FORMALIZED MATHEMATICS
Number 1, January 1990
Université Catholique de Louvain

Connected Spaces

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Summary. The following notions are defined: separated sets, connected spaces, connected sets, components of a topological space, the component of a point. The definition of the boundary of a set is also included. The singleton of a point of a topological space is redefined as a subset of the space. Some theorems about these notions are proved.

The articles [3], [4], [1], [2], and [5] provide the notation and terminology for this paper. For simplicity we adopt the following convention: GX, GY will have the type TopSpace; A, $A1$, B, $B1$, C will have the type Subset **of** GX. The arguments of the notions defined below are the following: GX which is an object of the type TopSpace; A, B which are objects of the type Subset **of** GX. The predicate

$$A, B \text{ are\_separated} \qquad \text{is defined by} \qquad \mathrm{Cl}\, A \cap B = \emptyset(GX) \;\&\; A \cap \mathrm{Cl}\, B = \emptyset(GX).$$

The following propositions are true:

$$A, B \text{ are\_separated } \textbf{implies } B, A \text{ are\_separated}, \tag{1}$$

$$A, B \text{ are\_separated } \textbf{implies } A \cap B = \emptyset(GX), \tag{2}$$

$$\Omega\,(GX) = A \cup B \;\&\; A \text{ is\_closed} \;\&\; B \text{ is\_closed} \;\&\; A \cap B = \emptyset(GX) \;\textbf{implies } A, B \text{ are\_separated}, \tag{3}$$

$$\Omega\,(GX) = A \cup B \;\&\; A \text{ is\_open} \;\&\; B \text{ is\_open} \;\&\; A \cap B = \emptyset(GX) \;\textbf{implies } A, B \text{ are\_separated}, \tag{4}$$

$$\Omega\,(GX) = A \cup B \;\&\; A, B \text{ are\_separated} \;\textbf{implies } A \text{ is\_open\_closed} \;\&\; B \text{ is\_open\_closed}, \tag{5}$$

[1]Supported by RPBP.III-24.C1.

ISSN 0777-4028

(6) $$\textbf{for } X' \textbf{ being } \text{SubSpace } \textbf{of } GX,\ P1,Q1 \textbf{ being } \text{Subset } \textbf{of } GX,\ P,Q \textbf{ being } \text{Subset } \textbf{of } X' \textbf{ st } P = P1 \ \&\ Q = Q1 \textbf{ holds } P,Q \text{ are\_separated } \textbf{implies } P1,Q1 \text{ are\_separated},$$

(7) $$\textbf{for } X' \textbf{ being } \text{SubSpace } \textbf{of } GX,\ P,Q \textbf{ being } \text{Subset } \textbf{of } GX,\ P1,Q1 \textbf{ being } \text{Subset } \textbf{of } X' \textbf{ st } P = P1 \ \&\ Q = Q1 \ \&\ P \cup Q \subseteq \Omega(X') \textbf{ holds } P,Q \text{ are\_separated } \textbf{implies } P1,Q1 \text{ are\_separated},$$

(8) $$A,B \text{ are\_separated } \&\ A1 \subseteq A \ \&\ B1 \subseteq B \textbf{ implies } A1,B1 \text{ are\_separated},$$

(9) $$A,B \text{ are\_separated } \&\ A,C \text{ are\_separated } \textbf{implies } A, B \cup C \text{ are\_separated},$$

(10) $$A \text{ is\_closed } \&\ B \text{ is\_closed } \textbf{or } A \text{ is\_open } \&\ B \text{ is\_open } \textbf{implies } A \setminus B, B \setminus A \text{ are\_separated}.$$

Let GX have the type TopSpace. The predicate

$$GX \text{ is\_connected}$$

is defined by

$$\textbf{for } A,B \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ st } \Omega(GX) = A \cup B \ \&\ A,B \text{ are\_separated } \textbf{holds } A = \emptyset(GX) \textbf{ or } B = \emptyset(GX).$$

