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Functions and Their Basic Properties

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Summary. The definitions of the mode Function and the graph of a function are introduced. The graph of a function is defined to be identical with the function. The following concepts are also defined: the domain of a function, the range of a function, the identity function, the composition of functions, the 1-1 function, the inverse function, the restriction of a function, the image and the inverse image. Certain basic facts about functions and the notions defined in the article are proved.

The notation and terminology used here are introduced in the papers [1] and [2]. For simplicity we adopt the following convention: X, $X1$, $X2$, Y, $Y1$, $Y2$ have the type set; p, x, $x1$, $x2$, y, $y1$, $y2$, z have the type Any. The mode

$$\text{Function},$$

which widens to the type Any, is defined by

$$\mathbf{ex}\, F\ \mathbf{being}\ \text{set}\ \mathbf{st}\ \mathbf{it} = F\ \&\ (\mathbf{for}\, p\ \mathbf{st}\ p \in F\ \mathbf{ex}\, x{,}y\ \mathbf{st}\ \langle x, y\rangle = p)$$
$$\&\ \mathbf{for}\, x{,}y1{,}y2\ \mathbf{st}\ \langle x, y1\rangle \in F\ \&\ \langle x, y2\rangle \in F\ \mathbf{holds}\ y1 = y2.$$

In the sequel f, g, h will have the type Function. Let us consider f. The functor

$$\text{graph}\, f,$$

yields the type set and is defined by

$$f = \mathbf{it}\,.$$

Next we state several propositions:

(1) $$\text{graph}\, f = f,$$

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$$\begin{gathered}\textbf{for}\, F \,\textbf{being}\, \text{set}\, \textbf{st} \\ (\textbf{for}\, p \,\textbf{st}\, p \in F \,\textbf{ex}\, x{,}y \,\textbf{st}\, \langle x, y\rangle = p) \\ \&\ \textbf{for}\, x{,}y1{,}y2 \,\textbf{st}\, \langle x, y1\rangle \in F \,\&\, \langle x, y2\rangle \in F \,\textbf{holds}\, y1 = y2 \\ \textbf{ex}\, f \,\textbf{being}\, \text{Function}\, \textbf{st}\, \text{graph}\, f = F,\end{gathered} \tag{2}$$

$$p \in \text{graph}\, f \,\textbf{implies}\, \textbf{ex}\, x{,}y \,\textbf{st}\, \langle x, y\rangle = p, \tag{3}$$

$$\langle x, y1\rangle \in \text{graph}\, f \,\&\, \langle x, y2\rangle \in \text{graph}\, f \,\textbf{implies}\, y1 = y2, \tag{4}$$

$$\text{graph}\, f = \text{graph}\, g \,\textbf{implies}\, f = g. \tag{5}$$

The scheme *GraphFunc* concerns a constant $\mathcal{A}$ that has the type set and a binary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex}\, f \,\textbf{st}\, \textbf{for}\, x{,}y \,\textbf{holds}\, \langle x, y\rangle \in \text{graph}\, f \,\textbf{iff}\, x \in \mathcal{A} \,\&\, \mathcal{P}[x, y]$$

provided the parameters satisfy the following condition:

- $\textbf{for}\, x{,}y1{,}y2 \,\textbf{st}\, \mathcal{P}[x, y1] \,\&\, \mathcal{P}[x, y2] \,\textbf{holds}\, y1 = y2.$

Let us consider f. The functor

$$\text{dom}\, f,$$

yields the type set and is defined by

$$\textbf{for}\, x \,\textbf{holds}\, x \in \textbf{it} \,\textbf{iff}\, \textbf{ex}\, y \,\textbf{st}\, \langle x, y\rangle \in \text{graph}\, f.$$

One can prove the following proposition

$$X = \text{dom}\, f \,\textbf{iff}\, \textbf{for}\, x \,\textbf{holds}\, x \in X \,\textbf{iff}\, \textbf{ex}\, y \,\textbf{st}\, \langle x, y\rangle \in \text{graph}\, f. \tag{6}$$

Let us consider f, x. Assume that the following holds

$$x \in \text{dom}\, f.$$

The functor

$$f\,.x,$$

yields the type Any and is defined by

$$\langle x, \textbf{it}\rangle \in \text{graph}\, f.$$

The following three propositions are true:

$$x \in \text{dom}\, f \,\textbf{implies}\, (y = f\,.x \,\textbf{iff}\, \langle x, y\rangle \in \text{graph}\, f), \tag{7}$$

$$\langle x, y\rangle \in \text{graph}\, f \,\textbf{iff}\, x \in \text{dom}\, f \,\&\, y = f\,.x, \tag{8}$$

$$X = \text{dom}\, f \,\&\, X = \text{dom}\, g \,\&\, (\textbf{for}\, x \,\textbf{st}\, x \in X \,\textbf{holds}\, f\,.x = g\,.x) \,\textbf{implies}\, f = g. \tag{9}$$

