neq q_2$,
(x) $q_1 \neq q_3$,
(xi) o, p_1 and q_1 are not collinear,
(xii) o, p_1 and p_2 are collinear,
(xiii) o, p_1 and p_3 are collinear,
(xiv) o, q_1 and q_2 are collinear,
(xv) o, q_1 and q_3 are collinear,
(xvi) p_1, q_2 and r_3 are collinear,
(xvii) q_1, p_2 and r_3 are collinear,
(xviii) p_1, q_3 and r_2 are collinear,
(xix) p_3, q_1 and r_2 are collinear,
(xx) p_2, q_3 and r_1 are collinear,
(xxi) p_3, q_2 and r_1 are collinear.

Then r_1, r_2 and r_3 are collinear.

The following four propositions are true:

(10) Let C_1 be a 3 dimensional projective space defined in terms of collinearity. Then C_1 is a Pappian 3 dimensional projective space defined in terms of collinearity if and only if for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear.

(11) Suppose that

(i) there exist u, v, w, u_1 such that for all a, b, c, d such that $((a \cdot u + b \cdot v) + c \cdot w) + d \cdot u_1 = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$ and for every y there exist a, b, c, d such that $y = ((a \cdot u + b \cdot v) + c \cdot w) + d \cdot u_1$.

Then the projective space over V is a Pappian 3 dimensional projective space defined in terms of collinearity.

(12) Let C_1 be a collinearity structure. Then C_1 is a Pappian 3 dimensional projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(ii) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(iii) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) for all elements p, p_1, p_2, r, r_1 of the points of C_1 such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of C_1 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(v) there exist elements p, p_1, q, q_1 of the points of C_1 such that for no element r of the points of C_1 holds p, p_1 and r are collinear and q, q_1 and r are collinear,

(vi) for every elements p, p_1, q, q_1, r_2 of the points of C_1 there exist elements r, r_1 of the points of C_1 such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear,

(vii) for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear.

(13) For every C_1 being a collinearity structure holds C_1 is a Pappian 3 dimensional projective space defined in terms of collinearity if and only if C_1 is a Pappian at least 3 dimensional projective space defined in terms of collinearity and for every elements p, p_1, q, q_1, r_2 of the points of C_1 there exist elements r, r_1 of the points of C_1 such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear.

A Fanoian 3 dimensional projective space defined in terms of collinearity is called a Fano-Pappian 3 dimensional projective space defined in terms of collinearity if:

(Def.4) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of the points of it . Suppose that

- (i) $o \neq p_2$,
- (ii) $o \neq p_3$,
- (iii) $p_2 \neq p_3$,
- (iv) $p_1 \neq p_2$,
- (v) $p_1 \neq p_3$,
- (vi) $o \neq q_2$,
- (vii) $o \neq q_3$,
- (viii) $q_2 \neq q_3$,
- (ix) $q_1 \neq q_2$,
- (x) $q_1 \neq q_3$,
- (xi) o, p_1 and q_1 are not collinear,
- (xii) o, p_1 and p_2 are collinear,
- (xiii) o, p_1 and p_3 are collinear,
- (xiv) o, q_1 and q_2 are collinear,
- (xv) o, q_1 and q_3 are collinear,
- (xvi) p_1, q_2 and r_3 are collinear,
- (xvii) q_1, p_2 and r_3 are collinear,
- (xviii) p_1, q_3 and r_2 are collinear,
- (xix) p_3, q_1 and r_2 are collinear,
- (xx) p_2, q_3 and r_1 are collinear,
- (xxi) p_3, q_2 and r_1 are collinear.

Then r_1, r_2 and r_3 are collinear.

The following propositions are true:

(14) Let C_1 be a Fanoian 3 dimensional projective space defined in terms of collinearity. Then C_1 is a Fano-Pappian 3 dimensional projective space defined in terms of collinearity if and only if for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are

collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear.

(15) Suppose that

(i) there exist u, v, w, u_1 such that for all a, b, c, d such that $((a \cdot u + b \cdot v) + c \cdot w) + d \cdot u_1 = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$ and for every y there exist a, b, c, d such that $y = ((a \cdot u + b \cdot v) + c \cdot w) + d \cdot u_1$.

Then the projective space over V is a Fano-Pappian 3 dimensional projective space defined in terms of collinearity.

(16) Let C_1 be a collinearity structure. Then C_1 is a Fano-Pappian 3 dimensional projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(ii) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(iii) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) for all elements p, p_1, p_2, r, r_1 of the points of C_1 such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of C_1 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(v) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear,

(vi) there exist elements p, p_1, q, q_1 of the points of C_1 such that for no element r of the points of C_1 holds p, p_1 and r are collinear and q, q_1 and r are collinear,

(vii) for every elements p, p_1, q, q_1, r_2 of the points of C_1 there exist elements r, r_1 of the points of C_1 such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear,

(viii) for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear holds r_1, r_2 and r_3 are collinear.

(17) Let C_1 be a collinearity structure. Then C_1 is a Fano-Pappian 3 dimensional projective space defined in terms of collinearity if and only if the

following conditions are satisfied:

(i) C_1 is a Pappian 3 dimensional projective space defined in terms of collinearity,

(ii) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

(18) For every C_1 being a collinearity structure holds C_1 is a Fano-Pappian 3 dimensional projective space defined in terms of collinearity if and only if C_1 is a Fano-Pappian at least 3 dimensional projective space defined in terms of collinearity and for every elements p, p_1, q, q_1, r_2 of the points of C_1 there exist elements r, r_1 of the points of C_1 such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear.

References

[1] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[2] Wojciech Leończuk and Krzysztof Prażmowski. A construction of analytical projective space. *Formalized Mathematics*, 1(**4**):761–766, 1990.

[3] Wojciech Leończuk and Krzysztof Prażmowski. Projective spaces - part II. *Formalized Mathematics*, 1(**5**):901–907, 1990.

[4] Wojciech Leończuk and Krzysztof Prażmowski. Projective spaces - part V. *Formalized Mathematics*, 1(**5**):929–938, 1990.

[5] Wojciech Skaba. The collinearity structure. *Formalized Mathematics*, 1(**4**):657–659, 1990.

Received August 10, 1990

FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Some Elementary Notions of the Theory of Petri Nets

Waldemar Korczyński[1]
Warsaw University
Białystok

Summary. Some fundamental notions of the theory of Petri nets are described in Mizar formalism. A Petri net is defined as a triple of the form ⟨places, transitions, flow⟩ with places and transitions being disjoint sets and flow being a relation included in places × transitions.

MML Identifier: `NET_1`.

The notation and terminology used here have been introduced in the following articles: [1], and [2]. In the sequel x, y will be arbitrary. We consider nets which are systems

⟨places, transitions, a flow relation⟩,

where the places constitute a set, the transitions constitute a set, and the flow relation is a binary relation. In the sequel N is a net. Let N be a net. We say that N is a Petri net if and only if:

(Def.1) (the places of N)$\cap$ (the transitions of N) $= \emptyset$ and the flow relation of $N \subseteq$ [: the places of N, the transitions of N :] $\cup$ [: the transitions of N, the places of N :].

Let N be a net. The functor Elements(N) yielding a set is defined as follows:

(Def.2) Elements(N) = (the places of N)$\cup$ (the transitions of N).

We now state several propositions:

(1) For every N and for every x such that Elements(N) $\neq \emptyset$ holds x is an element of Elements(N) if and only if $x \in$ Elements(N).

(2) For every N and for every x such that the places of $N \neq \emptyset$ holds x is an element of the places of N if and only if $x \in$ the places of N.

[1]Supported by RPBP.III-24.C1.

ISSN 0777–4028

(3) For every N and for every x such that the transitions of $N \neq \emptyset$ holds x is an element of the transitions of N if and only if $x \in$ the transitions of N.

(4) For every N holds the places of $N \subseteq$ Elements(N).

(5) For every N holds the transitions of $N \subseteq$ Elements(N).

Let N be a net. A set is said to be an element of N if:

(Def.3) it = Elements(N).

Next we state several propositions:

(6) For every N and for every x holds $x \in$ Elements(N) if and only if $x \in$ the places of N or $x \in$ the transitions of N.

(7) For every N and for every x such that Elements(N) $\neq \emptyset$ holds if x is an element of Elements(N), then x is an element of the places of N or x is an element of the transitions of N.

(8) For every N and for every x such that x is an element of the places of N and the places of $N \neq \emptyset$ holds x is an element of Elements(N).

(9) For every N and for every x such that x is an element of the transitions of N and the transitions of $N \neq \emptyset$ holds x is an element of Elements(N).

(10) $\langle \emptyset, \emptyset, \varnothing \rangle$ is a Petri net.

A net is said to be a Petri net if:

(Def.4) it is a Petri net.

We now state several propositions:

(11) For every Petri net N holds it is not true that: $x \in$ the places of N and $x \in$ the transitions of N.

(12) For every Petri net N and for all x, y such that $\langle x, y \rangle \in$ the flow relation of N and $x \in$ the transitions of N holds $y \in$ the places of N.

(13) For every Petri net N and for all x, y such that $\langle x, y \rangle \in$ the flow relation of N and $y \in$ the transitions of N holds $x \in$ the places of N.

(14) For every Petri net N and for all x, y such that $\langle x, y \rangle \in$ the flow relation of N and $x \in$ the places of N holds $y \in$ the transitions of N.

(15) For every Petri net N and for all x, y such that $\langle x, y \rangle \in$ the flow relation of N and $y \in$ the places of N holds $x \in$ the transitions of N.

We now define two new predicates. Let N be a Petri net, and let us consider x, y. We say that x is a pre-element of y in N if and only if:

(Def.5) $\langle y, x \rangle \in$ the flow relation of N and $x \in$ the transitions of N.

We say that x is a post-element of y in N if and only if:

(Def.6) $\langle x, y \rangle \in$ the flow relation of N and $x \in$ the transitions of N.

We now define two new functors. Let N be a net, and let x be an element of Elements(N). The functor Pre(N, x) yielding a set is defined by:

(Def.7) $y \in$ Pre(N, x) if and only if $y \in$ Elements(N) and $\langle y, x \rangle \in$ the flow relation of N.

The functor $\mathrm{Post}(N,x)$ yielding a set is defined by:

(Def.8) $y \in \mathrm{Post}(N,x)$ if and only if $y \in \mathrm{Elements}(N)$ and $\langle x,y\rangle \in$ the flow relation of N.

Next we state several propositions:

(16) For every Petri net N and for every element x of $\mathrm{Elements}(N)$ holds $\mathrm{Pre}(N,x) \subseteq \mathrm{Elements}(N)$.

(17) For every Petri net N and for every element x of $\mathrm{Elements}(N)$ holds $\mathrm{Pre}(N,x) \in 2^{\mathrm{Elements}(N)}$.

(18) For every Petri net N and for every element x of $\mathrm{Elements}(N)$ holds $\mathrm{Post}(N,x) \subseteq \mathrm{Elements}(N)$.

(19) For every Petri net N and for every element x of $\mathrm{Elements}(N)$ holds $\mathrm{Post}(N,x) \in 2^{\mathrm{Elements}(N)}$.

(20) For every Petri net N and for every element y of $\mathrm{Elements}(N)$ such that $y \in$ the transitions of N holds $x \in \mathrm{Pre}(N,y)$ if and only if y is a pre-element of x in N.

(21) For every Petri net N and for every element y of $\mathrm{Elements}(N)$ such that $y \in$ the transitions of N holds $x \in \mathrm{Post}(N,y)$ if and only if y is a post-element of x in N.

Let N be a Petri net, and let x be an element of $\mathrm{Elements}(N)$. Let us assume that $\mathrm{Elements}(N) \neq \emptyset$. The functor $\mathrm{enter}(N,x)$ yielding a set is defined by:

(Def.9) if $x \in$ the places of N, then $\mathrm{enter}(N,x) = \{x\}$ but if $x \in$ the transitions of N, then $\mathrm{enter}(N,x) = \mathrm{Pre}(N,x)$.

We now state three propositions:

(22) For every Petri net N and for every element x of $\mathrm{Elements}(N)$ such that $\mathrm{Elements}(N) \neq \emptyset$ holds $\mathrm{enter}(N,x) = \{x\}$ or $\mathrm{enter}(N,x) = \mathrm{Pre}(N,x)$.

(23) For every Petri net N and for every element x of $\mathrm{Elements}(N)$ such that $\mathrm{Elements}(N) \neq \emptyset$ holds $\mathrm{enter}(N,x) \subseteq \mathrm{Elements}(N)$.

(24) For every Petri net N and for every element x of $\mathrm{Elements}(N)$ such that $\mathrm{Elements}(N) \neq \emptyset$ holds $\mathrm{enter}(N,x) \in 2^{\mathrm{Elements}(N)}$.

Let N be a Petri net, and let x be an element of $\mathrm{Elements}(N)$. Let us assume that $\mathrm{Elements}(N) \neq \emptyset$. The functor $\mathrm{exit}(N,x)$ yields a set and is defined by:

(Def.10) if $x \in$ the places of N, then $\mathrm{exit}(N,x) = \{x\}$ but if $x \in$ the transitions of N, then $\mathrm{exit}(N,x) = \mathrm{Post}(N,x)$.

We now state three propositions:

(25) For every Petri net N and for every element x of $\mathrm{Elements}(N)$ such that $\mathrm{Elements}(N) \neq \emptyset$ holds $\mathrm{exit}(N,x) = \{x\}$ or $\mathrm{exit}(N,x) = \mathrm{Post}(N,x)$.

(26) For every Petri net N and for every element x of $\mathrm{Elements}(N)$ such that $\mathrm{Elements}(N) \neq \emptyset$ holds $\mathrm{exit}(N,x) \subseteq \mathrm{Elements}(N)$.

(27) For every Petri net N and for every element x of $\mathrm{Elements}(N)$ such that $\mathrm{Elements}(N) \neq \emptyset$ holds $\mathrm{exit}(N,x) \in 2^{\mathrm{Elements}(N)}$.

Let N be a Petri net, and let x be an element of $\mathrm{Elements}(N)$. Let us assume that $\mathrm{Elements}(N) \neq \emptyset$. The functor $\mathrm{field}(N, x)$ yielding a set is defined as follows:

(Def.11) $\mathrm{field}(N, x) = \mathrm{enter}(N, x) \cup \mathrm{exit}(N, x)$.