One can prove the following propositions:

(11) $$GX \text{ is\_connected } \textbf{iff for } A,B \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ st } \Omega(GX) = A \cup B \ \&\ A \neq \emptyset(GX) \ \&\ B \neq \emptyset(GX) \ \&\ A \text{ is\_closed } \&\ B \text{ is\_closed } \textbf{holds } A \cap B \neq \emptyset(GX),$$

(12) $$GX \text{ is\_connected } \textbf{iff for } A,B \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ st } \Omega(GX) = A \cup B \ \&\ A \neq \emptyset(GX) \ \&\ B \neq \emptyset(GX) \ \&\ A \text{ is\_open } \&\ B \text{ is\_open } \textbf{holds } A \cap B \neq \emptyset(GX),$$

(13) $$GX \text{ is\_connected } \textbf{iff for } A \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ st } A \neq \emptyset(GX) \ \&\ A \neq \Omega(GX) \textbf{ holds } (\mathrm{Cl}\, A) \cap \mathrm{Cl}\,(\Omega(GX) \setminus A) \neq \emptyset(GX),$$

(14) $$GX \text{ is\_connected } \textbf{iff for } A \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ st } A \text{ is\_open\_closed } \textbf{holds } A = \emptyset(GX) \textbf{ or } A = \Omega(GX),$$

(15) $$\textbf{for } F \textbf{ being } \text{map } \textbf{of } GX,GY \textbf{ st } F \text{ is\_continuous } \&\ F^{\circ}(\Omega(GX)) = \Omega(GY) \ \&\ GX \text{ is\_connected } \textbf{holds } GY \text{ is\_connected}.$$

The arguments of the notions defined below are the following: GX which is an object of the type TopSpace; A which is an object of the type Subset **of** GX. The predicate

$$A \text{ is\_connected} \quad \text{is defined by} \quad GX \mid A \text{ is\_connected}.$$

One can prove the following propositions:

(16) $A \neq \emptyset(GX)$ **implies** (A is_connected **iff for** P,Q **being** Subset **of** GX **st** $A = P \cup Q$ & P, Q are_separated **holds** $P = \emptyset(GX)$ **or** $Q = \emptyset(GX)$),

(17) A is_connected & $A \subseteq B \cup C$ & B, C are_separated **implies** $A \subseteq B$ **or** $A \subseteq C$,

(18) A is_connected & B is_connected & **not** A, B are_separated **implies** $A \cup B$ is_connected,

(19) $C \neq \emptyset(GX)$ & C is_connected & $C \subseteq A$ & $A \subseteq \mathrm{Cl}\, C$ **implies** A is_connected,

(20) $A \neq \emptyset(GX)$ & A is_connected **implies** $\mathrm{Cl}\, A$ is_connected,

(21) GX is_connected & $A \neq \emptyset(GX)$ & A is_connected & $\Omega\,(GX) \setminus A = B \cup C$ & B, C are_separated **implies** $A \cup B$ is_connected & $A \cup C$ is_connected,

(22) $\Omega\,(GX) \setminus A = B \cup C$ & B, C are_separated & A is_closed **implies** $A \cup B$ is_closed & $A \cup C$ is_closed,

(23) C is_connected & $C \cap A \neq \emptyset(GX)$ & $C \setminus A \neq \emptyset(GX)$ **implies** $C \cap \mathrm{Fr}\, A \neq \emptyset(GX)$,

(24) **for** X' **being** SubSpace **of** GX, A **being** Subset **of** GX, B **being** Subset **of** X' **st** $A \neq \emptyset(GX)$ & $A = B$ **holds** A is_connected **iff** B is_connected,

(25) $A \cap B \neq \emptyset(GX)$ & A is_closed & B is_closed **implies** ($A \cup B$ is_connected & $A \cap B$ is_connected **implies** A is_connected & B is_connected),