Let us consider f. The functor

$$\operatorname{rng} f,$$

with values of the type set, is defined by

$$\textbf{for}\, y\ \textbf{holds}\ y \in \textbf{it}\ \textbf{iff}\ \textbf{ex}\, x\ \textbf{st}\ x \in \operatorname{dom} f\ \&\ y = f\,.x.$$

One can prove the following propositions:

$$Y = \operatorname{rng} f\ \textbf{iff}\ \textbf{for}\, y\ \textbf{holds}\ y \in Y\ \textbf{iff}\ \textbf{ex}\, x\ \textbf{st}\ x \in \operatorname{dom} f\ \&\ y = f\,.x, \tag{10}$$

$$y \in \operatorname{rng} f\ \textbf{iff}\ \textbf{ex}\, x\ \textbf{st}\ x \in \operatorname{dom} f\ \&\ y = f\,.x, \tag{11}$$

$$x \in \operatorname{dom} f\ \textbf{implies}\ f\,.x \in \operatorname{rng} f, \tag{12}$$

$$\operatorname{dom} f = \emptyset\ \textbf{iff}\ \operatorname{rng} f = \emptyset, \tag{13}$$

$$\operatorname{dom} f = \{x\}\ \textbf{implies}\ \operatorname{rng} f = \{f\,.x\}. \tag{14}$$

Now we present two schemes. The scheme *FuncEx* concerns a constant $\mathcal{A}$ that has the type set and a binary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex}\, f\ \textbf{st}\ \operatorname{dom} f = \mathcal{A}\ \&\ \textbf{for}\, x\ \textbf{st}\ x \in \mathcal{A}\ \textbf{holds}\ \mathcal{P}[x, f\,.x]$$

provided the parameters satisfy the following conditions:

- $$\textbf{for}\, x{,}y1{,}y2\ \textbf{st}\ x \in \mathcal{A}\ \&\ \mathcal{P}[x, y1]\ \&\ \mathcal{P}[x, y2]\ \textbf{holds}\ y1 = y2,$$
- $$\textbf{for}\, x\ \textbf{st}\ x \in \mathcal{A}\ \textbf{ex}\, y\ \textbf{st}\ \mathcal{P}[x, y].$$

The scheme *Lambda* concerns a constant $\mathcal{A}$ that has the type set and a unary functor $\mathcal{F}$ and states that the following holds

$$\textbf{ex}\, f\ \textbf{being}\ \text{Function}\ \textbf{st}\ \operatorname{dom} f = \mathcal{A}\ \&\ \textbf{for}\, x\ \textbf{st}\ x \in \mathcal{A}\ \textbf{holds}\ f\,.x = \mathcal{F}(x)$$

for all values of the parameters.

Next we state several propositions:

$$X \neq \emptyset\ \textbf{implies}\ \textbf{for}\, y\ \textbf{ex}\, f\ \textbf{st}\ \operatorname{dom} f = X\ \&\ \operatorname{rng} f = \{y\}, \tag{15}$$

$$(\textbf{for}\, f{,}g\ \textbf{st}\ \operatorname{dom} f = X\ \&\ \operatorname{dom} g = X\ \textbf{holds}\ f = g)\ \textbf{implies}\ X = \emptyset, \tag{16}$$

$$\operatorname{dom} f = \operatorname{dom} g\ \&\ \operatorname{rng} f = \{y\}\ \&\ \operatorname{rng} g = \{y\}\ \textbf{implies}\ f = g, \tag{17}$$

$$Y \neq \emptyset\ \textbf{or}\ X = \emptyset\ \textbf{implies}\ \textbf{ex}\, f\ \textbf{st}\ X = \operatorname{dom} f\ \&\ \operatorname{rng} f \subseteq Y, \tag{18}$$

$$(\textbf{for}\, y\ \textbf{st}\ y \in Y\ \textbf{ex}\, x\ \textbf{st}\ x \in \operatorname{dom} f\ \&\ y = f\,.x)\ \textbf{implies}\ Y \subseteq \operatorname{rng} f. \tag{19}$$