We now define two new functors. Let N be a net, and let x be an element of the transitions of N. The functor $\mathrm{Prec}(N, x)$ yielding a set is defined by:

(Def.12) $y \in \mathrm{Prec}(N, x)$ if and only if $y \in$ the places of N and $\langle y, x \rangle \in$ the flow relation of N.

The functor $\mathrm{Postc}(N, x)$ yielding a set is defined as follows:

(Def.13) $y \in \mathrm{Postc}(N, x)$ if and only if $y \in$ the places of N and $\langle x, y \rangle \in$ the flow relation of N.

We now define two new functors. Let N be a Petri net, and let X be a set. Let us assume that $X \subseteq \mathrm{Elements}(N)$. The functor $\mathrm{Entr}(N, X)$ yields a set and is defined by:

(Def.14) $x \in \mathrm{Entr}(N, X)$ if and only if $x \in 2^{\mathrm{Elements}(N)}$ and there exists an element y of $\mathrm{Elements}(N)$ such that $y \in X$ and $x = \mathrm{enter}(N, y)$.

The functor $\mathrm{Ext}(N, X)$ yielding a set is defined by:

(Def.15) $x \in \mathrm{Ext}(N, X)$ if and only if $x \in 2^{\mathrm{Elements}(N)}$ and there exists an element y of $\mathrm{Elements}(N)$ such that $y \in X$ and $x = \mathrm{exit}(N, y)$.

Next we state two propositions:

(28) For every Petri net N and for every element x of $\mathrm{Elements}(N)$ and for every set X such that $\mathrm{Elements}(N) \neq \emptyset$ and $X \subseteq \mathrm{Elements}(N)$ and $x \in X$ holds $\mathrm{enter}(N, x) \in \mathrm{Entr}(N, X)$.

(29) For every Petri net N and for every element x of $\mathrm{Elements}(N)$ and for every set X such that $\mathrm{Elements}(N) \neq \emptyset$ and $X \subseteq \mathrm{Elements}(N)$ and $x \in X$ holds $\mathrm{exit}(N, x) \in \mathrm{Ext}(N, X)$.

We now define two new functors. Let N be a Petri net, and let X be a set. Let us assume that $X \subseteq \mathrm{Elements}(N)$. The functor $\mathrm{Input}(N, X)$ yields a set and is defined by:

(Def.16) $\mathrm{Input}(N, X) = \bigcup \mathrm{Entr}(N, X)$.

The functor $\mathrm{Output}(N, X)$ yielding a set is defined by:

(Def.17) $\mathrm{Output}(N, X) = \bigcup \mathrm{Ext}(N, X)$.

The following four propositions are true:

(30) For every Petri net N and for every x and for every set X such that $\mathrm{Elements}(N) \neq \emptyset$ and $X \subseteq \mathrm{Elements}(N)$ holds $x \in \mathrm{Input}(N, X)$ if and only if there exists an element y of $\mathrm{Elements}(N)$ such that $y \in X$ and $x \in \mathrm{enter}(N, y)$.

(31) For every Petri net N and for every x and for every set X such that $\mathrm{Elements}(N) \neq \emptyset$ and $X \subseteq \mathrm{Elements}(N)$ holds $x \in \mathrm{Output}(N, X)$ if and only if there exists an element y of $\mathrm{Elements}(N)$ such that $y \in X$ and $x \in \mathrm{exit}(N, y)$.

(32) Let N be a Petri net. Then for every subset X of Elements(N) and for every element x of Elements(N) such that Elements(N) $\neq \emptyset$ holds $x \in$ Input(N, X) if and only if $x \in X$ and $x \in$ the places of N or there exists an element y of Elements(N) such that $y \in X$ and $y \in$ the transitions of N and y is a pre-element of x in N.

(33) Let N be a Petri net. Then for every subset X of Elements(N) and for every element x of Elements(N) such that Elements(N) $\neq \emptyset$ holds $x \in$ Output(N, X) if and only if $x \in X$ and $x \in$ the places of N or there exists an element y of Elements(N) such that $y \in X$ and $y \in$ the transitions of N and y is a post-element of x in N.

References

[1] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[2] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received August 10, 1990

FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Classes of Conjugation. Normal Subgroups

Wojciech A. Trybulec[1]
Warsaw University

Summary. Theorems that were not proved in [8] and in [9] are discussed. In the article we define notion of conjugation for elements, subsets and subgroups of a group. We define the classes of conjugation. Normal subgroups of a group and normalizator of a subset of a group or of a subgroup are introduced. We also define the set of all subgroups of a group. An auxiliary theorem that belongs rather to [1] is proved.

MML Identifier: `GROUP_3`.

The papers [3], [10], [5], [2], [8], [9], [6], [4], and [7] provide the notation and terminology for this paper. For simplicity we follow a convention: x, y are arbitrary, X denotes a set, G denotes a group, a, b, c, d, g, h denote elements of G, A, B, C, D denote subsets of G, H, H_1, H_2, H_3 denote subgroups of G, n denotes a natural number, and i denotes an integer. Next we state a number of propositions:

(1) $(a \cdot b) \cdot b^{-1} = a$ and $(a \cdot b^{-1}) \cdot b = a$ and $(b^{-1} \cdot b) \cdot a = a$ and $(b \cdot b^{-1}) \cdot a = a$ and $a \cdot (b \cdot b^{-1}) = a$ and $a \cdot (b^{-1} \cdot b) = a$ and $b^{-1} \cdot (b \cdot a) = a$ and $b \cdot (b^{-1} \cdot a) = a$.

(2) G is an Abelian group if and only if the operation of G is commutative.

(3) $\{\mathbf{1}\}_G$ is an Abelian group.

(4) If $A \subseteq B$ and $C \subseteq D$, then $A \cdot C \subseteq B \cdot D$.

(5) If $A \subseteq B$, then $a \cdot A \subseteq a \cdot B$ and $A \cdot a \subseteq B \cdot a$.

(6) If H_1 is a subgroup of H_2, then $a \cdot H_1 \subseteq a \cdot H_2$ and $H_1 \cdot a \subseteq H_2 \cdot a$.

(7) $a \cdot H = \{a\} \cdot H$.

(8) $H \cdot a = H \cdot \{a\}$.

(9) $(a \cdot A) \cdot H = a \cdot (A \cdot H)$.

(10) $(A \cdot a) \cdot H = A \cdot (a \cdot H)$.

(11) $(a \cdot H) \cdot A = a \cdot (H \cdot A)$.

(12) $(A \cdot H) \cdot a = A \cdot (H \cdot a)$.

[1]Supported by RPBP.III-24.C1

ISSN 0777–4028

(13) $(H \cdot a) \cdot A = H \cdot (a \cdot A)$.

(14) $(H \cdot A) \cdot a = H \cdot (A \cdot a)$.

(15) $(H_1 \cdot a) \cdot H_2 = H_1 \cdot (a \cdot H_2)$.

Let us consider G. The functor $\mathrm{SubGr}\, G$ yielding a non-empty set is defined by:

(Def.1) $x \in \mathrm{SubGr}\, G$ if and only if x is a subgroup of G.

In the sequel D denotes a non-empty set. Next we state four propositions:

(16) If for every x holds $x \in D$ if and only if x is a subgroup of G, then $D = \mathrm{SubGr}\, G$.

(17) $x \in \mathrm{SubGr}\, G$ if and only if x is a subgroup of G.

(18) $G \in \mathrm{SubGr}\, G$.

(19) If G is finite, then $\mathrm{SubGr}\, G$ is finite.

Let us consider G, a, b. The functor a^b yielding an element of G is defined as follows:

(Def.2) $a^b = (b^{-1} \cdot a) \cdot b$.

One can prove the following propositions:

(20) $a^b = (b^{-1} \cdot a) \cdot b$ and $a^b = b^{-1} \cdot (a \cdot b)$.

(21) If $a^g = b^g$, then $a = b$.

(22) $(1_G)^a = 1_G$.

(23) If $a^b = 1_G$, then $a = 1_G$.

(24) $a^{1_G} = a$.

(25) $a^a = a$.

(26) $(a^a)^{-1} = a$ and $(a^{-1})^a = a^{-1}$.

(27) $a^b = a$ if and only if $a \cdot b = b \cdot a$.

(28) $(a \cdot b)^g = a^g \cdot b^g$.

(29) $(a^g)^h = a^{g \cdot h}$.

(30) $((a^b)^b)^{-1} = a$ and $((a^b)^{-1})^b = a$.

(31) $a^b = c$ if and only if $a = (c^b)^{-1}$.

(32) $(a^{-1})^b = (a^b)^{-1}$.

(33) $(a^n)^b = (a^b)^n$.

(34) $(a^i)^b = (a^b)^i$.

(35) If G is an Abelian group, then $a^b = a$.

(36) If for all a, b holds $a^b = a$, then G is an Abelian group.

Let us consider G, A, B. The functor A^B yielding a subset of G is defined as follows:

(Def.3) $A^B = \{g^h : g \in A \wedge h \in B\}$.

We now state a number of propositions:

(37) $A^B = \{g^h : g \in A \wedge h \in B\}$.

(38) $x \in A^B$ if and only if there exist g, h such that $x = g^h$ and $g \in A$ and $h \in B$.

(39) $A^B \neq \emptyset$ if and only if $A \neq \emptyset$ and $B \neq \emptyset$.

(40) $A^B \subseteq (B^{-1} \cdot A) \cdot B$.

(41) $(A \cdot B)^C \subseteq A^C \cdot B^C$.

(42) $(A^B)^C = A^{B \cdot C}$.

(43) $(A^{-1})^B = (A^B)^{-1}$.

(44) $\{a\}^{\{b\}} = \{a^b\}$.

(45) $\{a\}^{\{b,c\}} = \{a^b, a^c\}$.

(46) $\{a, b\}^{\{c\}} = \{a^c, b^c\}$.

(47) $\{a, b\}^{\{c,d\}} = \{a^c, a^d, b^c, b^d\}$.

We now define two new functors. Let us consider G, A, g. The functor A^g yields a subset of G and is defined as follows:

(Def.4) $A^g = A^{\{g\}}$.

The functor g^A yields a subset of G and is defined by:

(Def.5) $g^A = \{g\}^A$.

Next we state a number of propositions:

(48) $A^g = A^{\{g\}}$.

(49) $g^A = \{g\}^A$.

(50) $x \in A^g$ if and only if there exists h such that $x = h^g$ and $h \in A$.

(51) $x \in g^A$ if and only if there exists h such that $x = g^h$ and $h \in A$.

(52) $g^A \subseteq (A^{-1} \cdot g) \cdot A$.

(53) $(A^B)^g = A^{B \cdot g}$.

(54) $(A^g)^B = A^{g \cdot B}$.

(55) $(g^A)^B = g^{A \cdot B}$.

(56) $(A^a)^b = A^{a \cdot b}$.

(57) $(a^A)^b = a^{A \cdot b}$.

(58) $(a^b)^A = a^{b \cdot A}$.

(59) $A^g = (g^{-1} \cdot A) \cdot g$.

(60) $(A \cdot B)^a \subseteq A^a \cdot B^a$.

(61) $A^{1_G} = A$.

(62) If $A \neq \emptyset$, then $(1_G)^A = \{1_G\}$.

(63) $((A^a)^a)^{-1} = A$ and $((A^a)^{-1})^a = A$.

(64) $A = B^g$ if and only if $B = (A^g)^{-1}$.

(65) G is an Abelian group if and only if for all A, B such that $B \neq \emptyset$ holds $A^B = A$.

(66) G is an Abelian group if and only if for all A, g holds $A^g = A$.

(67) G is an Abelian group if and only if for all A, g such that $A \neq \emptyset$ holds $g^A = \{g\}$.

Let us consider G, H, a. The functor H^a yielding a subgroup of G is defined by:

(Def.6) the carrier of $H^a = \overline{H}^a$.

The following propositions are true:

(68) If the carrier of $H_1 = \overline{H}^a$, then $H_1 = H^a$.

(69) The carrier of $H^a = \overline{H}^a$.

(70) $x \in H^a$ if and only if there exists g such that $x = g^a$ and $g \in H$.

(71) The carrier of $H^a = (a^{-1} \cdot H) \cdot a$.

(72) $(H^a)^b = H^{a \cdot b}$.

(73) $H^{1_G} = H$.

(74) $((H^a)^a)^{-1} = H$ and $((H^a)^{-1})^a = H$.

(75) $H_1 = H_2^a$ if and only if $H_2 = (H_1^a)^{-1}$.

(76) $(H_1 \cap H_2)^a = H_1^a \cap H_2^a$.

(77) $\mathrm{Ord}(H) = \mathrm{Ord}(H^a)$.

(78) H is finite if and only if H^a is finite.

(79) If H is finite, then $\mathrm{ord}(H) = \mathrm{ord}(H^a)$.

(80) $\{\mathbf{1}\}_G^a = \{\mathbf{1}\}_G$.

(81) If $H^a = \{\mathbf{1}\}_G$, then $H = \{\mathbf{1}\}_G$.

(82) ${\Omega_G}^a = G$.

(83) If $H^a = G$, then $H = G$.

(84) $|\bullet : H| = |\bullet : H^a|$.

(85) If the left cosets of H is finite, then $|\bullet : H|_{\mathbb{N}} = |\bullet : H^a|_{\mathbb{N}}$.

(86) If G is an Abelian group, then for all H, a holds $H^a = H$.

Let us consider G, a, b. We say that a and b are conjugated if and only if:

(Def.7) there exists g such that $a = b^g$.

We now state several propositions:

(87) a and b are conjugated if and only if there exists g such that $a = b^g$.

(88) a and b are conjugated if and only if there exists g such that $b = a^g$.

(89) a and a are conjugated.

(90) If a and b are conjugated, then b and a are conjugated.

(91) If a and b are conjugated and b and c are conjugated, then a and c are conjugated.

(92) If a and 1_G are conjugated or 1_G and a are conjugated, then $a = 1_G$.

(93) $a^{\overline{\Omega_G}} = \{b : a \text{ and } b \text{ are conjugated }\}$.

Let us consider G, a. The functor $a^\bullet$ yielding a subset of G is defined by:

(Def.8) $a^\bullet = a^{\overline{\Omega_G}}$.

We now state several propositions:

(94) $a^\bullet = a^{\overline{\Omega_G}}$.

(95) $x \in a^\bullet$ if and only if there exists b such that $b = x$ and a and b are conjugated.

