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Basic Functions and Operations on Functions

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Summary. We define the following mappings: the characteristic function of a subset of a set, the inclusion function (injection or embedding), the projections from a Cartesian product onto its arguments and diagonal function (inclusion of a set into its Cartesian square). Some operations on functions are also defined: the products of two functions (the complex function and the more general product-function), the function induced on power sets by the image and inverse-image. Some simple propositions related to the introduced notions are proved.

The terminology and notation used in this paper are introduced in the following papers: [3], [4], [1], and [2]. For simplicity we adopt the following convention: x, y, z, $z1$, $z2$ denote objects of the type Any; A, B, V, X, $X1$, $X2$, Y, $Y1$, $Y2$, Z denote objects of the type set; C, $C1$, $C2$, D, $D1$, $D2$ denote objects of the type DOMAIN. We now state several propositions:

(1) $A \subseteq Y$ **implies** $\mathrm{id}\, A = (\mathrm{id}\, Y) \mid A$,

(2) **for** f,g **being** Function **st** $X \subseteq \mathrm{dom}\,(g \cdot f)$ **holds** $f\,^{\circ}\, X \subseteq \mathrm{dom}\, g$,

(3) **for** f,g **being** Function **st** $X \subseteq \mathrm{dom}\, f$ & $f\,^{\circ}\, X \subseteq \mathrm{dom}\, g$ **holds** $X \subseteq \mathrm{dom}\,(g \cdot f)$,

(4) **for** f,g **being** Function **st** $Y \subseteq \mathrm{rng}\,(g \cdot f)$ & g is_one-to-one **holds** $g\,^{-1}\, Y \subseteq \mathrm{rng}\, f$,

(5) **for** f,g **being** Function **st** $Y \subseteq \mathrm{rng}\, g$ & $g\,^{-1}\, Y \subseteq \mathrm{rng}\, f$ **holds** $Y \subseteq \mathrm{rng}\,(g \cdot f)$.

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In the article we present several logical schemes. The scheme *FuncEx_3* concerns a constant $\mathcal{A}$ that has the type set, a constant $\mathcal{B}$ that has the type set and a ternary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex}\, f \textbf{ being } \mathrm{Function}$$
$$\textbf{st } \mathrm{dom}\, f = [:\mathcal{A},\mathcal{B}:] \;\&\; \textbf{for}\, x,y \textbf{ st } x \in \mathcal{A} \;\&\; y \in \mathcal{B} \textbf{ holds } \mathcal{P}[x, y, f\,.\langle x, y\rangle]$$

provided the parameters satisfy the following conditions:

- $\textbf{for}\, x,y,z1,z2 \textbf{ st } x \in \mathcal{A} \;\&\; y \in \mathcal{B} \;\&\; \mathcal{P}[x, y, z1] \;\&\; \mathcal{P}[x, y, z2] \textbf{ holds } z1 = z2,$
- $\textbf{for}\, x,y \textbf{ st } x \in \mathcal{A} \;\&\; y \in \mathcal{B} \textbf{ ex}\, z \textbf{ st } \mathcal{P}[x, y, z].$

The scheme *Lambda_3* concerns a constant $\mathcal{A}$ that has the type set, a constant $\mathcal{B}$ that has the type set and a binary functor $\mathcal{F}$ and states that the following holds

$$\textbf{ex}\, f \textbf{ being } \mathrm{Function}$$
$$\textbf{st } \mathrm{dom}\, f = [:\mathcal{A},\mathcal{B}:] \;\&\; \textbf{for}\, x,y \textbf{ st } x \in \mathcal{A} \;\&\; y \in \mathcal{B} \textbf{ holds } f\,.\langle x, y\rangle = \mathcal{F}(x, y)$$

for all values of the parameters.

We now state a proposition

$$\textbf{for}\, f,g \textbf{ being } \mathrm{Function} \textbf{ st}$$
$$\mathrm{dom}\, f = [:X, Y:]$$
$$\&\; \mathrm{dom}\, g = [:X, Y:] \;\&\; \textbf{for}\, x,y \textbf{ st } x \in X \;\&\; y \in Y \textbf{ holds } f\,.\langle x, y\rangle = g\,.\langle x, y\rangle$$
$$\textbf{holds } f = g. \tag{6}$$

Let f have the type Function. The functor

$$^{\circ} f,$$

yields the type Function and is defined by

$$\mathrm{dom}\,\textbf{it} = \mathrm{bool}\,\mathrm{dom}\, f \;\&\; \textbf{for}\, X \textbf{ st } X \in \mathrm{bool}\,\mathrm{dom}\, f \textbf{ holds } \textbf{it}.X = f \,^{\circ}\, X.$$