Let us consider f, g. The functor

$$g \cdot f,$$

yields the type Function and is defined by

$$(\textbf{for}\, x\ \textbf{holds}\ x \in \operatorname{dom}\textbf{it}\ \textbf{iff}\ x \in \operatorname{dom} f\ \&\ f\,.x \in \operatorname{dom} g)$$
$$\&\ \textbf{for}\, x\ \textbf{st}\ x \in \operatorname{dom}\textbf{it}\ \textbf{holds}\ \textbf{it}.x = g\,.(f\,.x).$$

The following propositions are true:

(20) $$h = g \cdot f\ \textbf{iff}\ (\textbf{for}\, x\ \textbf{holds}\ x \in \operatorname{dom} h\ \textbf{iff}\ x \in \operatorname{dom} f\ \&\ f\,.x \in \operatorname{dom} g)$$
$$\&\ \textbf{for}\, x\ \textbf{st}\ x \in \operatorname{dom} h\ \textbf{holds}\ h.x = g\,.(f\,.x),$$

(21) $$x \in \operatorname{dom}(g \cdot f)\ \textbf{iff}\ x \in \operatorname{dom} f\ \&\ f\,.x \in \operatorname{dom} g,$$

(22) $$x \in \operatorname{dom}(g \cdot f)\ \textbf{implies}\ (g \cdot f).x = g\,.(f\,.x),$$

(23) $$x \in \operatorname{dom} f\ \&\ f\,.x \in \operatorname{dom} g\ \textbf{implies}\ (g \cdot f).x = g\,.(f\,.x),$$

(24) $$\operatorname{dom}(g \cdot f) \subseteq \operatorname{dom} f,$$

(25) $$z \in \operatorname{rng}(g \cdot f)\ \textbf{implies}\ z \in \operatorname{rng} g,$$

(26) $$\operatorname{rng}(g \cdot f) \subseteq \operatorname{rng} g,$$

(27) $$\operatorname{rng} f \subseteq \operatorname{dom} g\ \textbf{iff}\ \operatorname{dom}(g \cdot f) = \operatorname{dom} f,$$

(28) $$\operatorname{dom} g \subseteq \operatorname{rng} f\ \textbf{implies}\ \operatorname{rng}(g \cdot f) = \operatorname{rng} g,$$

(29) $$\operatorname{rng} f = \operatorname{dom} g\ \textbf{implies}\ \operatorname{dom}(g \cdot f) = \operatorname{dom} f\ \&\ \operatorname{rng}(g \cdot f) = \operatorname{rng} g,$$

(30) $$h \cdot (g \cdot f) = (h \cdot g) \cdot f,$$

(31) $$\operatorname{rng} f \subseteq \operatorname{dom} g\ \&\ x \in \operatorname{dom} f\ \textbf{implies}\ (g \cdot f).x = g\,.(f\,.x),$$

(32) $$\operatorname{rng} f = \operatorname{dom} g\ \&\ x \in \operatorname{dom} f\ \textbf{implies}\ (g \cdot f).x = g\,.(f\,.x),$$

(33) $$\operatorname{rng} f \subseteq Y\ \&\ (\textbf{for}\, g,h\ \textbf{st}\ \operatorname{dom} g = Y\ \&\ \operatorname{dom} h = Y\ \&\ g \cdot f = h \cdot f\ \textbf{holds}\ g = h)$$
$$\textbf{implies}\ Y = \operatorname{rng} f.$$

Let us consider X. The functor

$$\operatorname{id} X,$$

with values of the type Function, is defined by

$$\operatorname{dom}\textbf{it} = X\ \&\ \textbf{for}\, x\ \textbf{st}\ x \in X\ \textbf{holds}\ \textbf{it}.x = x.$$

Next we state a number of propositions:

(34) $$f = \operatorname{id} X\ \textbf{iff}\ \operatorname{dom} f = X\ \&\ \textbf{for}\, x\ \textbf{st}\ x \in X\ \textbf{holds}\ f\,.x = x,$$

(35) $$x \in X\ \textbf{implies}\ (\operatorname{id} X).x = x,$$

$$\operatorname{dom} \operatorname{id} X = X \ \& \ \operatorname{rng} \operatorname{id} X = X, \tag{36}$$

$$\operatorname{dom}(f \cdot (\operatorname{id} X)) = \operatorname{dom} f \cap X, \tag{37}$$

$$x \in \operatorname{dom} f \cap X \ \textbf{implies}\ f.x = (f \cdot (\operatorname{id} X)).x, \tag{38}$$

$$\operatorname{dom} f \subseteq X \ \textbf{implies}\ f \cdot (\operatorname{id} X) = f, \tag{39}$$