(96) $a \in b^\bullet$ if and only if a and b are conjugated.

(97) $a^g \in a^\bullet$.

(98) $a \in a^\bullet$.

(99) If $a \in b^\bullet$, then $b \in a^\bullet$.

(100) $a^\bullet = b^\bullet$ if and only if $a^\bullet$ meets $b^\bullet$.

(101) $a^\bullet = \{1_G\}$ if and only if $a = 1_G$.

(102) $a^\bullet \cdot A = A \cdot a^\bullet$.

Let us consider G, A, B. We say that A and B are conjugated if and only if:

(Def.9) there exists g such that $A = B^g$.

We now state several propositions:

(103) A and B are conjugated if and only if there exists g such that $A = B^g$.

(104) A and B are conjugated if and only if there exists g such that $B = A^g$.

(105) A and A are conjugated.

(106) If A and B are conjugated, then B and A are conjugated.

(107) If A and B are conjugated and B and C are conjugated, then A and C are conjugated.

(108) $\{a\}$ and $\{b\}$ are conjugated if and only if a and b are conjugated.

(109) If A and $\overline{H_1}$ are conjugated, then there exists H_2 such that the carrier of $H_2 = A$.

Let us consider G, A. The functor $A^\bullet$ yielding a family of subsets of the carrier of G is defined as follows:

(Def.10) $A^\bullet = \{B : A \text{ and } B \text{ are conjugated }\}$.

The following propositions are true:

(110) $A^\bullet = \{B : A \text{ and } B \text{ are conjugated }\}$.

(111) $x \in A^\bullet$ if and only if there exists B such that $x = B$ and A and B are conjugated.

(112) If $x \in A^\bullet$, then x is a subset of G.

(113) $A \in B^\bullet$ if and only if A and B are conjugated.

(114) $A^g \in A^\bullet$.

(115) $A \in A^\bullet$.

(116) If $A \in B^\bullet$, then $B \in A^\bullet$.

(117) $A^\bullet = B^\bullet$ if and only if $A^\bullet$ meets $B^\bullet$.

(118) $\{a\}^\bullet = \{\{b\} : b \in a^\bullet\}$.

(119) If G is finite, then $A^\bullet$ is finite.

Let us consider G, H_1, H_2. We say that H_1 and H_2 are conjugated if and only if:

(Def.11) there exists g such that $H_1 = H_2^g$.

The following propositions are true:

(120) H_1 and H_2 are conjugated if and only if there exists g such that $H_1 = H_2^g$.

(121) H_1 and H_2 are conjugated if and only if there exists g such that $H_2 = H_1^g$.

(122) H_1 and H_1 are conjugated.

(123) If H_1 and H_2 are conjugated, then H_2 and H_1 are conjugated.

(124) If H_1 and H_2 are conjugated and H_2 and H_3 are conjugated, then H_1 and H_3 are conjugated.

In the sequel L denotes a subset of $\mathrm{SubGr}\, G$. Let us consider G, H. The functor $H^\bullet$ yielding a subset of $\mathrm{SubGr}\, G$ is defined as follows:

(Def.12) $x \in H^\bullet$ if and only if there exists H_1 such that $x = H_1$ and H and H_1 are conjugated.

One can prove the following propositions:

(125) If for every x holds $x \in L$ if and only if there exists H such that $x = H$ and H_1 and H are conjugated, then $L = H_1^\bullet$.

(126) $x \in H_1^\bullet$ if and only if there exists H_2 such that $x = H_2$ and H_1 and H_2 are conjugated.

(127) If $x \in H^\bullet$, then x is a subgroup of G.

(128) $H_1 \in H_2^\bullet$ if and only if H_1 and H_2 are conjugated.

(129) $H^g \in H^\bullet$.

(130) $H \in H^\bullet$.

(131) If $H_1 \in H_2^\bullet$, then $H_2 \in H_1^\bullet$.

(132) $H_1^\bullet = H_2^\bullet$ if and only if $H_1^\bullet$ meets $H_2^\bullet$.

(133) If G is finite, then $H^\bullet$ is finite.

(134) H_1 and H_2 are conjugated if and only if $\overline{H_1}$ and $\overline{H_2}$ are conjugated.

Let us consider G. A subgroup of G is called a normal subgroup of G if:

(Def.13) for every a holds $\mathrm{it}^a = \mathrm{it}$.

One can prove the following proposition

(135) If for every a holds $H = H^a$, then H is a normal subgroup of G.

In the sequel N, N_1, N_2 will denote ha normal subgroups of G. We now state a number of propositions:

(136) $N^a = N$.

(137) $\{\mathbf{1}\}_G$ is a normal subgroup of G and Ω_G is a normal subgroup of G.

(138) $N_1 \cap N_2$ is a normal subgroup of G.

(139) If G is an Abelian group, then H is a normal subgroup of G.

(140) H is a normal subgroup of G if and only if for every a holds $a \cdot H = H \cdot a$.

(141) H is a normal subgroup of G if and only if for every a holds $a \cdot H \subseteq H \cdot a$.

(142) H is a normal subgroup of G if and only if for every a holds $H \cdot a \subseteq a \cdot H$.

(143) H is a normal subgroup of G if and only if for every A holds $A \cdot H = H \cdot A$.

(144) H is a normal subgroup of G if and only if for every a holds H is a subgroup of H^a.

(145) H is a normal subgroup of G if and only if for every a holds H^a is a subgroup of H.

(146) H is a normal subgroup of G if and only if $H^\bullet = \{H\}$.

(147) H is a normal subgroup of G if and only if for every a such that $a \in H$ holds $a^\bullet \subseteq \overline{H}$.

(148) $\overline{N_1} \cdot \overline{N_2} = \overline{N_2} \cdot \overline{N_1}$.

(149) There exists N such that the carrier of $N = \overline{N_1} \cdot \overline{N_2}$.

(150) The left cosets of $N =$ the right cosets of N.

(151) If the left cosets of H is finite and $|\bullet : H|_{\mathbb{N}} = 2$, then H is a normal subgroup of G.

Let us consider G, A. The functor $\mathrm{N}(A)$ yielding a subgroup of G is defined by:

(Def.14) the carrier of $\mathrm{N}(A) = \{h : A^h = A\}$.

We now state several propositions:

(152) If the carrier of $H = \{h : A^h = A\}$, then $H = \mathrm{N}(A)$.

(153) The carrier of $\mathrm{N}(A) = \{h : A^h = A\}$.

(154) $x \in \mathrm{N}(A)$ if and only if there exists h such that $x = h$ and $A^h = A$.

(155) $\overline{\overline{A^\bullet}} = |\bullet : \mathrm{N}(A)|$.

(156) If $A^\bullet$ is finite or the left cosets of $\mathrm{N}(A)$ is finite, then $\mathrm{card}\, A^\bullet = |\bullet : \mathrm{N}(A)|_{\mathbb{N}}$.

(157) $\overline{\overline{a^\bullet}} = |\bullet : \mathrm{N}(\{a\})|$.

(158) If $a^\bullet$ is finite or the left cosets of $\mathrm{N}(\{a\})$ is finite, then $\mathrm{card}\, a^\bullet = |\bullet : \mathrm{N}(\{a\})|_{\mathbb{N}}$.

Let us consider G, H. The functor $\mathrm{N}(H)$ yields a subgroup of G and is defined as follows:

(Def.15) $\mathrm{N}(H) = \mathrm{N}(\overline{H})$.

We now state several propositions:

(159) $\mathrm{N}(H) = \mathrm{N}(\overline{H})$.

(160) $x \in \mathrm{N}(H)$ if and only if there exists h such that $x = h$ and $H^h = H$.

(161) $\overline{\overline{H^\bullet}} = |\bullet : \mathrm{N}(H)|$.

(162) If $H^\bullet$ is finite or the left cosets of $\mathrm{N}(H)$ is finite, then $\mathrm{card}\, H^\bullet = |\bullet : \mathrm{N}(H)|_{\mathbb{N}}$.

(163) H is a normal subgroup of G if and only if $\mathrm{N}(H) = G$.

(164) $\mathrm{N}(\{\mathbf{1}\}_G) = G$.

(165) $\mathrm{N}(\Omega_G) = G$.

(166) If X is finite and $\mathrm{card}\, X = 2$, then there exist x, y such that $x \neq y$ and $X = \{x, y\}$.

References

[1] Grzegorz Bancerek. Cardinal arithmetics. *Formalized Mathematics*, 1(**3**):543–547, 1990.

[2] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[4] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[5] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[6] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[7] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.

[8] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.

[9] Wojciech A. Trybulec. Subgroup and cosets of subgroups. *Formalized Mathematics*, 1(**5**):855–864, 1990.

[10] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

Received August 10, 1990

FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Replacing of Variables in Formulas of ZF Theory

Grzegorz Bancerek[1]
Warsaw University
Białystok

Summary. Part one is a supplement to papers [1], [2], and [3]. It deals with concepts of selector functions, atomic, negative, conjunctive formulas and etc., subformulas, free variables, satisfiability and models (it is shown that axioms of the predicate and the quantifier calculus are satisfied in an arbitrary set). In part two there are introduced notions of variables occurring in a formula and replacing of variables in a formula.

MML Identifier: `ZF_LANG1`.

The terminology and notation used in this paper have been introduced in the following articles: [9], [8], [5], [6], [4], [7], [1], and [2]. For simplicity we adopt the following rules: p, p_1, p_2, q, r, F, G, G_1, G_2, H, H_1, H_2 will be ZF-formulae, x, x_1, x_2, y, y_1, y_2, z, z_1, z_2, s, t will be variables, a will be arbitrary, and X will be a set. Next we state a number of propositions:

(1) $\mathrm{Var}_1(x{=}y) = x$ and $\mathrm{Var}_2(x{=}y) = y$.

(2) $\mathrm{Var}_1(x \epsilon y) = x$ and $\mathrm{Var}_2(x \epsilon y) = y$.

(3) $\mathrm{Arg}(\neg p) = p$.

(4) $\mathrm{LeftArg}(p \wedge q) = p$ and $\mathrm{RightArg}(p \wedge q) = q$.

(5) $\mathrm{LeftArg}(p \vee q) = p$ and $\mathrm{RightArg}(p \vee q) = q$.

(6) $\mathrm{Antecedent}(p \Rightarrow q) = p$ and $\mathrm{Consequent}(p \Rightarrow q) = q$.

(7) $\mathrm{LeftSide}(p \Leftrightarrow q) = p$ and $\mathrm{RightSide}(p \Leftrightarrow q) = q$.

(8) $\mathrm{Bound}(\forall_x p) = x$ and $\mathrm{Scope}(\forall_x p) = p$.

(9) $\mathrm{Bound}(\exists_x p) = x$ and $\mathrm{Scope}(\exists_x p) = p$.

(10) $p \vee q = \neg p \Rightarrow q$.

(11) If $\forall_{x,y} p = \forall_z q$, then $x = z$ and $\forall_y p = q$.

[1]Supported by RPBP III-24.C1.

ISSN 0777–4028

(12) If $\exists_{x,y}p = \exists_z q$, then $x = z$ and $\exists_y p = q$.

(13) $\forall_{x,y}p$ is universal and $\mathrm{Bound}(\forall_{x,y}p) = x$ and $\mathrm{Scope}(\forall_{x,y}p) = \forall_y p$.

(14) $\exists_{x,y}p$ is existential and $\mathrm{Bound}(\exists_{x,y}p) = x$ and $\mathrm{Scope}(\exists_{x,y}p) = \exists_y p$.

(15) $\forall_{x,y,z}p = \forall_x(\forall_y(\forall_z p))$ and $\forall_{x,y,z}p = \forall_{x,y}(\forall_z p)$.

(16) If $\forall_{x_1,y_1}p_1 = \forall_{x_2,y_2}p_2$, then $x_1 = x_2$ and $y_1 = y_2$ and $p_1 = p_2$.

(17) If $\forall_{x_1,y_1,z_1}p_1 = \forall_{x_2,y_2,z_2}p_2$, then $x_1 = x_2$ and $y_1 = y_2$ and $z_1 = z_2$ and $p_1 = p_2$.

(18) If $\forall_{x,y,z}p = \forall_t q$, then $x = t$ and $\forall_{y,z}p = q$.

(19) If $\forall_{x,y,z}p = \forall_{t,s}q$, then $x = t$ and $y = s$ and $\forall_z p = q$.

(20) If $\exists_{x_1,y_1}p_1 = \exists_{x_2,y_2}p_2$, then $x_1 = x_2$ and $y_1 = y_2$ and $p_1 = p_2$.

(21) $\exists_{x,y,z}p = \exists_x(\exists_y(\exists_z p))$ and $\exists_{x,y,z}p = \exists_{x,y}(\exists_z p)$.

(22) If $\exists_{x_1,y_1,z_1}p_1 = \exists_{x_2,y_2,z_2}p_2$, then $x_1 = x_2$ and $y_1 = y_2$ and $z_1 = z_2$ and $p_1 = p_2$.

(23) If $\exists_{x,y,z}p = \exists_t q$, then $x = t$ and $\exists_{y,z}p = q$.

(24) If $\exists_{x,y,z}p = \exists_{t,s}q$, then $x = t$ and $y = s$ and $\exists_z p = q$.

(25) $\forall_{x,y,z}p$ is universal and $\mathrm{Bound}(\forall_{x,y,z}p) = x$ and $\mathrm{Scope}(\forall_{x,y,z}p) = \forall_{y,z}p$.

(26) $\exists_{x,y,z}p$ is existential and $\mathrm{Bound}(\exists_{x,y,z}p) = x$ and $\mathrm{Scope}(\exists_{x,y,z}p) = \exists_{y,z}p$.

(27) If H is disjunctive, then $\mathrm{LeftArg}(H) = \mathrm{Arg}(\mathrm{LeftArg}(\mathrm{Arg}(H)))$.

(28) If H is disjunctive, then $\mathrm{RightArg}(H) = \mathrm{Arg}(\mathrm{RightArg}(\mathrm{Arg}(H)))$.

(29) If H is conditional, then $\mathrm{Antecedent}(H) = \mathrm{LeftArg}(\mathrm{Arg}(H))$.

(30) If H is conditional, then $\mathrm{Consequent}(H) = \mathrm{Arg}(\mathrm{RightArg}(\mathrm{Arg}(H)))$.