The following propositions are true:

$$\textbf{for}\, f,g \textbf{ being } \mathrm{Function} \textbf{ holds } g = {}^{\circ} f$$
$$\textbf{iff } \mathrm{dom}\, g = \mathrm{bool}\,\mathrm{dom}\, f \;\&\; \textbf{for}\, X \textbf{ st } X \in \mathrm{bool}\,\mathrm{dom}\, f \textbf{ holds } g.X = f \,^{\circ}\, X, \tag{7}$$

$$\textbf{for}\, f \textbf{ being } \mathrm{Function} \textbf{ st } X \in \mathrm{dom}\,({}^{\circ} f) \textbf{ holds } ({}^{\circ} f).X = f \,^{\circ}\, X, \tag{8}$$

$$\textbf{for}\, f \textbf{ being } \mathrm{Function} \textbf{ holds } ({}^{\circ} f).\emptyset = \emptyset, \tag{9}$$

$$\textbf{for}\, f \textbf{ being } \mathrm{Function} \textbf{ holds } \mathrm{rng}\,({}^{\circ} f) \subseteq \mathrm{bool}\,\mathrm{rng}\, f, \tag{10}$$

$$\textbf{for}\, f \textbf{ being } \mathrm{Function}$$
$$\textbf{holds } Y \in ({}^{\circ} f) \,^{\circ}\, A \textbf{ iff ex}\, X \textbf{ st } X \in \mathrm{dom}\,({}^{\circ} f) \;\&\; X \in A \;\&\; Y = ({}^{\circ} f).X, \tag{11}$$

$$\textbf{for } f \textbf{ being } \text{Function } \textbf{holds } ({}^{\circ} f)\, {}^{\circ}\, A \subseteq \text{bool}\,\text{rng}\, f, \tag{12}$$

$$\textbf{for } f \textbf{ being } \text{Function } \textbf{holds } ({}^{\circ} f)\, {}^{-1}\, B \subseteq \text{bool}\,\text{dom}\, f, \tag{13}$$

$$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D \textbf{ holds } ({}^{\circ} f)\, {}^{-1}\, B \subseteq \text{bool}\, X, \tag{14}$$

$$\textbf{for } f \textbf{ being } \text{Function } \textbf{holds } \bigcup(({}^{\circ} f)\, {}^{\circ}\, A) \subseteq f\, {}^{\circ}\, (\bigcup A), \tag{15}$$

$$\textbf{for } f \textbf{ being } \text{Function } \textbf{st } A \subseteq \text{bool}\,\text{dom}\, f \textbf{ holds } f\, {}^{\circ}\, (\bigcup A) = \bigcup(({}^{\circ} f)\, {}^{\circ}\, A), \tag{16}$$

$$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D \textbf{ st } A \subseteq \text{bool}\, X \textbf{ holds } f\, {}^{\circ}\, (\bigcup A) = \bigcup(({}^{\circ} f)\, {}^{\circ}\, A), \tag{17}$$

$$\textbf{for } f \textbf{ being } \text{Function } \textbf{holds } \bigcup(({}^{\circ} f)\, {}^{-1}\, B) \subseteq f\, {}^{-1}\, (\bigcup B), \tag{18}$$

$$\textbf{for } f \textbf{ being } \text{Function } \textbf{st } B \subseteq \text{bool}\,\text{rng}\, f \textbf{ holds } f\, {}^{-1}\, (\bigcup B) = \bigcup(({}^{\circ} f)\, {}^{-1}\, B), \tag{19}$$

$$\textbf{for } f, g \textbf{ being } \text{Function } \textbf{holds } {}^{\circ}\, (g \cdot f) = {}^{\circ} g \cdot {}^{\circ} f, \tag{20}$$

$$\textbf{for } f \textbf{ being } \text{Function } \textbf{holds } {}^{\circ} f \textbf{ is } \text{Function } \textbf{of } \text{bool}\,\text{dom}\, f, \text{bool}\,\text{rng}\, f, \tag{21}$$

$$\begin{gathered}\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \\ \textbf{st } Y = \emptyset \textbf{ implies } X = \emptyset \textbf{ holds } {}^{\circ} f \textbf{ is } \text{Function } \textbf{of } \text{bool}\, X, \text{bool}\, Y.\end{gathered} \tag{22}$$

The arguments of the notions defined below are the following: X, D which are objects of the type reserved above; f which is an object of the type Function **of** X, D. Let us note that it makes sense to consider the following functor on a restricted area. Then

$${}^{\circ} f \qquad \text{is} \qquad \text{Function } \textbf{of } \text{bool}\, X, \text{bool}\, D.$$