$$x \in \operatorname{dom}((\operatorname{id} Y) \cdot f) \ \textbf{iff}\ x \in \operatorname{dom} f \ \& \ f.x \in Y, \tag{40}$$

$$\operatorname{rng} f \subseteq Y \ \textbf{implies}\ (\operatorname{id} Y) \cdot f = f, \tag{41}$$

$$f \cdot (\operatorname{id} \operatorname{dom} f) = f \ \& \ (\operatorname{id} \operatorname{rng} f) \cdot f = f, \tag{42}$$

$$(\operatorname{id} X) \cdot (\operatorname{id} Y) = \operatorname{id}(X \cap Y), \tag{43}$$

$$\operatorname{dom} f = X \ \& \ \operatorname{rng} f = X \ \& \ \operatorname{dom} g = X \ \& \ g \cdot f = f \ \textbf{implies}\ g = \operatorname{id} X. \tag{44}$$

Let us consider f. The predicate

$$f \ \text{is\_one-to-one}$$

is defined by

$$\textbf{for}\ x1,x2\ \textbf{st}\ x1 \in \operatorname{dom} f \ \& \ x2 \in \operatorname{dom} f \ \& \ f.x1 = f.x2\ \textbf{holds}\ x1 = x2.$$

One can prove the following propositions:

$$\begin{gathered} f \ \text{is\_one-to-one} \\ \textbf{iff for}\ x1,x2\ \textbf{st}\ x1 \in \operatorname{dom} f \ \& \ x2 \in \operatorname{dom} f \ \& \ f.x1 = f.x2\ \textbf{holds}\ x1 = x2, \end{gathered} \tag{45}$$

$$f \ \text{is\_one-to-one} \ \& \ g \ \text{is\_one-to-one}\ \textbf{implies}\ g \cdot f \ \text{is\_one-to-one}, \tag{46}$$

$$g \cdot f \ \text{is\_one-to-one} \ \& \ \operatorname{rng} f \subseteq \operatorname{dom} g \ \textbf{implies}\ f \ \text{is\_one-to-one}, \tag{47}$$

$$g \cdot f \ \text{is\_one-to-one} \ \& \ \operatorname{rng} f = \operatorname{dom} g \ \textbf{implies}\ f \ \text{is\_one-to-one} \ \& \ g \ \text{is\_one-to-one}, \tag{48}$$

$$\begin{gathered} f \ \text{is\_one-to-one}\ \textbf{iff for}\ g,h\ \textbf{st} \\ \operatorname{rng} g \subseteq \operatorname{dom} f \ \& \ \operatorname{rng} h \subseteq \operatorname{dom} f \ \& \ \operatorname{dom} g = \operatorname{dom} h \ \& \ f \cdot g = f \cdot h\ \textbf{holds}\ g = h, \end{gathered} \tag{49}$$

$$\begin{gathered} \operatorname{dom} f = X \ \& \ \operatorname{dom} g = X \ \& \ \operatorname{rng} g \subseteq X \ \& \ f \ \text{is\_one-to-one} \ \& \ f \cdot g = f \\ \textbf{implies}\ g = \operatorname{id} X, \end{gathered} \tag{50}$$

$$\operatorname{rng}(g \cdot f) = \operatorname{rng} g \ \& \ g \ \text{is\_one-to-one}\ \textbf{implies}\ \operatorname{dom} g \subseteq \operatorname{rng} f, \tag{51}$$

$$\operatorname{id} X \ \text{is\_one-to-one}, \tag{52}$$

$$(\textbf{ex}\ g\ \textbf{st}\ g \cdot f = \operatorname{id} \operatorname{dom} f)\ \textbf{implies}\ f \ \text{is\_one-to-one}. \tag{53}$$

Let us consider f. Assume that the following holds

$$f \text{ is\_one-to-one}.$$

The functor

$$f^{-1},$$

with values of the type Function, is defined by

$$\operatorname{dom} \mathbf{it} = \operatorname{rng} f \ \& \ \mathbf{for}\ y,x\ \mathbf{holds}\ y \in \operatorname{rng} f \ \& \ x = \mathbf{it}.y \ \mathbf{iff}\ x \in \operatorname{dom} f \ \& \ y = f.x.$$