(31) If H is biconditional, then $\mathrm{LeftSide}(H) = \mathrm{Antecedent}(\mathrm{LeftArg}(H))$ and $\mathrm{LeftSide}(H) = \mathrm{Consequent}(\mathrm{RightArg}(H))$.

(32) If H is biconditional, then $\mathrm{RightSide}(H) = \mathrm{Consequent}(\mathrm{LeftArg}(H))$ and $\mathrm{RightSide}(H) = \mathrm{Antecedent}(\mathrm{RightArg}(H))$.

(33) If H is existential, then $\mathrm{Bound}(H) = \mathrm{Bound}(\mathrm{Arg}(H))$ and $\mathrm{Scope}(H) = \mathrm{Arg}(\mathrm{Scope}(\mathrm{Arg}(H)))$.

(34) $\mathrm{Arg}(F \vee G) = \neg F \wedge \neg G$ and $\mathrm{Antecedent}(F \vee G) = \neg F$ and $\mathrm{Consequent}(F \vee G) = G$.

(35) $\mathrm{Arg}(F \Rightarrow G) = F \wedge \neg G$.

(36) $\mathrm{LeftArg}(F \Leftrightarrow G) = F \Rightarrow G$ and $\mathrm{RightArg}(F \Leftrightarrow G) = G \Rightarrow F$.

(37) $\mathrm{Arg}(\exists_x H) = \forall_x \neg H$.

(38) If H is disjunctive, then H is conditional and H is negative and $\mathrm{Arg}(H)$ is conjunctive and $\mathrm{LeftArg}(\mathrm{Arg}(H))$ is negative and $\mathrm{RightArg}(\mathrm{Arg}(H))$ is negative.

(39) If H is conditional, then H is negative and $\mathrm{Arg}(H)$ is conjunctive and $\mathrm{RightArg}(\mathrm{Arg}(H))$ is negative.

(40) If H is biconditional, then H is conjunctive and $\mathrm{LeftArg}(H)$ is conditional and $\mathrm{RightArg}(H)$ is conditional.

(41) If H is existential, then H is negative and $\mathrm{Arg}(H)$ is universal and $\mathrm{Scope}(\mathrm{Arg}(H))$ is negative.

(42) It is not true that: H is an equality and H is a membership or H is negative or H is conjunctive or H is universal and it is not true that: H is a membership and H is negative or H is conjunctive or H is universal and it is not true that: H is negative and H is conjunctive or H is universal and it is not true that: H is conjunctive and H is universal.

(43) If F is a subformula of G, then $\mathrm{len}\, F \leq \mathrm{len}\, G$.

(44) Suppose F is a proper subformula of G and G is a subformula of H or F is a subformula of G and G is a proper subformula of H or F is a subformula of G and G is an immediate constituent of H or F is an immediate constituent of G and G is a subformula of H or F is a proper subformula of G and G is an immediate constituent of H or F is an immediate constituent of G and G is a proper subformula of H. Then F is a proper subformula of H.

(45) H is not a proper subformula of H.

(46) H is not an immediate constituent of H.

(47) It is not true that: G is a proper subformula of H and H is a subformula of G.

(48) It is not true that: G is a proper subformula of H and H is a proper subformula of G.

(49) It is not true that: G is a subformula of H and H is an immediate constituent of G.

(50) It is not true that: G is a proper subformula of H and H is an immediate constituent of G.

(51) If $\neg F$ is a subformula of H, then F is a proper subformula of H.

(52) If $F \wedge G$ is a subformula of H, then F is a proper subformula of H and G is a proper subformula of H.

(53) If $\forall_x H$ is a subformula of F, then H is a proper subformula of F.

(54) $F \wedge \neg G$ is a proper subformula of $F \Rightarrow G$ and F is a proper subformula of $F \Rightarrow G$ and $\neg G$ is a proper subformula of $F \Rightarrow G$ and G is a proper subformula of $F \Rightarrow G$.

(55) $\neg F \wedge \neg G$ is a proper subformula of $F \vee G$ and $\neg F$ is a proper subformula of $F \vee G$ and $\neg G$ is a proper subformula of $F \vee G$ and F is a proper subformula of $F \vee G$ and G is a proper subformula of $F \vee G$.

(56) $\forall_x \neg H$ is a proper subformula of $\exists_x H$ and $\neg H$ is a proper subformula of $\exists_x H$.

(57) G is a subformula of H if and only if $G \in \mathrm{Subformulae}\, H$.

(58) If $G \in \mathrm{Subformulae}\, H$, then $\mathrm{Subformulae}\, G \subseteq \mathrm{Subformulae}\, H$.

(59) $H \in \mathrm{Subformulae}\, H$.

(60) $\mathrm{Subformulae}\, F \Rightarrow G = (\mathrm{Subformulae}\, F \cup \mathrm{Subformulae}\, G) \cup \{\neg G, F \wedge \neg G, F \Rightarrow G\}$.

(61) Subformulae $F \vee G = (\text{Subformulae}\, F \cup \text{Subformulae}\, G) \cup \{\neg G, \neg F, \neg F \wedge \neg G, F \vee G\}$.

(62) Subformulae $F \Leftrightarrow G = (\text{Subformulae}\, F \cup \text{Subformulae}\, G) \cup \{\neg G, F \wedge \neg G, F \Rightarrow G, \neg F, G \wedge \neg F, G \Rightarrow F, F \Leftrightarrow G\}$.

(63) $\text{Free}(x{=}y) = \{x, y\}$.

(64) $\text{Free}(x \epsilon y) = \{x, y\}$.

(65) $\text{Free}(\neg p) = \text{Free}\, p$.

(66) $\text{Free}(p \wedge q) = \text{Free}\, p \cup \text{Free}\, q$.

(67) $\text{Free}(\forall_x p) = \text{Free}\, p \setminus \{x\}$.

(68) $\text{Free}(p \vee q) = \text{Free}\, p \cup \text{Free}\, q$.

(69) $\text{Free}(p \Rightarrow q) = \text{Free}\, p \cup \text{Free}\, q$.

(70) $\text{Free}(p \Leftrightarrow q) = \text{Free}\, p \cup \text{Free}\, q$.

(71) $\text{Free}(\exists_x p) = \text{Free}\, p \setminus \{x\}$.

(72) $\text{Free}(\forall_{x,y} p) = \text{Free}\, p \setminus \{x, y\}$.

(73) $\text{Free}(\forall_{x,y,z} p) = \text{Free}\, p \setminus \{x, y, z\}$.

(74) $\text{Free}(\exists_{x,y} p) = \text{Free}\, p \setminus \{x, y\}$.

(75) $\text{Free}(\exists_{x,y,z} p) = \text{Free}\, p \setminus \{x, y, z\}$.

The scheme *ZF_Induction* deals with a unary predicate $\mathcal{P}$, and states that:
for every H holds $\mathcal{P}[H]$
provided the parameter satisfies the following conditions:

- for all x_1, x_2 holds $\mathcal{P}[x_1{=}x_2]$ and $\mathcal{P}[x_1 \epsilon x_2]$,
- for every H such that $\mathcal{P}[H]$ holds $\mathcal{P}[\neg H]$,
- for all H_1, H_2 such that $\mathcal{P}[H_1]$ and $\mathcal{P}[H_2]$ holds $\mathcal{P}[H_1 \wedge H_2]$,
- for all H, x such that $\mathcal{P}[H]$ holds $\mathcal{P}[\forall_x H]$.

For simplicity we adopt the following rules: M, E will denote non-empty families of sets, e will denote an element of E, m, m' will denote elements of M, f, g will denote functions from VAR into E, and v, v' will denote functions from VAR into M. Let us consider E, f, x, e. The functor $f(\frac{x}{e})$ yields a function from VAR into E and is defined by:

(Def.1) $(f(\frac{x}{e}))(x) = e$ and for every y such that $(f(\frac{x}{e}))(y) \neq f(y)$ holds $x = y$.

The following proposition is true

(76) $g = f(\frac{x}{e})$ if and only if $g(x) = e$ and for every y such that $g(y) \neq f(y)$ holds $x = y$.

Let D, D_1, D_2 be non-empty sets, and let f be a function from D into D_1. Let us assume that $D_1 \subseteq D_2$. The functor $D_2[f]$ yields a function from D into D_2 and is defined as follows:

(Def.2) $D_2[f] = f$.

Next we state several propositions:

(77) For all non-empty sets D, D_1, D_2 and for every function f from D into D_1 such that $D_1 \subseteq D_2$ holds $D_2[f] = f$.

(78) $(v(\frac{x}{m'}))(\frac{x}{m}) = v(\frac{x}{m})$ and $v(\frac{x}{v(x)}) = v$.

(79) If $x \neq y$, then $(v(\frac{x}{m}))(\frac{y}{m'}) = (v(\frac{y}{m'}))(\frac{x}{m})$.

(80) $M, v \models \forall_x H$ if and only if for every m holds $M, v(\frac{x}{m}) \models H$.

(81) $M, v \models \forall_x H$ if and only if $M, v(\frac{x}{m}) \models \forall_x H$.

(82) $M, v \models \exists_x H$ if and only if there exists m such that $M, v(\frac{x}{m}) \models H$.

(83) $M, v \models \exists_x H$ if and only if $M, v(\frac{x}{m}) \models \exists_x H$.

(84) For all v, v' such that for every x such that $x \in \mathrm{Free}\, H$ holds $v'(x) = v(x)$ holds if $M, v \models H$, then $M, v' \models H$.

(85) $\mathrm{Free}\, H$ is finite.

In the sequel i, j will denote natural numbers. The following propositions are true:

(86) If $x_i = x_j$, then $i = j$.

(87) There exists i such that $x = x_i$.

(88) x is a natural number and $x \in \mathbb{N}$.

(89) $M, v \models x{=}x$.

(90) $M \models x{=}x$.

(91) $M, v \not\models x \epsilon x$.

(92) $M \not\models x \epsilon x$ and $M \models \neg x \epsilon x$.

(93) $M \models x{=}y$ if and only if $x = y$ or there exists a such that $\{a\} = M$.

(94) $M \models \neg x \epsilon y$ if and only if $x = y$ or for every X such that $X \in M$ holds $X \cap M = \emptyset$.

(95) If H is an equality, then $M, v \models H$ if and only if $v(\mathrm{Var}_1(H)) = v(\mathrm{Var}_2(H))$.

(96) If H is a membership, then $M, v \models H$ if and only if $v(\mathrm{Var}_1(H)) \in v(\mathrm{Var}_2(H))$.

(97) If H is negative, then $M, v \models H$ if and only if $M, v \not\models \mathrm{Arg}(H)$.

(98) If H is conjunctive, then $M, v \models H$ if and only if $M, v \models \mathrm{LeftArg}(H)$ and $M, v \models \mathrm{RightArg}(H)$.

(99) If H is universal, then $M, v \models H$ if and only if for every m holds $M, v(\frac{\mathrm{Bound}(H)}{m}) \models \mathrm{Scope}(H)$.

(100) If H is disjunctive, then $M, v \models H$ if and only if $M, v \models \mathrm{LeftArg}(H)$ or $M, v \models \mathrm{RightArg}(H)$.

(101) If H is conditional, then $M, v \models H$ if and only if if
$M, v \models \mathrm{Antecedent}(H)$,
then $M, v \models \mathrm{Consequent}(H)$.

(102) If H is biconditional, then $M, v \models H$ if and only if $M, v \models \mathrm{LeftSide}(H)$ if and only if $M, v \models \mathrm{RightSide}(H)$.

(103) If H is existential, then $M, v \models H$ if and only if there exists m such that $M, v(\frac{\mathrm{Bound}(H)}{m}) \models \mathrm{Scope}(H)$.

(104) $M \models \exists_x H$ if and only if for every v there exists m such that $M, v(\frac{x}{m}) \models H$.

(105) If $M \models H$, then $M \models \exists_x H$.

(106) $M \models H$ if and only if $M \models \forall_{x,y} H$.

(107) If $M \models H$, then $M \models \exists_{x,y} H$.

(108) $M \models H$ if and only if $M \models \forall_{x,y,z} H$.

(109) If $M \models H$, then $M \models \exists_{x,y,z} H$.

(110) $M, v \models (p \Leftrightarrow q) \Rightarrow (p \Rightarrow q)$ and $M \models (p \Leftrightarrow q) \Rightarrow (p \Rightarrow q)$.

(111) $M, v \models (p \Leftrightarrow q) \Rightarrow (q \Rightarrow p)$ and $M \models (p \Leftrightarrow q) \Rightarrow (q \Rightarrow p)$.

(112) $M \models (p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$.

(113) If $M, v \models p \Rightarrow q$ and $M, v \models q \Rightarrow r$, then $M, v \models p \Rightarrow r$.

(114) If $M \models p \Rightarrow q$ and $M \models q \Rightarrow r$, then $M \models p \Rightarrow r$.

(115) $M, v \models (p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$ and $M \models (p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$.

(116) $M, v \models p \Rightarrow (q \Rightarrow p)$ and $M \models p \Rightarrow (q \Rightarrow p)$.

(117) $M, v \models (p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$ and $M \models (p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$.

(118) $M, v \models p \wedge q \Rightarrow p$ and $M \models p \wedge q \Rightarrow p$.

(119) $M, v \models p \wedge q \Rightarrow q$ and $M \models p \wedge q \Rightarrow q$.

(120) $M, v \models p \wedge q \Rightarrow q \wedge p$ and $M \models p \wedge q \Rightarrow q \wedge p$.

(121) $M, v \models p \Rightarrow p \wedge p$ and $M \models p \Rightarrow p \wedge p$.

(122) $M, v \models (p \Rightarrow q) \Rightarrow ((p \Rightarrow r) \Rightarrow (p \Rightarrow q \wedge r))$ and $M \models (p \Rightarrow q) \Rightarrow ((p \Rightarrow r) \Rightarrow (p \Rightarrow q \wedge r))$.

(123) $M, v \models p \Rightarrow p \vee q$ and $M \models p \Rightarrow p \vee q$.

(124) $M, v \models q \Rightarrow p \vee q$ and $M \models q \Rightarrow p \vee q$.

(125) $M, v \models p \vee q \Rightarrow q \vee p$ and $M \models p \vee q \Rightarrow q \vee p$.

(126) $M, v \models p \Rightarrow p \vee p$ and $M \models p \Rightarrow p \vee p$.

(127) $M, v \models (p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r))$ and $M \models (p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r))$.