(26) **for** F **being** Subset-Family **of** GX **st** (**for** A **being** Subset **of** GX **st** $A \in F$ **holds** A is_connected) & **ex** A **being** Subset **of** GX **st** $A \neq \emptyset(GX)$ & $A \in F$ & **for** B **being** Subset **of** GX **st** $B \in F$ & $B \neq A$ **holds not** A, B are_separated **holds** $\bigcup F$ is_connected,

(27) **for** F **being** Subset-Family **of** GX **st** (**for** A **being** Subset **of** GX **st** $A \in F$ **holds** A is_connected) & $\bigcap F \neq \emptyset(GX)$ **holds** $\bigcup F$ is_connected,

(28) $\Omega\,(GX)$ is_connected **iff** GX is_connected .

The arguments of the notions defined below are the following: GX which is an object of the type TopSpace; x which is an object of the type Point **of** GX. Let us note that it makes sense to consider the following functor on a restricted area. Then

$\{x\}$ is Subset **of** GX.

We now state a proposition

(29) **for** x **being** Point **of** GX **holds** $\{x\}$ is_connected .

The arguments of the notions defined below are the following: GX which is an object of the type TopSpace; x, y which are objects of the type Point **of** GX. The predicate

x, y are_joined

is defined by

ex C **being** Subset **of** GX **st** C is_connected & $x \in C$ & $y \in C$.

We now state four propositions:

(30) (**ex** x **being** Point **of** GX **st for** y **being** Point **of** GX **holds** x, y are_joined) **implies** GX is_connected ,

(31) (**ex** x **being** Point **of** GX **st for** y **being** Point **of** GX **holds** x, y are_joined) **iff for** x,y **being** Point **of** GX **holds** x, y are_joined ,

(32) (**for** x,y **being** Point **of** GX **holds** x, y are_joined) **implies** GX is_connected ,

(33) **for** x **being** Point **of** GX, F **being** Subset-Family **of** GX **st for** A **being** Subset **of** GX **holds** $A \in F$ **iff** A is_connected & $x \in A$ **holds** $F \neq \emptyset$.

The arguments of the notions defined below are the following: GX which is an object of the type TopSpace; A which is an object of the type Subset **of** GX. The predicate

A is_a_component_of GX

is defined by

A is_connected
& **for** B **being** Subset **of** GX **st** B is_connected **holds** $A \subseteq B$ **implies** $A = B$.

The following propositions are true:

(34) A is_a_component_of GX **implies** $A \neq \emptyset(GX)$,

(35) $$A \text{ is\_a\_component\_of } GX \textbf{ implies } A \text{ is\_closed},$$

(36) $$A \text{ is\_a\_component\_of } GX \mathrel{\&} B \text{ is\_a\_component\_of } GX$$
$$\textbf{implies } A = B \textbf{ or } (A \neq B \textbf{ implies } A, B \text{ are\_separated}),$$

(37) $$A \text{ is\_a\_component\_of } GX \mathrel{\&} B \text{ is\_a\_component\_of } GX$$
$$\textbf{implies } A = B \textbf{ or } (A \neq B \textbf{ implies } A \cap B = \emptyset(GX)),$$

(38) $$C \text{ is\_connected } \textbf{implies for } S \textbf{ being } \text{Subset } \textbf{of } GX$$
$$\textbf{st } S \text{ is\_a\_component\_of } GX \textbf{ holds } C \cap S = \emptyset(GX) \textbf{ or } C \subseteq S.$$

The arguments of the notions defined below are the following: GX which is an object of the type TopSpace; A, B which are objects of the type Subset **of** GX. The predicate

$$B \text{ is\_a\_component\_of } A$$

is defined by

$$\textbf{ex } B1 \textbf{ being } \text{Subset } \textbf{of } GX \mid A \textbf{ st } B1 = B \mathrel{\&} B1 \text{ is\_a\_component\_of } (GX \mid A).$$