We now state a number of propositions:

$$f \text{ is\_one-to-one } \mathbf{implies}\ (g = f^{-1}\ \mathbf{iff}$$
$$\operatorname{dom} g = \operatorname{rng} f \ \& \ \mathbf{for}\ y,x\ \mathbf{holds}\ y \in \operatorname{rng} f \ \& \ x = g.y \ \mathbf{iff}\ x \in \operatorname{dom} f \ \& \ y = f.x), \tag{54}$$

$$f \text{ is\_one-to-one } \mathbf{implies}\ \operatorname{rng} f = \operatorname{dom}(f^{-1}) \ \& \ \operatorname{dom} f = \operatorname{rng}(f^{-1}), \tag{55}$$

$$f \text{ is\_one-to-one} \ \& \ x \in \operatorname{dom} f \ \mathbf{implies}\ x = (f^{-1}).(f.x) \ \& \ x = (f^{-1} \cdot f).x, \tag{56}$$

$$f \text{ is\_one-to-one} \ \& \ y \in \operatorname{rng} f \ \mathbf{implies}\ y = f.((f^{-1}).y) \ \& \ y = (f \cdot f^{-1}).y, \tag{57}$$

$$f \text{ is\_one-to-one } \mathbf{implies}\ \operatorname{dom}(f^{-1} \cdot f) = \operatorname{dom} f \ \& \ \operatorname{rng}(f^{-1} \cdot f) = \operatorname{dom} f, \tag{58}$$

$$f \text{ is\_one-to-one } \mathbf{implies}\ \operatorname{dom}(f \cdot f^{-1}) = \operatorname{rng} f \ \& \ \operatorname{rng}(f \cdot f^{-1}) = \operatorname{rng} f, \tag{59}$$

$$f \text{ is\_one-to-one} \ \& \ \operatorname{dom} f = \operatorname{rng} g \ \& \ \operatorname{rng} f = \operatorname{dom} g$$
$$\& \ (\mathbf{for}\ x,y\ \mathbf{st}\ x \in \operatorname{dom} f \ \& \ y \in \operatorname{dom} g \ \mathbf{holds}\ f.x = y \ \mathbf{iff}\ g.y = x)$$
$$\mathbf{implies}\ g = f^{-1}, \tag{60}$$

$$f \text{ is\_one-to-one } \mathbf{implies}\ f^{-1} \cdot f = \operatorname{id} \operatorname{dom} f \ \& \ f \cdot f^{-1} = \operatorname{id} \operatorname{rng} f, \tag{61}$$

$$f \text{ is\_one-to-one } \mathbf{implies}\ f^{-1} \text{ is\_one-to-one}, \tag{62}$$

$$f \text{ is\_one-to-one} \ \& \ \operatorname{rng} f = \operatorname{dom} g \ \& \ g \cdot f = \operatorname{id} \operatorname{dom} f \ \mathbf{implies}\ g = f^{-1}, \tag{63}$$

$$f \text{ is\_one-to-one} \ \& \ \operatorname{rng} g = \operatorname{dom} f \ \& \ f \cdot g = \operatorname{id} \operatorname{rng} f \ \mathbf{implies}\ g = f^{-1}, \tag{64}$$

$$f \text{ is\_one-to-one } \mathbf{implies}\ (f^{-1})^{-1} = f, \tag{65}$$

$$f \text{ is\_one-to-one} \ \& \ g \text{ is\_one-to-one } \mathbf{implies}\ (g \cdot f)^{-1} = f^{-1} \cdot g^{-1}, \tag{66}$$

$$(\operatorname{id} X)^{-1} = (\operatorname{id} X). \tag{67}$$

Let us consider f, X. The functor

$$f \mid X,$$

yields the type Function and is defined by

$$\operatorname{dom} \mathbf{it} = \operatorname{dom} f \cap X \ \& \ \mathbf{for}\ x\ \mathbf{st}\ x \in \operatorname{dom} \mathbf{it}\ \mathbf{holds}\ \mathbf{it}.x = f.x.$$

We now state a number of propositions:

$$g = f \mid X \textbf{ iff } \operatorname{dom} g = \operatorname{dom} f \cap X \;\&\; \textbf{for } x \textbf{ st } x \in \operatorname{dom} g \textbf{ holds } g.x = f.x, \tag{68}$$

$$\operatorname{dom}(f \mid X) = \operatorname{dom} f \cap X, \tag{69}$$

$$x \in \operatorname{dom}(f \mid X) \textbf{ implies } (f \mid X).x = f.x, \tag{70}$$

$$x \in \operatorname{dom} f \cap X \textbf{ implies } (f \mid X).x = f.x, \tag{71}$$

$$x \in \operatorname{dom} f \;\&\; x \in X \textbf{ implies } (f \mid X).x = f.x, \tag{72}$$

$$x \in \operatorname{dom} f \;\&\; x \in X \textbf{ implies } f.x \in \operatorname{rng}(f \mid X), \tag{73}$$