(128) $M, v \models (p \Rightarrow r) \wedge (q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r)$ and $M \models (p \Rightarrow r) \wedge (q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r)$.

(129) $M, v \models (p \Rightarrow \neg q) \Rightarrow (q \Rightarrow \neg p)$ and $M \models (p \Rightarrow \neg q) \Rightarrow (q \Rightarrow \neg p)$.

(130) $M, v \models \neg p \Rightarrow (p \Rightarrow q)$ and $M \models \neg p \Rightarrow (p \Rightarrow q)$.

(131) $M, v \models (p \Rightarrow q) \wedge (p \Rightarrow \neg q) \Rightarrow \neg p$ and $M \models (p \Rightarrow q) \wedge (p \Rightarrow \neg q) \Rightarrow \neg p$.

(132) If $M, v \models p \Rightarrow q$ and $M, v \models p$, then $M, v \models q$.

(133) If $M \models p \Rightarrow q$ and $M \models p$, then $M \models q$.

(134) $M, v \models \neg(p \wedge q) \Rightarrow \neg p \vee \neg q$ and $M \models \neg(p \wedge q) \Rightarrow \neg p \vee \neg q$.

(135) $M, v \models \neg p \vee \neg q \Rightarrow \neg(p \wedge q)$ and $M \models \neg p \vee \neg q \Rightarrow \neg(p \wedge q)$.

(136) $M, v \models \neg(p \vee q) \Rightarrow \neg p \wedge \neg q$ and $M \models \neg(p \vee q) \Rightarrow \neg p \wedge \neg q$.

(137) $M, v \models \neg p \wedge \neg q \Rightarrow \neg(p \vee q)$ and $M \models \neg p \wedge \neg q \Rightarrow \neg(p \vee q)$.

(138) $M \models (\forall_x H) \Rightarrow H$.

(139) $M \models H \Rightarrow (\exists_x H)$.

(140) If $x \notin \mathrm{Free}\, H_1$, then $M \models (\forall_x H_1 \Rightarrow H_2) \Rightarrow (H_1 \Rightarrow (\forall_x H_2))$.

(141) If $x \notin \mathrm{Free}\, H_1$ and $M \models H_1 \Rightarrow H_2$, then $M \models H_1 \Rightarrow (\forall_x H_2)$.

(142) If $x \notin \mathrm{Free}\, H_2$, then $M \models (\forall_x H_1 \Rightarrow H_2) \Rightarrow ((\exists_x H_1) \Rightarrow H_2)$.

(143) If $x \notin \mathrm{Free}\, H_2$ and $M \models H_1 \Rightarrow H_2$, then $M \models (\exists_x H_1) \Rightarrow H_2$.

(144) If $M \models H_1 \Rightarrow (\forall_x H_2)$, then $M \models H_1 \Rightarrow H_2$.

(145) If $M \models (\exists_x H_1) \Rightarrow H_2$, then $M \models H_1 \Rightarrow H_2$.

(146) $\mathrm{WFF} \subseteq 2^{[\mathbb{N},\mathbb{N}]}$.

Let us consider H. The functor Var_H yields a set and is defined by:

(Def.3) $\mathrm{Var}_H = \mathrm{rng}\, H \setminus \{0, 1, 2, 3, 4\}$.

We now state a number of propositions:

(147) $\mathrm{Var}_H = \mathrm{rng}\, H \setminus \{0, 1, 2, 3, 4\}$.

(148) $x \neq 0$ and $x \neq 1$ and $x \neq 2$ and $x \neq 3$ and $x \neq 4$.

(149) $x \notin \{0, 1, 2, 3, 4\}$.

(150) If $a \in \mathrm{Var}_H$, then $a \neq 0$ and $a \neq 1$ and $a \neq 2$ and $a \neq 3$ and $a \neq 4$.

(151) $\mathrm{Var}_{x=y} = \{x, y\}$.

(152) $\mathrm{Var}_{x \epsilon y} = \{x, y\}$.

(153) $\mathrm{Var}_{\neg H} = \mathrm{Var}_H$.

(154) $\mathrm{Var}_{H_1 \wedge H_2} = \mathrm{Var}_{H_1} \cup \mathrm{Var}_{H_2}$.

(155) $\mathrm{Var}_{\forall_x H} = \mathrm{Var}_H \cup \{x\}$.

(156) $\mathrm{Var}_{H_1 \vee H_2} = \mathrm{Var}_{H_1} \cup \mathrm{Var}_{H_2}$.

(157) $\mathrm{Var}_{H_1 \Rightarrow H_2} = \mathrm{Var}_{H_1} \cup \mathrm{Var}_{H_2}$.

(158) $\mathrm{Var}_{H_1 \Leftrightarrow H_2} = \mathrm{Var}_{H_1} \cup \mathrm{Var}_{H_2}$.

(159) $\mathrm{Var}_{\exists_x H} = \mathrm{Var}_H \cup \{x\}$.

(160) $\mathrm{Var}_{\forall_{x,y} H} = \mathrm{Var}_H \cup \{x, y\}$.

(161) $\mathrm{Var}_{\exists_{x,y} H} = \mathrm{Var}_H \cup \{x, y\}$.

(162) $\mathrm{Var}_{\forall_{x,y,z} H} = \mathrm{Var}_H \cup \{x, y, z\}$.

(163) $\mathrm{Var}_{\exists_{x,y,z} H} = \mathrm{Var}_H \cup \{x, y, z\}$.

(164) $\mathrm{Free}\, H \subseteq \mathrm{Var}_H$.

Let us consider H. Then Var_H is a non-empty subset of VAR.

Let us consider H, x, y. The functor $H(\frac{x}{y})$ yields a function and is defined by:

(Def.4) $\mathrm{dom}(H(\frac{x}{y})) = \mathrm{dom}\, H$ and for every a such that $a \in \mathrm{dom}\, H$ holds if $H(a) = x$, then $(H(\frac{x}{y}))(a) = y$ but if $H(a) \neq x$, then $(H(\frac{x}{y}))(a) = H(a)$.

One can prove the following propositions:

(165) For every function f holds $f = H(\frac{x}{y})$ if and only if $\mathrm{dom}\, f = \mathrm{dom}\, H$ and for every a such that $a \in \mathrm{dom}\, H$ holds if $H(a) = x$, then $f(a) = y$ but if $H(a) \neq x$, then $f(a) = H(a)$.

(166) $x_1{=}x_2(\frac{y_1}{y_2}) = z_1{=}z_2$ if and only if $x_1 \neq y_1$ and $x_2 \neq y_1$ and $z_1 = x_1$ and $z_2 = x_2$ or $x_1 = y_1$ and $x_2 \neq y_1$ and $z_1 = y_2$ and $z_2 = x_2$ or $x_1 \neq y_1$ and $x_2 = y_1$ and $z_1 = x_1$ and $z_2 = y_2$ or $x_1 = y_1$ and $x_2 = y_1$ and $z_1 = y_2$ and $z_2 = y_2$.

(167) There exist z_1, z_2 such that $x_1{=}x_2(\frac{y_1}{y_2}) = z_1{=}z_2$.

(168) $x_1 \epsilon x_2(\frac{y_1}{y_2}) = z_1 \epsilon z_2$ if and only if $x_1 \neq y_1$ and $x_2 \neq y_1$ and $z_1 = x_1$ and $z_2 = x_2$ or $x_1 = y_1$ and $x_2 \neq y_1$ and $z_1 = y_2$ and $z_2 = x_2$ or $x_1 \neq y_1$ and $x_2 = y_1$ and $z_1 = x_1$ and $z_2 = y_2$ or $x_1 = y_1$ and $x_2 = y_1$ and $z_1 = y_2$ and $z_2 = y_2$.

(169) There exist z_1, z_2 such that $x_1 \epsilon x_2(\frac{y_1}{y_2}) = z_1 \epsilon z_2$.

(170) $\neg F = (\neg H)(\frac{x}{y})$ if and only if $F = H(\frac{x}{y})$.

(171) $H(\frac{x}{y}) \in \mathrm{WFF}$.

Let us consider H, x, y. Then $H(\frac{x}{y})$ is a ZF-formula.

The following propositions are true:

(172) $G_1 \wedge G_2 = (H_1 \wedge H_2)(\frac{x}{y})$ if and only if $G_1 = H_1(\frac{x}{y})$ and $G_2 = H_2(\frac{x}{y})$.

(173) If $z \neq x$, then $\forall_z G = (\forall_z H)(\frac{x}{y})$ if and only if $G = H(\frac{x}{y})$.

(174) $\forall_y G = (\forall_x H)(\frac{x}{y})$ if and only if $G = H(\frac{x}{y})$.

(175) $G_1 \vee G_2 = (H_1 \vee H_2)(\frac{x}{y})$ if and only if $G_1 = H_1(\frac{x}{y})$ and $G_2 = H_2(\frac{x}{y})$.

(176) $G_1 \Rightarrow G_2 = (H_1 \Rightarrow H_2)(\frac{x}{y})$ if and only if $G_1 = H_1(\frac{x}{y})$ and $G_2 = H_2(\frac{x}{y})$.

(177) $G_1 \Leftrightarrow G_2 = (H_1 \Leftrightarrow H_2)(\frac{x}{y})$ if and only if $G_1 = H_1(\frac{x}{y})$ and $G_2 = H_2(\frac{x}{y})$.

(178) If $z \neq x$, then $\exists_z G = (\exists_z H)(\frac{x}{y})$ if and only if $G = H(\frac{x}{y})$.

(179) $\exists_y G = (\exists_x H)(\frac{x}{y})$ if and only if $G = H(\frac{x}{y})$.

(180) H is an equality if and only if $H(\frac{x}{y})$ is an equality.

(181) H is a membership if and only if $H(\frac{x}{y})$ is a membership.

(182) H is negative if and only if $H(\frac{x}{y})$ is negative.

(183) H is conjunctive if and only if $H(\frac{x}{y})$ is conjunctive.

(184) H is universal if and only if $H(\frac{x}{y})$ is universal.

(185) If H is negative, then $\mathrm{Arg}(H(\frac{x}{y})) = \mathrm{Arg}(H)(\frac{x}{y})$.

(186) If H is conjunctive, then $\mathrm{LeftArg}(H(\frac{x}{y})) = \mathrm{LeftArg}(H)(\frac{x}{y})$ and $\mathrm{RightArg}(H(\frac{x}{y})) = \mathrm{RightArg}(H)(\frac{x}{y})$.

(187) If H is universal, then $\mathrm{Scope}(H(\frac{x}{y})) = \mathrm{Scope}(H)(\frac{x}{y})$ but if $\mathrm{Bound}(H) = x$, then $\mathrm{Bound}(H(\frac{x}{y})) = y$ but if $\mathrm{Bound}(H) \neq x$, then $\mathrm{Bound}(H(\frac{x}{y})) = \mathrm{Bound}(H)$.

(188) H is disjunctive if and only if $H(\frac{x}{y})$ is disjunctive.

(189) H is conditional if and only if $H(\frac{x}{y})$ is conditional.

(190) If H is biconditional, then $H(\frac{x}{y})$ is biconditional.

(191) H is existential if and only if $H(\frac{x}{y})$ is existential.

(192) If H is disjunctive, then $\mathrm{LeftArg}(H(\frac{x}{y})) = \mathrm{LeftArg}(H)(\frac{x}{y})$ and $\mathrm{RightArg}(H(\frac{x}{y})) = \mathrm{RightArg}(H)(\frac{x}{y})$.

(193) If H is conditional, then $\mathrm{Antecedent}(H(\frac{x}{y})) = \mathrm{Antecedent}(H)(\frac{x}{y})$ and $\mathrm{Consequent}(H(\frac{x}{y})) = \mathrm{Consequent}(H)(\frac{x}{y})$.

(194) If H is biconditional, then $\mathrm{LeftSide}(H(\frac{x}{y})) = \mathrm{LeftSide}(H)(\frac{x}{y})$ and $\mathrm{RightSide}(H(\frac{x}{y})) = \mathrm{RightSide}(H)(\frac{x}{y})$.

(195) If H is existential, then $\mathrm{Scope}(H(\frac{x}{y})) = \mathrm{Scope}(H)(\frac{x}{y})$ but if $\mathrm{Bound}(H) = x$, then $\mathrm{Bound}(H(\frac{x}{y})) = y$ but if $\mathrm{Bound}(H) \neq x$, then $\mathrm{Bound}(H(\frac{x}{y})) = \mathrm{Bound}(H)$.

(196) If $x \notin \mathrm{Var}_H$, then $H(\frac{x}{y}) = H$.

(197) $H(\frac{x}{x}) = H$.

(198) If $x \neq y$, then $x \notin \mathrm{Var}_{H(\frac{x}{y})}$.

(199) If $x \in \mathrm{Var}_H$, then $y \in \mathrm{Var}_{H(\frac{x}{y})}$.

(200) If $x \neq y$, then $(H(\frac{x}{y}))(\frac{x}{z}) = H(\frac{x}{y})$.

(201) $\mathrm{Var}_{H(\frac{x}{y})} \subseteq (\mathrm{Var}_H \setminus \{x\}) \cup \{y\}$.

References

[1] Grzegorz Bancerek. A model of ZF set theory language. *Formalized Mathematics*, 1(**1**):131–145, 1990.

[2] Grzegorz Bancerek. Models and satisfiability. *Formalized Mathematics*, 1(**1**):191–199, 1990.

[3] Grzegorz Bancerek. Properties of zf models. *Formalized Mathematics*, 1(**2**):277–280, 1990.

[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[7] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[8] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.

[9] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

Received August 10, 1990

FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

The Reflection Theorem

Grzegorz Bancerek[1]
Warsaw University
Białystok

Summary. The goal is show that the reflection theorem holds. The theorem is as usual in the Morse-Kelley theory of classes (MK). That theory works with universal class which consists of all sets and every class is a subclass of it. In this paper (and in another Mizar articles) we work in Tarski-Grothendieck (TG) theory (see [16]) which ensures the existence of sets that have properties like universal class (i.e. this theory is stronger than MK). The sets are introduced in [14] and some concepts of MK are modeled. The concepts are: the class On of all ordinal numbers belonging to the universe, subclasses, transfinite sequences of non-empty elements of universe, etc. The reflection theorem states that if A_ξ is an increasing and continuous transfinite sequence of non-empty sets and class $A = \bigcup_{\xi \in On} A_\xi$, then for every formula H there is a strictly increasing continuous mapping $F : On \to On$ such that if $\varkappa$ is a critical number of F (i.e. $F(\varkappa) = \varkappa > 0$) and $f \in A_\varkappa^{\mathbf{VAR}}$, then $A, f \models H \equiv A_\varkappa, f \models H$. The proof is based on [13]. Besides, in the article it is shown that every universal class is a model of ZF set theory if ω (the first infinite ordinal number) belongs to it. Some propositions concerning ordinal numbers and sequences of them are also present.