We now state a proposition

(39) $$GX \text{ is\_connected} \mathrel{\&} A \neq \Omega\,(GX)$$
$$\mathrel{\&} A \neq \emptyset(GX) \mathrel{\&} A \text{ is\_connected} \mathrel{\&} C \text{ is\_a\_component\_of } (\Omega\,(GX) \setminus A)$$
$$\textbf{implies } (\Omega\,(GX) \setminus C) \text{ is\_connected}.$$

The arguments of the notions defined below are the following: GX which is an object of the type TopSpace; x which is an object of the type Point **of** GX. The functor

$$\text{skl}\, x,$$

with values of the type Subset **of** GX, is defined by

$$\textbf{ex } F \textbf{ being } \text{Subset-Family } \textbf{of } GX$$
$$\textbf{st } (\textbf{for } A \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ holds } A \in F \textbf{ iff } A \text{ is\_connected} \mathrel{\&} x \in A) \mathrel{\&} \bigcup F = \textbf{it}.$$

In the sequel x has the type Point **of** GX. One can prove the following propositions:

(40) $$x \in \text{skl}\, x,$$

(41) $$\text{skl}\, x \text{ is\_connected},$$

(42) $$C \text{ is\_connected } \textbf{implies } (\text{skl}\, x \subseteq C \textbf{ implies } C = \text{skl}\, x),$$

(43) $$A \text{ is\_a\_component\_of } GX \textbf{ iff ex } x \textbf{ being } \text{Point } \textbf{of } GX \textbf{ st } A = \text{skl}\, x,$$

(44) $$A \text{ is\_a\_component\_of } GX \mathrel{\&} x \in A \textbf{ implies } A = \text{skl}\, x,$$

(45) $$\textbf{for } S \textbf{ being } \text{Subset } \textbf{of } GX \; \textbf{st } S = \text{skl}\, x \textbf{ for } p \textbf{ being } \text{Point } \textbf{of } GX \textbf{ st } p \neq x \;\&\; p \in S \textbf{ holds } \text{skl}\, p = S,$$

(46) $$\textbf{for } F \textbf{ being } \text{Subset-Family } \textbf{of } GX \textbf{ st } \textbf{for } A \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ holds } A \in F \textbf{ iff } A \text{ is\_a\_component\_of } GX \; \textbf{holds } F \text{ is\_a\_cover\_of } GX,$$

(47) $$A, B \text{ are\_separated } \textbf{iff } \text{Cl}\, A \cap B = \emptyset(GX) \;\&\; A \cap \text{Cl}\, B = \emptyset(GX),$$

(48) $$GX \text{ is\_connected } \textbf{iff for } A,B \textbf{ being } \text{Subset } \textbf{of } GX \; \textbf{st } \Omega\,(GX) = A \cup B \;\&\; A, B \text{ are\_separated } \textbf{holds } A = \emptyset(GX) \textbf{ or } B = \emptyset(GX),$$

(49) $$A \text{ is\_connected } \textbf{iff } GX \mid A \text{ is\_connected}\,,$$

(50) $$A \text{ is\_a\_component\_of } GX \textbf{ iff } A \text{ is\_connected} \;\&\; \textbf{for } B \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ st } B \text{ is\_connected } \textbf{holds } A \subseteq B \textbf{ implies } A = B,$$

(51) $$B \text{ is\_a\_component\_of } A \textbf{ iff } \textbf{ex } B1 \textbf{ being } \text{Subset } \textbf{of } GX \mid A \textbf{ st } B1 = B \;\&\; B1 \text{ is\_a\_component\_of } (GX \mid A),$$

(52) $$B = \text{skl}\, x \textbf{ iff ex } F \textbf{ being } \text{Subset-Family } \textbf{of } GX \textbf{ st } (\textbf{for } A \textbf{ being } \text{Subset } \textbf{of } GX \textbf{ holds } A \in F \textbf{ iff } A \text{ is\_connected} \;\&\; x \in A) \;\&\; \bigcup F = B.$$

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Received May 6, 1989