$$X \subseteq \operatorname{dom} f \textbf{ implies } \operatorname{dom}(f \mid X) = X, \tag{74}$$

$$\operatorname{dom}(f \mid X) \subseteq X, \tag{75}$$

$$\operatorname{dom}(f \mid X) \subseteq \operatorname{dom} f \;\&\; \operatorname{rng}(f \mid X) \subseteq \operatorname{rng} f, \tag{76}$$

$$f \mid X = f \cdot (\operatorname{id} X), \tag{77}$$

$$\operatorname{dom} f \subseteq X \textbf{ implies } f \mid X = f, \tag{78}$$

$$f \mid (\operatorname{dom} f) = f, \tag{79}$$

$$(f \mid X) \mid Y = f \mid (X \cap Y), \tag{80}$$

$$(f \mid X) \mid X = f \mid X, \tag{81}$$

$$X \subseteq Y \textbf{ implies } (f \mid X) \mid Y = f \mid X \;\&\; (f \mid Y) \mid X = f \mid X, \tag{82}$$

$$(g \cdot f) \mid X = g \cdot (f \mid X), \tag{83}$$

$$f \text{ is\_one-to-one} \textbf{ implies } f \mid X \text{ is\_one-to-one}. \tag{84}$$

Let us consider Y, f. The functor

$$Y \mid f,$$

with values of the type Function, is defined by

$$\begin{gathered}(\textbf{for } x \textbf{ holds } x \in \operatorname{dom} \textbf{it} \textbf{ iff } x \in \operatorname{dom} f \;\&\; f.x \in Y) \\ \&\; \textbf{for } x \textbf{ st } x \in \operatorname{dom} \textbf{it} \textbf{ holds } \textbf{it}.x = f.x.\end{gathered}$$

We now state a number of propositions:

$$\begin{gathered}g = Y \mid f \textbf{ iff } (\textbf{for } x \textbf{ holds } x \in \operatorname{dom} g \textbf{ iff } x \in \operatorname{dom} f \;\&\; f.x \in Y) \\ \&\; \textbf{for } x \textbf{ st } x \in \operatorname{dom} g \textbf{ holds } g.x = f.x,\end{gathered} \tag{85}$$

$$x \in \operatorname{dom}(Y \mid f) \textbf{ iff } x \in \operatorname{dom} f \;\&\; f.x \in Y, \tag{86}$$

$$x \in \operatorname{dom}(Y \mid f) \textbf{ implies } (Y \mid f).x = f.x, \tag{87}$$

$$\operatorname{rng}(Y \mid f) \subseteq Y, \tag{88}$$

$$\operatorname{dom}(Y \mid f) \subseteq \operatorname{dom} f \;\&\; \operatorname{rng}(Y \mid f) \subseteq \operatorname{rng} f, \tag{89}$$

$$\operatorname{rng}(Y \mid f) = \operatorname{rng} f \cap Y, \tag{90}$$

$$Y \subseteq \operatorname{rng} f \textbf{ implies } \operatorname{rng}(Y \mid f) = Y, \tag{91}$$

$$Y \mid f = (\operatorname{id} Y) \cdot f, \tag{92}$$

$$\operatorname{rng} f \subseteq Y \textbf{ implies } Y \mid f = f, \tag{93}$$

$$(\operatorname{rng} f) \mid f = f, \tag{94}$$

$$Y \mid (X \mid f) = (Y \cap X) \mid f, \tag{95}$$

$$Y \mid (Y \mid f) = Y \mid f, \tag{96}$$

$$X \subseteq Y \textbf{ implies } Y \mid (X \mid f) = X \mid f \;\&\; X \mid (Y \mid f) = X \mid f, \tag{97}$$

$$Y \mid (g \cdot f) = (Y \mid g) \cdot f, \tag{98}$$

$$f \text{ is\_one-to-one} \textbf{ implies } Y \mid f \text{ is\_one-to-one}, \tag{99}$$

$$(Y \mid f) \mid X = Y \mid (f \mid X). \tag{100}$$

Let us consider f, X. The functor

$$f \circ X,$$

yields the type set and is defined by

$$\textbf{for } y \textbf{ holds } y \in \textbf{it} \textbf{ iff } \textbf{ex } x \textbf{ st } x \in \operatorname{dom} f \;\&\; x \in X \;\&\; y = f.x.$$