MML Identifier: ZF_REFLE.

The notation and terminology used in this paper have been introduced in the following articles: [16], [15], [11], [12], [4], [5], [6], [10], [8], [1], [3], [9], [14], [2], and [7]. In the sequel W is a universal class, H is a ZF-formula, x is arbitrary, and X is a set. We now state several propositions:

(1) $W \models$ the axiom of extensionality.

(2) $W \models$ the axiom of pairs.

(3) $W \models$ the axiom of unions.

(4) If $\omega \in W$, then $W \models$ the axiom of infinity.

(5) $W \models$ the axiom of power sets.

[1]Supported by RPBP III-24.C1.

ISSN 0777–4028

(6) For every H such that $\{x_0, x_1, x_2\}$ misses $\mathrm{Free}\, H$ holds $W \models$ the axiom of substitution for H.

(7) If $\omega \in W$, then W is a model of ZF.

For simplicity we follow the rules: E denotes a non-empty family of sets, F denotes a function, f denotes a function from VAR into E, A, B, C denote ordinal numbers, a, b denote ordinals of W, p_1 denotes a transfinite sequence of ordinals of W, and H denotes a ZF-formula. Let us consider A, B. Let us note that one can characterize the predicate $A \subseteq B$ by the following (equivalent) condition:

(Def.1) for every C such that $C \in A$ holds $C \in B$.

In this article we present several logical schemes. The scheme *ALFA* deals with a non-empty set $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists F such that $\mathrm{dom}\, F = \mathcal{A}$ and for every element d of $\mathcal{A}$ there exists A such that $A = F(d)$ and $\mathcal{P}[d, A]$ and for every B such that $\mathcal{P}[d, B]$ holds $A \subseteq B$

provided the parameters meet the following condition:

- for every element d of $\mathcal{A}$ there exists A such that $\mathcal{P}[d, A]$.

The scheme *ALFA'Universe* deals with a universal class $\mathcal{A}$, a non-empty set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

there exists F such that $\mathrm{dom}\, F = \mathcal{B}$ and for every element d of $\mathcal{B}$ there exists an ordinal a of $\mathcal{A}$ such that $a = F(d)$ and $\mathcal{P}[d, a]$ and for every ordinal b of $\mathcal{A}$ such that $\mathcal{P}[d, b]$ holds $a \subseteq b$

provided the following condition is met:

- for every element d of $\mathcal{B}$ there exists an ordinal a of $\mathcal{A}$ such that $\mathcal{P}[d, a]$.

One can prove the following proposition

(8) x is an ordinal of W if and only if $x \in \mathrm{On}\, W$.

In the sequel p_2 is a sequence of ordinal numbers. Now we present three schemes. The scheme *OrdSeqOfUnivEx* deals with a universal class $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a transfinite sequence p_1 of ordinals of $\mathcal{A}$ such that for every ordinal a of $\mathcal{A}$ holds $\mathcal{P}[a, p_1(a)]$

provided the following conditions are satisfied:

- for all ordinals a, b_1, b_2 of $\mathcal{A}$ such that $\mathcal{P}[a, b_1]$ and $\mathcal{P}[a, b_2]$ holds $b_1 = b_2$,
- for every ordinal a of $\mathcal{A}$ there exists an ordinal b of $\mathcal{A}$ such that $\mathcal{P}[a, b]$.

The scheme *UOS_Exist* concerns a universal class $\mathcal{A}$, an ordinal $\mathcal{B}$ of $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding an ordinal of $\mathcal{A}$, and a binary functor $\mathcal{G}$ yielding an ordinal of $\mathcal{A}$ and states that:

there exists a transfinite sequence p_1 of ordinals of $\mathcal{A}$ such that $p_1(\mathbf{0}_{\mathcal{A}}) = \mathcal{B}$ and for all ordinals a, b of $\mathcal{A}$ such that $b = p_1(a)$ holds $p_1(\mathrm{succ}\, a) = \mathcal{F}(a, b)$ and for every ordinal a of $\mathcal{A}$ and for every sequence p_2 of ordinal numbers such that $a \neq \mathbf{0}_{\mathcal{A}}$ and a is a limit ordinal number and $p_2 = p_1 \restriction a$ holds $p_1(a) = \mathcal{G}(a, p_2)$

for all values of the parameters.

The scheme *Universe_Ind* concerns a universal class $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

for every ordinal a of $\mathcal{A}$ holds $\mathcal{P}[a]$

provided the parameters have the following properties:

- $\mathcal{P}[\mathbf{0}_{\mathcal{A}}]$,
- for every ordinal a of $\mathcal{A}$ such that $\mathcal{P}[a]$ holds $\mathcal{P}[\operatorname{succ} a]$,
- for every ordinal a of $\mathcal{A}$ such that $a \neq \mathbf{0}_{\mathcal{A}}$ and a is a limit ordinal number and for every ordinal b of $\mathcal{A}$ such that $b \in a$ holds $\mathcal{P}[b]$ holds $\mathcal{P}[a]$.

Let f be a function, and let W be a universal class, and let a be an ordinal of W. The functor $\bigcup_a f$ yields a set and is defined as follows:

(Def.2) $\bigcup_a f = \bigcup(W \restriction (f \restriction \mathbf{R}_a))$.

We now state several propositions:

(9) $\bigcup_a f = \bigcup(W \restriction (f \restriction \mathbf{R}_a))$.

(10) For every transfinite sequence L and for every A holds $L \restriction \mathbf{R}_A$ is a transfinite sequence.

(11) For every sequence L of ordinal numbers and for every A holds $L \restriction \mathbf{R}_A$ is a sequence of ordinal numbers.

(12) $\bigcup p_2$ is an ordinal number.

(13) $\bigcup(X \restriction p_2)$ is an ordinal number.

(14) $\operatorname{On} \mathbf{R}_A = A$.

(15) $p_2 \restriction \mathbf{R}_A = p_2 \restriction A$.

Let p_1 be a sequence of ordinal numbers, and let W be a universal class, and let a be an ordinal of W. Then $\bigcup_a p_1$ is an ordinal of W.

Next we state the proposition

(17)[2] For every transfinite sequence p_1 of ordinals of W holds $\bigcup_a p_1 = \bigcup(p_1 \restriction a)$ and $\bigcup_a(p_1 \restriction a) = \bigcup(p_1 \restriction a)$.

Let W be a universal class, and let a, b be ordinals of W. Then $a \cup b$ is an ordinal of W.

Let us consider W. A non-empty family of sets is said to be a non-empty set from W if:

(Def.3) it $\in W$.

Let us consider W. A non-empty family of sets is said to be a subclass of W if:

(Def.4) it $\subseteq W$.

Let us consider W. A transfinite sequence of elements of W is called a transfinite sequence of non-empty sets from W if:

(Def.5) $\operatorname{dom} \mathrm{it} = \operatorname{On} W$ and $\emptyset \notin \operatorname{rng} \mathrm{it}$.

[2]The proposition (16) became obvious.

We now state four propositions:

(18) E is a non-empty set from W if and only if $E \in W$.

(19) E is a subclass of W if and only if $E \subseteq W$.

(20) For every transfinite sequence T of elements of W holds T is a transfinite sequence of non-empty sets from W if and only if $\operatorname{dom} T = \operatorname{On} W$ and $\emptyset \notin \operatorname{rng} T$.

(21) For every non-empty set D from W holds D is a subclass of W.

Let us consider W, and let L be a transfinite sequence of non-empty sets from W. Then $\bigcup L$ is a subclass of W. Let us consider a. Then $L(a)$ is a non-empty set from W.

In the sequel L is a transfinite sequence of non-empty sets from W and f is a function from VAR into $L(a)$. Next we state several propositions:

(22) If $X \in W$, then $\overline{\overline{X}} < \overline{\overline{W}}$.

(23) $a \in \operatorname{dom} L$.

(24) $L(a) \subseteq \bigcup L$.

(25) $\mathbb{N} \approx \mathrm{VAR}$ and $\overline{\overline{\mathrm{VAR}}} = \overline{\overline{\mathbb{N}}}$.

(26) $\bigcup(\operatorname{On} X)$ is an ordinal number.

(27) $\sup X \subseteq \operatorname{succ}(\bigcup(\operatorname{On} X))$.

(28) If $X \in W$, then $\sup X \in W$.

The following proposition is true

(29) Suppose $\omega \in W$ and for all a, b such that $a \in b$ holds $L(a) \subseteq L(b)$ and for every a such that $a \neq \mathbf{0}$ and a is a limit ordinal number holds $L(a) = \bigcup(L \restriction a)$. Then for every H there exists p_1 such that p_1 is increasing and p_1 is continuous and for every a such that $p_1(a) = a$ and $\mathbf{0} \neq a$ for every f holds $\bigcup L, \bigcup L[f] \models H$ if and only if $L(a), f \models H$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[2] Grzegorz Bancerek. Increasing and continuous ordinal sequences. *Formalized Mathematics*, 1(**4**):711–714, 1990.

[3] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.

[4] Grzegorz Bancerek. A model of ZF set theory language. *Formalized Mathematics*, 1(**1**):131–145, 1990.

[5] Grzegorz Bancerek. Models and satisfiability. *Formalized Mathematics*, 1(**1**):191–199, 1990.

[6] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[7] Grzegorz Bancerek. Replacing of variables in formulas of ZF theory. *Formalized Mathematics*, 1(**5**):963–972, 1990.

[8] Grzegorz Bancerek. Sequences of ordinal numbers. *Formalized Mathematics*, 1(**2**):281–290, 1990.

[9] Grzegorz Bancerek. Tarski's classes and ranks. *Formalized Mathematics*, 1(**3**):563–567, 1990.

[10] Grzegorz Bancerek. Zermelo theorem and axiom of choice. *Formalized Mathematics*, 1(**2**):265–267, 1990.

[11] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[12] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[13] Andrzej Mostowski. *Constructible Sets with Applications.* North Holland, 1969.

[14] Bogdan Nowak and Grzegorz Bancerek. Universal classes. *Formalized Mathematics*, 1(**3**):595–600, 1990.

[15] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.

[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

Received August 10, 1990

FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Binary Operations on Finite Sequences

Wojciech A. Trybulec[1]
Warsaw University

Summary. We generalize the semigroup operation on finite sequences introduced in [6] for binary operations that have a unity or for non-empty finite sequences.

MML Identifier: `FINSOP_1`.

The papers [9], [4], [5], [2], [3], [8], [6], [7], and [1] provide the notation and terminology for this paper. For simplicity we adopt the following convention: D denotes a non-empty set, d, d_1, d_2, d_3 denote elements of D, F, G, H denote finite sequences of elements of D, f denotes a function from $\mathbb{N}$ into D, g denotes a binary operation on D, k, n, l denote natural numbers, and P denotes a permutation of $\mathrm{Seg}(\mathrm{len}\, F)$. Let us consider D, n, d. Then $n \longmapsto d$ is a finite sequence of elements of D.

Let us consider D, F, g. Let us assume that g has a unity or $\mathrm{len}\, F \geq 1$. The functor $g \odot F$ yields an element of D and is defined by:

(Def.1) $g \odot F = \mathbf{1}_g$ if g has a unity and $\mathrm{len}\, F = 0$, there exists f such that $f(1) = F(1)$ and for every n such that $0 \neq n$ and $n < \mathrm{len}\, F$ holds $f(n+1) = g(f(n), F(n+1))$ and $g \odot F = f(\mathrm{len}\, F)$, otherwise.

One can prove the following propositions:

(1) If g has a unity and $\mathrm{len}\, F = 0$, then $g \odot F = \mathbf{1}_g$.

(2) Suppose $\mathrm{len}\, F \geq 1$. Then there exists f such that $f(1) = F(1)$ and for every n such that $0 \neq n$ and $n < \mathrm{len}\, F$ holds $f(n+1) = g(f(n), F(n+1))$ and $g \odot F = f(\mathrm{len}\, F)$.

(3) Suppose $\mathrm{len}\, F \geq 1$ and there exists f such that $f(1) = F(1)$ and for every n such that $0 \neq n$ and $n < \mathrm{len}\, F$ holds $f(n+1) = g(f(n), F(n+1))$ and $d = f(\mathrm{len}\, F)$. Then $d = g \odot F$.

(4) If g has a unity or $\mathrm{len}\, F \geq 1$ but g is associative and g is commutative, then $g \odot F = g \circledast F$.

[1]Supported by RPBP.III-24.C1

ISSN 0777–4028

(5) If g has a unity or $\operatorname{len} F \geq 1$, then $g \odot F \frown \langle d \rangle = g(g \odot F, d)$.

(6) If g is associative but g has a unity or $\operatorname{len} F \geq 1$ and $\operatorname{len} G \geq 1$, then $g \odot F \frown G = g(g \odot F, g \odot G)$.

(7) If g is associative but g has a unity or $\operatorname{len} F \geq 1$, then $g \odot \langle d \rangle \frown F = g(d, g \odot F)$.

(8) If g is commutative and g is associative but g has a unity or $\operatorname{len} F \geq 1$ and $G = F \cdot P$, then $g \odot F = g \odot G$.

(9) If g has a unity or $\operatorname{len} F \geq 1$ but g is associative and g is commutative and F is one-to-one and G is one-to-one and $\operatorname{rng} F = \operatorname{rng} G$, then $g \odot F = g \odot G$.

(10) Suppose g is associative and g is commutative but g has a unity or $\operatorname{len} F \geq 1$ and $\operatorname{len} F = \operatorname{len} G$ and $\operatorname{len} F = \operatorname{len} H$ and for every k such that $k \in \operatorname{Seg}(\operatorname{len} F)$ holds $F(k) = g(G(k), H(k))$. Then $g \odot F = g(g \odot G, g \odot H)$.

(11) If g has a unity, then $g \odot \varepsilon_D = \mathbf{1}_g$.

(12) $g \odot \langle d \rangle = d$.

(13) $g \odot \langle d_1, d_2 \rangle = g(d_1, d_2)$.