Let f have the type Function. The functor

$${}^{-1} f,$$

yields the type Function and is defined by

$$\text{dom}\, \textbf{it} = \text{bool}\,\text{rng}\, f\ \&\ \textbf{for } Y \textbf{ st } Y \in \text{bool}\,\text{rng}\, f \textbf{ holds } \textbf{it}.Y = f\, {}^{-1}\, Y.$$

We now state a number of propositions:

$$\begin{gathered}\textbf{for } g, f \textbf{ being } \text{Function } \textbf{holds} \\ g = {}^{-1} f \textbf{ iff } \text{dom}\, g = \text{bool}\,\text{rng}\, f\ \&\ \textbf{for } Y \textbf{ st } Y \in \text{bool}\,\text{rng}\, f \textbf{ holds } g.Y = f\, {}^{-1}\, Y,\end{gathered} \tag{23}$$

$$\textbf{for } f \textbf{ being } \text{Function } \textbf{st } Y \in \text{dom}\, ({}^{-1} f) \textbf{ holds } ({}^{-1} f).Y = f\, {}^{-1}\, Y, \tag{24}$$

$$\textbf{for } f \textbf{ being } \text{Function } \textbf{holds } \text{rng}\, ({}^{-1} f) \subseteq \text{bool}\,\text{dom}\, f, \tag{25}$$

$$\begin{gathered}\textbf{for } f \textbf{ being } \text{Function} \\ \textbf{holds } X \in ({}^{-1} f)\, {}^{\circ}\, A \textbf{ iff } \textbf{ex } Y \textbf{ st } Y \in \text{dom}\, ({}^{-1} f)\ \&\ Y \in A\ \&\ X = ({}^{-1} f).Y,\end{gathered} \tag{26}$$

(27) $\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ holds } ({}^{-1} f)\,{}^{\circ}\, B \subseteq \mathrm{bool}\,\mathrm{dom}\, f,$

(28) $\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ holds } ({}^{-1} f)\,{}^{-1}\, A \subseteq \mathrm{bool}\,\mathrm{rng}\, f,$

(29) $\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ holds } \bigcup(({}^{-1} f)\,{}^{\circ}\, B) \subseteq f\,{}^{-1}\,(\bigcup B),$

(30) $\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ st } B \subseteq \mathrm{bool}\,\mathrm{rng}\, f \textbf{ holds } \bigcup(({}^{-1} f)\,{}^{\circ}\, B) = f\,{}^{-1}\,(\bigcup B),$

(31) $\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ holds } \bigcup(({}^{-1} f)\,{}^{-1}\, A) \subseteq f\,{}^{\circ}\,(\bigcup A),$

(32) $\textbf{for } f \textbf{ being } \mathrm{Function}$
$\textbf{st } A \subseteq \mathrm{bool}\,\mathrm{dom}\, f \ \&\ f \ \mathrm{is\_one\text{-}to\text{-}one} \textbf{ holds } \bigcup(({}^{-1} f)\,{}^{-1}\, A) = f\,{}^{\circ}\,(\bigcup A),$

(33) $\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ holds } ({}^{-1} f)\,{}^{\circ}\, B \subseteq ({}^{\circ} f)\,{}^{-1}\, B,$

(34) $\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ st } f \ \mathrm{is\_one\text{-}to\text{-}one} \textbf{ holds } ({}^{-1} f)\,{}^{\circ}\, B = ({}^{\circ} f)\,{}^{-1}\, B,$

(35) $\textbf{for } f \textbf{ being } \mathrm{Function},\ A \textbf{ being } \mathrm{set}$
$\textbf{st } A \subseteq \mathrm{bool}\,\mathrm{dom}\, f \textbf{ holds } ({}^{-1} f)\,{}^{-1}\, A \subseteq ({}^{\circ} f)\,{}^{\circ}\, A,$

(36) $\textbf{for } f \textbf{ being } \mathrm{Function},\ A \textbf{ being } \mathrm{set}$
$\textbf{st } f \ \mathrm{is\_one\text{-}to\text{-}one} \textbf{ holds } ({}^{\circ} f)\,{}^{\circ}\, A \subseteq ({}^{-1} f)\,{}^{-1}\, A,$

(37) $\textbf{for } f \textbf{ being } \mathrm{Function},\ A \textbf{ being } \mathrm{set}$
$\textbf{st } f \ \mathrm{is\_one\text{-}to\text{-}one} \ \&\ A \subseteq \mathrm{bool}\,\mathrm{dom}\, f \textbf{ holds } ({}^{-1} f)\,{}^{-1}\, A = ({}^{\circ} f)\,{}^{\circ}\, A,$