The following propositions are true:

$$Y = f \circ X \textbf{ iff } \textbf{for } y \textbf{ holds } y \in Y \textbf{ iff } \textbf{ex } x \textbf{ st } x \in \operatorname{dom} f \;\&\; x \in X \;\&\; y = f.x, \tag{101}$$

$$y \in f \circ X \textbf{ iff } \textbf{ex } x \textbf{ st } x \in \operatorname{dom} f \;\&\; x \in X \;\&\; y = f.x, \tag{102}$$

$$f \circ X \subseteq \operatorname{rng} f, \tag{103}$$

$$f \circ (X) = f \circ (\operatorname{dom} f \cap X), \tag{104}$$

$$f \circ (\operatorname{dom} f) = \operatorname{rng} f, \tag{105}$$

$$f \circ X \subseteq f \circ (\operatorname{dom} f), \tag{106}$$

(107) $$\operatorname{rng}(f \mid X) = f^{\circ} X,$$

(108) $$f^{\circ} X = \emptyset \textbf{ iff } \operatorname{dom} f \cap X = \emptyset,$$

(109) $$f^{\circ} \emptyset = \emptyset,$$

(110) $$X \neq \emptyset \;\&\; X \subseteq \operatorname{dom} f \textbf{ implies } f^{\circ} X \neq \emptyset,$$

(111) $$X1 \subseteq X2 \textbf{ implies } f^{\circ} X1 \subseteq f^{\circ} X2,$$

(112) $$f^{\circ} (X1 \cup X2) = f^{\circ} X1 \cup f^{\circ} X2,$$

(113) $$f^{\circ} (X1 \cap X2) \subseteq f^{\circ} X1 \cap f^{\circ} X2,$$

(114) $$f^{\circ} X1 \setminus f^{\circ} X2 \subseteq f^{\circ} (X1 \setminus X2),$$

(115) $$(g \cdot f)^{\circ} X = g^{\circ} (f^{\circ} X),$$

(116) $$\operatorname{rng}(g \cdot f) = g^{\circ} (\operatorname{rng} f),$$

(117) $$x \in \operatorname{dom} f \textbf{ implies } f^{\circ} \{x\} = \{f.x\},$$

(118) $$x1 \in \operatorname{dom} f \;\&\; x2 \in \operatorname{dom} f \textbf{ implies } f^{\circ} \{x1,x2\} = \{f.x1,f.x2\},$$

(119) $$(f \mid Y)^{\circ} X \subseteq f^{\circ} X,$$

(120) $$(Y \mid f)^{\circ} X \subseteq f^{\circ} X,$$

(121) $$f \text{ is\_one-to-one} \textbf{ implies } f^{\circ} (X1 \cap X2) = f^{\circ} X1 \cap f^{\circ} X2,$$

(122) $$(\textbf{for } X1,X2 \textbf{ holds } f^{\circ} (X1 \cap X2) = f^{\circ} X1 \cap f^{\circ} X2) \\ \textbf{implies } f \text{ is\_one-to-one},$$

(123) $$f \text{ is\_one-to-one} \textbf{ implies } f^{\circ} (X1 \setminus X2) = f^{\circ} X1 \setminus f^{\circ} X2,$$

(124) $$(\textbf{for } X1,X2 \textbf{ holds } f^{\circ} (X1 \setminus X2) = f^{\circ} X1 \setminus f^{\circ} X2) \textbf{ implies } f \text{ is\_one-to-one},$$

(125) $$X \cap Y = \emptyset \;\&\; f \text{ is\_one-to-one} \textbf{ implies } f^{\circ} X \cap f^{\circ} Y = \emptyset,$$

(126) $$(Y \mid f)^{\circ} X = Y \cap f^{\circ} X.$$

Let us consider f, Y. The functor

$$f^{-1}\, Y,$$

yields the type set and is defined by

$$\textbf{for } x \textbf{ holds } x \in \textbf{it} \textbf{ iff } x \in \operatorname{dom} f \;\&\; f.x \in Y.$$

We now state a number of propositions:

(127) $$X = f^{-1}\,Y \ \textbf{iff}\ \textbf{for}\ x\ \textbf{holds}\ x \in X\ \textbf{iff}\ x \in \operatorname{dom} f\ \&\ f.x \in Y,$$

(128) $$x \in f^{-1}\,Y\ \textbf{iff}\ x \in \operatorname{dom} f\ \&\ f.x \in Y,$$

(129) $$f^{-1}\,Y \subseteq \operatorname{dom} f,$$

(130) $$f^{-1}\,Y = f^{-1}\,(\operatorname{rng} f \cap Y),$$

(131) $$f^{-1}\,(\operatorname{rng} f) = \operatorname{dom} f,$$

(132) $$f^{-1}\,\emptyset = \emptyset,$$

(133) $$f^{-1}\,Y = \emptyset\ \textbf{iff}\ \operatorname{rng} f \cap Y = \emptyset,$$