(14) If g is commutative, then $g \odot \langle d_1, d_2 \rangle = g \odot \langle d_2, d_1 \rangle$.

(15) $g \odot \langle d_1, d_2, d_3 \rangle = g(g(d_1, d_2), d_3)$.

(16) If g is commutative, then $g \odot \langle d_1, d_2, d_3 \rangle = g \odot \langle d_2, d_1, d_3 \rangle$.

(17) $g \odot (1 \longmapsto d) = d$.

(18) $g \odot (2 \longmapsto d) = g(d, d)$.

(19) If g is associative but g has a unity or $k \neq 0$ and $l \neq 0$, then $g \odot (k+l \longmapsto d) = g(g \odot (k \longmapsto d), g \odot (l \longmapsto d))$.

(20) If g is associative but g has a unity or $k \neq 0$ and $l \neq 0$, then $g \odot (k \cdot l \longmapsto d) = g \odot (l \longmapsto g \odot (k \longmapsto d))$.

(21) If $\operatorname{len} F = 1$, then $g \odot F = F(1)$.

(22) If $\operatorname{len} F = 2$, then $g \odot F = g(F(1), F(2))$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Czesław Byliński. Semigroup operations on finite subsets. *Formalized Mathematics*, 1(**4**):651–656, 1990.

[7] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[8] Andrzej Trybulec. Semilattice operations on finite subsets. *Formalized Mathematics*, 1(**2**):369–376, 1990.

[9] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

Received August 10, 1990

FORMALIZED MATHEMATICS
Vol.1, No.**5**, November–December 1990
Université Catholique de Louvain

Finite Join and Finite Meet, and Dual Lattices

Andrzej Trybulec[1]
Warsaw University
Białystok

Summary. The concepts of finite join and finite meet in a lattice are introduced. Some properties of the finite join are proved. After introducing the concept of dual lattice in view of dualism we obtain analogous properties of the meet. We prove these properties of binary operations in a lattice, which are usually included in axioms of the lattice theory. We also introduce the concept of Heyting lattice (a bounded lattice with relative pseudo-complements).

MML Identifier: `LATTICE2`.

The papers [10], [3], [4], [5], [8], [2], [11], [6], [9], [7], and [1] provide the notation and terminology for this paper. For simplicity we adopt the following convention: A denotes a set, C denotes a non-empty set, B denotes a subset of A, x denotes an element of A, and f, g denote functions from A into C. The following propositions are true:

(1) $f \restriction B$ is a function from B into C.

(2) $\mathrm{dom}(g \restriction B) = B$.

(3) $f \circ B = (f \restriction B) \circ B$.

(4) If $x \in B$, then $(f \restriction B)(x) = f(x)$.

(5) $f \restriction B = g \restriction B$ if and only if for every x such that $x \in B$ holds $g(x) = f(x)$.

(6) For every set B holds $f +\cdot g \restriction B$ is a function from A into C.

(7) $g \restriction B +\cdot f = f$.

(8) For all functions f, g such that $g \leq f$ holds $f +\cdot g = f$.

(9) $f +\cdot f \restriction B = f$.

(10) If for every x such that $x \in B$ holds $g(x) = f(x)$, then $f +\cdot g \restriction B = f$.

[1]Supported by RPBP.III-24.C1.

ISSN 0777–4028

In the sequel B will denote a finite subset of A. We now state several propositions:

(11) For every set X holds X is a finite subset of A if and only if $X \subseteq A$ and X is finite.

(12) $g \restriction B +\cdot f = f$.

(13) $\mathrm{dom}(g \restriction B) = B$.

(14) If for every x such that $x \in B$ holds $g(x) = f(x)$, then $f +\cdot g \restriction B = f$.

(15) $f \circ B = (f \restriction B) \circ B$.

(16) If $f \restriction B = g \restriction B$, then $f \circ B = g \circ B$.

Let D be a non-empty set, and let o, o' be binary operations on D. We say that o absorbs o' if and only if:

(Def.1) for all elements x, y of D holds $o(x, o'(x, y)) = x$.

In the sequel L will be a lattice structure. The following proposition is true

(17) If the join operation of L is commutative and the join operation of L is associative and the meet operation of L is commutative and the meet operation of L is associative and the join operation of L absorbs the meet operation of L and the meet operation of L absorbs the join operation of L, then L is a lattice.

Let L be a lattice structure. The functor L° yields a lattice structure and is defined by:

(Def.2) $L^\circ = \langle$ the carrier of L, the meet operation of L, the join operation of $L\rangle$.

One can prove the following propositions:

(18) The carrier of $L =$ the carrier of L° and the join operation of $L =$ the meet operation of L° and the meet operation of $L =$ the join operation of L°.

(19) $(L^\circ)^\circ = L$.

We follow the rules: L will be a lattice and a, b, u, v will be elements of the carrier of L. We now state a number of propositions:

(20) If for every v holds $u \sqcap v = u$, then $u = \perp_L$.

(21) If for every v holds $u \sqcup v = v$, then $u = \perp_L$.

(22) If for every v holds (the join operation of L)$(u, v) = v$, then $u = \perp_L$.

(23) If for every v holds $u \sqcup v = u$, then $u = \top_L$.

(24) If for every v holds $u \sqcap v = v$, then $u = \top_L$.

(25) If for every v holds (the meet operation of L)$(u, v) = v$, then $u = \top_L$.

(26) The join operation of L is idempotent.

(27) The join operation of L is commutative.

(28) If the join operation of L has a unity, then $\perp_L = \mathbf{1}_{\text{the join operation of } L}$.

(29) The join operation of L is associative.

(30) The meet operation of L is idempotent.

(31) The meet operation of L is commutative.

(32) The meet operation of L is associative.

(33) If the meet operation of L has a unity, then $\top_L = \mathbf{1}_{\text{the meet operation of } L}$.

(34) The join operation of L is distributive w.r.t. the join operation of L.

(35) If L is a distributive lattice, then the join operation of L is distributive w.r.t. the meet operation of L.

(36) If the join operation of L is distributive w.r.t. the meet operation of L, then L is a distributive lattice.

(37) If L is a distributive lattice, then the meet operation of L is distributive w.r.t. the join operation of L.

(38) If the meet operation of L is distributive w.r.t. the join operation of L, then L is a distributive lattice.

(39) The meet operation of L is distributive w.r.t. the meet operation of L.

(40) The join operation of L absorbs the meet operation of L.

(41) The meet operation of L absorbs the join operation of L.

We now define two new functors. Let A be a non-empty set, and let L be a lattice, and let B be a finite subset of A, and let f be a function from A into the carrier of L. The functor $\bigsqcup^{\mathrm{f}}_B f$ yields an element of the carrier of L and is defined as follows:

(Def.3) $\bigsqcup^{\mathrm{f}}_B f =$ (the join operation of L)- $\sum_B f$.

The functor $\bigsqcap^{\mathrm{f}}_B f$ yields an element of the carrier of L and is defined by:

(Def.4) $\bigsqcap^{\mathrm{f}}_B f =$ (the meet operation of L)- $\sum_B f$.

We now state the proposition

(42) For every non-empty set A and for every lattice L and for every finite subset B of A and for every function f from A into the carrier of L holds $\bigsqcup^{\mathrm{f}}_B f =$ (the join operation of L)- $\sum_B f$.

For simplicity we adopt the following convention: A will be a non-empty set, x will be an element of A, B will be a finite subset of A, and f, g will be functions from A into the carrier of L. Next we state several propositions:

(43) If $x \in B$, then $f(x) \sqsubseteq \bigsqcup^{\mathrm{f}}_B f$.

(44) If there exists x such that $x \in B$ and $u \sqsubseteq f(x)$, then $u \sqsubseteq \bigsqcup^{\mathrm{f}}_B f$.

(45) If for every x such that $x \in B$ holds $f(x) = u$ and $B \neq \emptyset$, then $\bigsqcup^{\mathrm{f}}_B f = u$.

(46) If $\bigsqcup^{\mathrm{f}}_B f \sqsubseteq u$, then for every x such that $x \in B$ holds $f(x) \sqsubseteq u$.

(47) If $B \neq \emptyset$ and for every x such that $x \in B$ holds $f(x) \sqsubseteq u$, then $\bigsqcup^{\mathrm{f}}_B f \sqsubseteq u$.

(48) If $B \neq \emptyset$ and for every x such that $x \in B$ holds $f(x) \sqsubseteq g(x)$, then $\bigsqcup^{\mathrm{f}}_B f \sqsubseteq \bigsqcup^{\mathrm{f}}_B g$.

(49) If $B \neq \emptyset$ and $f \restriction B = g \restriction B$, then $\bigsqcup^{\mathrm{f}}_B f = \bigsqcup^{\mathrm{f}}_B g$.

(50) If $B \neq \emptyset$, then $v \sqcup \bigsqcup^{\mathrm{f}}_B f = \bigsqcup^{\mathrm{f}}_B($ (the join operation of $L)^{\circ}(v, f))$.

Let L be a lattice. Then L° is a lattice.

We now state a number of propositions:

(51) For every lattice L and for every finite subset B of A and for every function f from A into the carrier of L and for every function f' from A into the carrier of L° such that $f = f'$ holds $\bigsqcup^{\mathrm{f}}_{B} f = \bigsqcap^{\mathrm{f}}_{B} f'$ and $\bigsqcap^{\mathrm{f}}_{B} f = \bigsqcup^{\mathrm{f}}_{B} f'$.

(52) For all elements a', b' of the carrier of L° such that $a = a'$ and $b = b'$ holds $a \sqcap b = a' \sqcup b'$ and $a \sqcup b = a' \sqcap b'$.

(53) If $a \sqsubseteq b$, then for all elements a', b' of the carrier of L° such that $a = a'$ and $b = b'$ holds $b' \sqsubseteq a'$.

(54) For all elements a', b' of the carrier of L° such that $a' \sqsubseteq b'$ and $a = a'$ and $b = b'$ holds $b \sqsubseteq a$.

(55) If $x \in B$, then $\bigsqcap^{\mathrm{f}}_{B} f \sqsubseteq f(x)$.

(56) If there exists x such that $x \in B$ and $f(x) \sqsubseteq u$, then $\bigsqcap^{\mathrm{f}}_{B} f \sqsubseteq u$.

(57) If for every x such that $x \in B$ holds $f(x) = u$ and $B \neq \emptyset$, then $\bigsqcap^{\mathrm{f}}_{B} f = u$.

(58) If $B \neq \emptyset$, then $v \sqcap \bigsqcap^{\mathrm{f}}_{B} f = \bigsqcap^{\mathrm{f}}_{B}($ (the meet operation of $L)^{\circ}(v, f))$.

(59) If $u \sqsubseteq \bigsqcap^{\mathrm{f}}_{B} f$, then for every x such that $x \in B$ holds $u \sqsubseteq f(x)$.

(60) If $B \neq \emptyset$ and $f \restriction B = g \restriction B$, then $\bigsqcap^{\mathrm{f}}_{B} f = \bigsqcap^{\mathrm{f}}_{B} g$.

(61) If $B \neq \emptyset$ and for every x such that $x \in B$ holds $u \sqsubseteq f(x)$, then $u \sqsubseteq \bigsqcap^{\mathrm{f}}_{B} f$.

(62) If $B \neq \emptyset$ and for every x such that $x \in B$ holds $f(x) \sqsubseteq g(x)$, then $\bigsqcap^{\mathrm{f}}_{B} f \sqsubseteq \bigsqcap^{\mathrm{f}}_{B} g$.

(63) For every lattice L holds L is a lower bound lattice if and only if L° is an upper bound lattice.

(64) For every lattice L holds L is an upper bound lattice if and only if L° is a lower bound lattice.

(65) L is a distributive lattice if and only if L° is a distributive lattice.

In the sequel L denotes a lower bound lattice, f, g denote functions from A into the carrier of L, and u denotes an element of the carrier of L. The following propositions are true:

(66) $\perp_{L}$ is a unity w.r.t. the join operation of L.

(67) The join operation of L has a unity.

(68) $\perp_{L} = \mathbf{1}_{\text{the join operation of } L}$.

(69) If $f \restriction B = g \restriction B$, then $\bigsqcup^{\mathrm{f}}_{B} f = \bigsqcup^{\mathrm{f}}_{B} g$.

(70) If for every x such that $x \in B$ holds $f(x) \sqsubseteq u$, then $\bigsqcup^{\mathrm{f}}_{B} f \sqsubseteq u$.

(71) If for every x such that $x \in B$ holds $f(x) \sqsubseteq g(x)$, then $\bigsqcup^{\mathrm{f}}_{B} f \sqsubseteq \bigsqcup^{\mathrm{f}}_{B} g$.

In the sequel L will denote an upper bound lattice, f, g will denote functions from A into the carrier of L, and u will denote an element of the carrier of L. The following propositions are true:

(72) $\top_{L}$ is a unity w.r.t. the meet operation of L.

(73) The meet operation of L has a unity.

(74) $\top_{L} = \mathbf{1}_{\text{the meet operation of } L}$.

(75) If $f \restriction B = g \restriction B$, then $\bigsqcap^{\mathrm{f}}_{B} f = \bigsqcap^{\mathrm{f}}_{B} g$.

(76) If for every x such that $x \in B$ holds $u \sqsubseteq f(x)$, then $u \sqsubseteq \bigsqcap^{\rm f}_B f$.

(77) If for every x such that $x \in B$ holds $f(x) \sqsubseteq g(x)$, then $\bigsqcap^{\rm f}_B f \sqsubseteq \bigsqcap^{\rm f}_B g$.

(78) For every lower bound lattice L holds $\perp_L = \top_{L^\circ}$.

(79) For every upper bound lattice L holds $\top_L = \perp_{L^\circ}$.

A lower bound lattice is called a distributive lower bounded lattice if:

(Def.5) it is a distributive lattice.

In the sequel L will denote a distributive lower bounded lattice, f, g will denote functions from A into the carrier of L, and u will denote an element of the carrier of L. We now state four propositions:

(80) The meet operation of L is distributive w.r.t. the join operation of L.

(81) (the meet operation of L)$(u, \bigsqcup^{\rm f}_B f) = \bigsqcup^{\rm f}_B($ (the meet operation of $L)^\circ(u, f))$.

(82) If for every x such that $x \in B$ holds $g(x) = u \sqcap f(x)$, then $u \sqcap \bigsqcup^{\rm f}_B f = \bigsqcup^{\rm f}_B g$.