(134) $$Y \subseteq \operatorname{rng} f\ \textbf{implies}\ (f^{-1}\,Y = \emptyset\ \textbf{iff}\ Y = \emptyset),$$

(135) $$Y1 \subseteq Y2\ \textbf{implies}\ f^{-1}\,Y1 \subseteq f^{-1}\,Y2,$$

(136) $$f^{-1}\,(Y1 \cup Y2) = f^{-1}\,Y1 \cup f^{-1}\,Y2,$$

(137) $$f^{-1}\,(Y1 \cap Y2) = f^{-1}\,Y1 \cap f^{-1}\,Y2,$$

(138) $$f^{-1}\,(Y1 \setminus Y2) = f^{-1}\,Y1 \setminus f^{-1}\,Y2,$$

(139) $$(f \mid X)^{-1}\,Y = X \cap (f^{-1}\,Y),$$

(140) $$(g \cdot f)^{-1}\,Y = f^{-1}\,(g^{-1}\,Y),$$

(141) $$\operatorname{dom}(g \cdot f) = f^{-1}\,(\operatorname{dom} g),$$

(142) $$y \in \operatorname{rng} f\ \textbf{iff}\ f^{-1}\,\{y\} \neq \emptyset,$$

(143) $$(\textbf{for}\ y\ \textbf{st}\ y \in Y\ \textbf{holds}\ f^{-1}\,\{y\} \neq \emptyset)\ \textbf{implies}\ Y \subseteq \operatorname{rng} f,$$

(144) $$(\textbf{for}\ y\ \textbf{st}\ y \in \operatorname{rng} f\ \textbf{ex}\ x\ \textbf{st}\ f^{-1}\,\{y\} = \{x\})\ \textbf{iff}\ f\ \text{is\_one-to-one},$$

(145) $$f^{\circ}\,(f^{-1}\,Y) \subseteq Y,$$

(146) $$X \subseteq \operatorname{dom} f\ \textbf{implies}\ X \subseteq f^{-1}\,(f^{\circ}\,X),$$

(147) $$Y \subseteq \operatorname{rng} f\ \textbf{implies}\ f^{\circ}\,(f^{-1}\,Y) = Y,$$

(148) $$f^{\circ}\,(f^{-1}\,Y) = Y \cap f^{\circ}\,(\operatorname{dom} f),$$

(149) $$f^{\circ}\,(X \cap f^{-1}\,Y) \subseteq (f^{\circ}\,X) \cap Y,$$

(150) $$f^{\circ}\,(X \cap f^{-1}\,Y) = (f^{\circ}\,X) \cap Y,$$

(151) $$X \cap f^{-1} Y \subseteq f^{-1} (f^{\circ} X \cap Y),$$

(152) $$f \text{ is\_one-to-one } \textbf{implies } f^{-1} (f^{\circ} X) \subseteq X,$$

(153) $$(\textbf{for } X \textbf{ holds } f^{-1} (f^{\circ} X) \subseteq X) \textbf{ implies } f \text{ is\_one-to-one},$$

(154) $$f \text{ is\_one-to-one } \textbf{implies } f^{\circ} X = (f^{-1})^{-1} X,$$

(155) $$f \text{ is\_one-to-one } \textbf{implies } f^{-1} Y = (f^{-1})^{\circ} Y,$$

(156) $$Y = \operatorname{rng} f \ \& \ \operatorname{dom} g = Y \ \& \ \operatorname{dom} h = Y \ \& \ g \cdot f = h \cdot f \textbf{ implies } g = h,$$

(157) $$f^{\circ} X1 \subseteq f^{\circ} X2 \ \& \ X1 \subseteq \operatorname{dom} f \ \& \ f \text{ is\_one-to-one } \textbf{implies } X1 \subseteq X2,$$

(158) $$f^{-1} Y1 \subseteq f^{-1} Y2 \ \& \ Y1 \subseteq \operatorname{rng} f \textbf{ implies } Y1 \subseteq Y2,$$

(159) $$f \text{ is\_one-to-one } \textbf{iff for } y \textbf{ ex } x \textbf{ st } f^{-1} \{y\} \subseteq \{x\},$$

(160) $$\operatorname{rng} f \subseteq \operatorname{dom} g \textbf{ implies } f^{-1} X \subseteq (g \cdot f)^{-1} (g^{\circ} X).$$

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