(83) $u \sqcap \bigsqcup^{\rm f}_B f = \bigsqcup^{\rm f}_B($ (the meet operation of $L)^\circ(u, f))$.

A lower bound lattice is said to be a Heyting lattice if:

(Def.6) it is a implicative lattice.

Next we state the proposition

(84) For every lower bound lattice L holds L is a Heyting lattice if and only if for every elements x, z of the carrier of L there exists an element y of the carrier of L such that $x \sqcap y \sqsubseteq z$ and for every element v of the carrier of L such that $x \sqcap v \sqsubseteq z$ holds $v \sqsubseteq y$.

Let L be a lattice. We say that L is finite if and only if:

(Def.7) the carrier of L is finite.

We now state several propositions:

(85) For every lattice L holds L is finite if and only if L° is finite.

(86) For every lattice L such that L is finite holds L is a lower bound lattice.

(87) For every lattice L such that L is finite holds L is an upper bound lattice.

(88) For every lattice L such that L is finite holds L is a bound lattice.

(89) For every distributive lattice L such that L is finite holds L is a Heyting lattice.

References

[1] Grzegorz Bancerek. Filters - part I. *Formalized Mathematics*, 1(**5**):813–819, 1990.

[2] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.

[6] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[7] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[8] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[9] Andrzej Trybulec. Semilattice operations on finite subsets. *Formalized Mathematics*, 1(**2**):369–376, 1990.

[10] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[11] Stanisław Żukowski. Introduction to lattice theory. *Formalized Mathematics*, 1(**1**):215–222, 1990.

Received August 10, 1990

FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Consequences of the Reflection Theorem

Grzegorz Bancerek[1]
Warsaw University
Białystok

Summary. Some consequences of the reflection theorem are discussed. To formulate them the notions of elementary equivalence and subsystems, and of models for a set of formulae are introduced. Besides, the concept of cofinality of a ordinal number with second one is used. The consequences of the reflection theorem (it is sometimes called the Scott-Scarpellini lemma) are: (i) If A_ξ is a transfinite sequence as in the reflection theorem (see [9]) and $A = \bigcup_{\xi \in On} A_\xi$, then there is an increasing and continuous mapping ϕ from On into On such that for every critical number κ the set A_κ is an elementary subsystem of A ($A_\kappa \prec A$). (ii) There is an increasing continuous mapping $\phi : On \to On$ such that $\mathbf{R}_\kappa \prec V$ for each of its critical numbers κ (V is the universal class and On is the class of all ordinals belonging to V). (iii) There are ordinal numbers α cofinal with ω for which $\mathbf{R}_\alpha$ are models of ZF set theory. (iv) For each set X from universe V there is a model of ZF M which belongs to V and has X as an element.

MML Identifier: ZFREFLE1.

The articles [18], [14], [15], [19], [17], [8], [13], [5], [6], [1], [11], [4], [2], [7], [12], [16], [3], [10], and [9] provide the terminology and notation for this paper. We follow a convention: H, S will be ZF-formulae, X, Y will be sets, and e, u will be arbitrary. Let M be a non-empty family of sets, and let F be a subset of WFF. The predicate $M \models F$ is defined by:

(Def.1) for every H such that $H \in F$ holds $M \models H$.

We now define two new predicates. Let M_1, M_2 be non-empty families of sets. The predicate $M_1 \equiv M_2$ is defined as follows:

(Def.2) for every H such that $\mathrm{Free}\, H = \emptyset$ holds $M_1 \models H$ if and only if $M_2 \models H$.

Let us notice that this predicate is reflexive and symmetric. The predicate $M_1 \prec M_2$ is defined as follows:

[1] Supported by RPBP III-24.C1.

ISSN 0777–4028

(Def.3) $M_1 \subseteq M_2$ and for every H and for every function v from VAR into M_1 holds $M_1, v \models H$ if and only if $M_2, M_2[v] \models H$.

Let us observe that the predicate introduced above is reflexive.

The set $\mathbf{Ax}_{\mathrm{ZF}}$ is defined by:

(Def.4) $e \in \mathbf{Ax}_{\mathrm{ZF}}$ if and only if $e \in$ WFF but $e =$ the axiom of extensionality or $e =$ the axiom of pairs or $e =$ the axiom of unions or $e =$ the axiom of infinity or $e =$ the axiom of power sets or there exists H such that $\{x_0, x_1, x_2\}$ misses Free H and $e =$ the axiom of substitution for H.

Let us note that it makes sense to consider the following constant. Then $\mathbf{Ax}_{\mathrm{ZF}}$ is a subset of WFF.

Let D be a non-empty set. Then $\emptyset_D$ is a subset of D.

For simplicity we follow a convention: M, M_1, M_2 will be non-empty families of sets, f will be a function, F, F_1, F_2 will be subsets of WFF, W will be a universal class, a, b will be ordinals of W, A, B, C will be ordinal numbers, L will be a transfinite sequence of non-empty sets from W, and p_1, x_1 will be transfinite sequences of ordinals of W. We now state a number of propositions:

(1) $M \models \emptyset_{\mathrm{WFF}}$.

(2) If $F_1 \subseteq F_2$ and $M \models F_2$, then $M \models F_1$.

(3) If $M \models F_1$ and $M \models F_2$, then $M \models F_1 \cup F_2$.

(4) If M is a model of ZF, then $M \models \mathbf{Ax}_{\mathrm{ZF}}$.

(5) If $M \models \mathbf{Ax}_{\mathrm{ZF}}$ and M is transitive, then M is a model of ZF.

(6) There exists S such that Free $S = \emptyset$ and for every M holds $M \models S$ if and only if $M \models H$.

(7) $M_1 \equiv M_2$ if and only if for every H holds $M_1 \models H$ if and only if $M_2 \models H$.

(8) $M_1 \equiv M_2$ if and only if for every F holds $M_1 \models F$ if and only if $M_2 \models F$.

(9) If $M_1 \prec M_2$, then $M_1 \equiv M_2$.

(10) If M_1 is a model of ZF and $M_1 \equiv M_2$ and M_2 is transitive, then M_2 is a model of ZF.

In this article we present several logical schemes. The scheme *NonUniqBoundFunc* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a function f such that dom $f = \mathcal{A}$ and rng $f \subseteq \mathcal{B}$ and for every e such that $e \in \mathcal{A}$ holds $\mathcal{P}[e, f(e)]$

provided the following requirement is met:

- for every e such that $e \in \mathcal{A}$ there exists u such that $u \in \mathcal{B}$ and $\mathcal{P}[e, u]$.

The scheme *NonUniqFuncEx* deals with a set $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a function f such that dom $f = \mathcal{A}$ and for every e such that $e \in \mathcal{A}$ holds $\mathcal{P}[e, f(e)]$

provided the following condition is met:

- for every e such that $e \in \mathcal{A}$ there exists u such that $\mathcal{P}[e, u]$.

The following propositions are true:

(11) If $X \subseteq W$ and $\overline{\overline{X}} < \overline{\overline{W}}$, then $X \in W$.

(12) If $\operatorname{dom} f \in W$ and $\operatorname{rng} f \subseteq W$, then $\operatorname{rng} f \in W$.

(13) If $X \approx Y$ or $\overline{\overline{X}} = \overline{\overline{Y}}$, then $2^X \approx 2^Y$ and $\overline{\overline{2^X}} = \overline{\overline{2^Y}}$.

(14) Let D be a non-empty set. Let P_1 be a function from D into $(\operatorname{On} W)^{\operatorname{On} W}$. Suppose $\overline{\overline{D}} < \overline{\overline{W}}$ and for every x_1 such that $x_1 \in \operatorname{rng} P_1$ holds x_1 is increasing and x_1 is continuous. Then there exists p_1 such that p_1 is increasing and p_1 is continuous and $p_1(\mathbf{0}_W) = \mathbf{0}_W$ and for every a holds $p_1(\operatorname{succ} a) = \sup(\{p_1(a)\} \cup \operatorname{uncurry} P_1{}^\circ[\!: D, \{\operatorname{succ} a\} :\!])$ and for every a such that $a \neq \mathbf{0}_W$ and a is a limit ordinal number holds $p_1(a) = \sup(p_1 \restriction a)$.

(15) For every sequence p_1 of ordinal numbers such that p_1 is increasing holds $C + p_1$ is increasing.

(16) For every sequence x_1 of ordinal numbers holds $(C + x_1) \restriction A = C + x_1 \restriction A$.

(17) For every sequence p_1 of ordinal numbers such that p_1 is increasing and p_1 is continuous holds $C + p_1$ is continuous.

Let A, B be ordinal numbers. We say that A is cofinal with B if and only if:

(Def.5) there exists a sequence x_1 of ordinal numbers such that $\operatorname{dom} x_1 = B$ and $\operatorname{rng} x_1 \subseteq A$ and x_1 is increasing and $A = \sup x_1$.

Let us notice that the predicate defined above is reflexive.

In the sequel p_2 will be a sequence of ordinal numbers. We now state a number of propositions:

(18) If p_2 is increasing and $A \subseteq B$ and $B \in \operatorname{dom} p_2$, then $p_2(A) \subseteq p_2(B)$.

(19) If $e \in \operatorname{rng} p_2$, then e is an ordinal number.

(20) $\operatorname{rng} p_2 \subseteq \sup p_2$.

(21) If A is cofinal with B and B is cofinal with C, then A is cofinal with C.

(22) If A is cofinal with B, then $B \subseteq A$.

(23) If A is cofinal with B and B is cofinal with A, then $A = B$.

(24) If $\operatorname{dom} p_2 \neq \mathbf{0}$ and $\operatorname{dom} p_2$ is a limit ordinal number and p_2 is increasing and A is the limit of p_2, then A is cofinal with $\operatorname{dom} p_2$.

(25) $\operatorname{succ} A$ is cofinal with $\mathbf{1}$.

(26) If A is cofinal with $\operatorname{succ} B$, then there exists C such that $A = \operatorname{succ} C$.

(27) If A is cofinal with B, then A is a limit ordinal number if and only if B is a limit ordinal number.

(28) If A is cofinal with $\mathbf{0}$, then $A = \mathbf{0}$.

(29) $\operatorname{On} W$ is not cofinal with a.

(30) If $\omega \in W$ and p_1 is increasing and p_1 is continuous, then there exists b such that $a \in b$ and $p_1(b) = b$.

(31) If $\omega \in W$ and p_1 is increasing and p_1 is continuous, then there exists a such that $b \in a$ and $p_1(a) = a$ and a is cofinal with ω.

(32) Suppose $\omega \in W$ and for all a, b such that $a \in b$ holds $L(a) \subseteq L(b)$ and for every a such that $a \neq \mathbf{0}$ and a is a limit ordinal number holds $L(a) = \bigcup(L \restriction a)$. Then there exists p_1 such that p_1 is increasing and p_1 is continuous and for every a such that $p_1(a) = a$ and $\mathbf{0} \neq a$ holds $L(a) \prec \bigcup L$.

(33) $\mathbf{R}_a \in W$.

(34) If $a \neq \mathbf{0}$, then $\mathbf{R}_a$ is a non-empty set from W.

(35) If $\omega \in W$, then there exists p_1 such that p_1 is increasing and p_1 is continuous and for all a, M such that $p_1(a) = a$ and $\mathbf{0} \neq a$ and $M = \mathbf{R}_a$ holds $M \prec W$.

(36) If $\omega \in W$, then there exist b, M such that $a \in b$ and $M = \mathbf{R}_b$ and $M \prec W$.

(37) If $\omega \in W$, then there exist a, M such that a is cofinal with ω and $M = \mathbf{R}_a$ and $M \prec W$.

(38) Suppose $\omega \in W$ and for all a, b such that $a \in b$ holds $L(a) \subseteq L(b)$ and for every a such that $a \neq \mathbf{0}$ and a is a limit ordinal number holds $L(a) = \bigcup(L \restriction a)$. Then there exists p_1 such that p_1 is increasing and p_1 is continuous and for every a such that $p_1(a) = a$ and $\mathbf{0} \neq a$ holds $L(a) \equiv \bigcup L$.

(39) If $\omega \in W$, then there exists p_1 such that p_1 is increasing and p_1 is continuous and for all a, M such that $p_1(a) = a$ and $\mathbf{0} \neq a$ and $M = \mathbf{R}_a$ holds $M \equiv W$.

(40) If $\omega \in W$, then there exist b, M such that $a \in b$ and $M = \mathbf{R}_b$ and $M \equiv W$.

(41) If $\omega \in W$, then there exist a, M such that a is cofinal with ω and $M = \mathbf{R}_a$ and $M \equiv W$.

(42) If $\omega \in W$, then there exist a, M such that a is cofinal with ω and $M = \mathbf{R}_a$ and M is a model of ZF.

(43) If $\omega \in W$ and $X \in W$, then there exists M such that $X \in M$ and $M \in W$ and M is a model of ZF.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[2] Grzegorz Bancerek. Curried and uncurried functions. *Formalized Mathematics*, 1(**3**):537–541, 1990.

[3] Grzegorz Bancerek. Increasing and continuous ordinal sequences. *Formalized Mathematics*, 1(**4**):711–714, 1990.

[4] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.

[5] Grzegorz Bancerek. A model of ZF set theory language. *Formalized Mathematics*, 1(**1**):131–145, 1990.

[6] Grzegorz Bancerek. Models and satisfiability. *Formalized Mathematics*, 1(**1**):191–199, 1990.

[7] Grzegorz Bancerek. Ordinal arithmetics. *Formalized Mathematics*, 1(**3**):515–519, 1990.

[8] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[9] Grzegorz Bancerek. The reflection theorem. *Formalized Mathematics*, 1(**5**):973–977, 1990.

[10] Grzegorz Bancerek. Replacing of variables in formulas of ZF theory. *Formalized Mathematics*, 1(**5**):963–972, 1990.

[11] Grzegorz Bancerek. Sequences of ordinal numbers. *Formalized Mathematics*, 1(**2**):281–290, 1990.

[12] Grzegorz Bancerek. Tarski's classes and ranks. *Formalized Mathematics*, 1(**3**):563–567, 1990.

[13] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[14] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[15] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[16] Bogdan Nowak and Grzegorz Bancerek. Universal classes. *Formalized Mathematics*, 1(**3**):595–600, 1990.

[17] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[18] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[19] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

Received August 13, 1990

Index of MML Identifiers

Contents

Form. Math. 1 (**5**)

Continued on inside back cover