(38) $\textbf{for } f{,}g \textbf{ being } \mathrm{Function} \textbf{ st } g \ \mathrm{is\_one\text{-}to\text{-}one} \textbf{ holds } {}^{-1}\,(g \cdot f) = {}^{-1} f \cdot {}^{-1} g,$

(39) $\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ holds } {}^{-1} f \textbf{ is } \mathrm{Function} \textbf{ of } \mathrm{bool}\,\mathrm{rng}\, f{,}\mathrm{bool}\,\mathrm{dom}\, f.$

Let us consider A, X. The functor

$$\chi\,(A, X),$$

yields the type Function and is defined by

$$\mathrm{dom}\,\textbf{it} = X$$
$$\&\ \textbf{for } x \textbf{ st } x \in X \textbf{ holds } (x \in A \textbf{ implies it}.x = 1)\ \&\ (\textbf{not } x \in A \textbf{ implies it}.x = 0).$$

We now state a number of propositions:

(40) $\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ holds } f = \chi\,(A, X) \textbf{ iff } \mathrm{dom}\, f = X \ \&\ \textbf{for } x$
$\textbf{st } x \in X \textbf{ holds } (x \in A \textbf{ implies } f.x = 1)\ \&\ (\textbf{not } x \in A \textbf{ implies } f.x = 0),$

(41) $A \subseteq X \ \&\ x \in A \textbf{ implies } \chi\,(A, X).x = 1,$

(42) $x \in X \ \&\ \chi\,(A, X).x = 1 \textbf{ implies } x \in A,$

(43) $$x \in X \setminus A \textbf{ implies } \chi(A, X).x = 0,$$

(44) $$x \in X \ \& \ \chi(A, X).x = 0 \textbf{ implies not } x \in A,$$

(45) $$x \in X \textbf{ implies } \chi(\emptyset, X).x = 0,$$

(46) $$x \in X \textbf{ implies } \chi(X, X).x = 1,$$

(47) $$A \subseteq X \ \& \ B \subseteq X \ \& \ \chi(A, X) = \chi(B, X) \textbf{ implies } A = B,$$

(48) $$\operatorname{rng} \chi(A, X) \subseteq \{0,1\},$$

(49) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, \{0,1\} \textbf{ holds } f = \chi(f^{-1}\{1\}, X).$$

Let us consider A, X. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\chi(A, X) \quad \text{is} \quad \text{Function } \textbf{of } X, \{0,1\}.$$

One can prove the following propositions:

(50) $$\textbf{for } d \textbf{ being } \text{Element } \textbf{of } D \textbf{ holds } \chi(A, D).d = 1 \textbf{ iff } d \in A,$$

(51) $$\textbf{for } d \textbf{ being } \text{Element } \textbf{of } D \textbf{ holds } \chi(A, D).d = 0 \textbf{ iff not } d \in A.$$

The arguments of the notions defined below are the following: Y which is an object of the type reserved above; A which is an object of the type Subset **of** Y. The functor

$$\operatorname{incl} A,$$

yields the type Function **of** A, Y and is defined by

$$\textbf{it} = \operatorname{id} A.$$

We now state several propositions:

(52) $$\textbf{for } A \textbf{ being } \text{Subset } \textbf{of } Y \textbf{ holds } \operatorname{incl} A = \operatorname{id} A,$$

(53) $$\textbf{for } A \textbf{ being } \text{Subset } \textbf{of } Y \textbf{ holds } \operatorname{incl} A = (\operatorname{id} Y) \mid A,$$

(54) $$\textbf{for } A \textbf{ being } \text{Subset } \textbf{of } Y \textbf{ holds } \operatorname{dom} \operatorname{incl} A = A \ \& \ \operatorname{rng} \operatorname{incl} A = A,$$

(55) $$\textbf{for } A \textbf{ being } \text{Subset } \textbf{of } Y \textbf{ st } x \in A \textbf{ holds } (\operatorname{incl} A).x = x,$$

(56) $$\textbf{for } A \textbf{ being } \text{Subset } \textbf{of } Y \textbf{ st } x \in A \textbf{ holds } \operatorname{incl}(A).x \in Y.$$

We now define two new functors. Let us consider X, Y. The functor

$$\pi_1(X, Y),$$

with values of the type Function, is defined by

$$\mathrm{dom}\,\mathbf{it} = [:X, Y:]\ \&\ \mathbf{for}\ x,y\ \mathbf{st}\ x \in X\ \&\ y \in Y\ \mathbf{holds}\ \mathbf{it}.\langle x, y\rangle = x.$$

The functor

$$\pi_2\,(X, Y),$$

yields the type Function and is defined by

$$\mathrm{dom}\,\mathbf{it} = [:X, Y:]\ \&\ \mathbf{for}\ x,y\ \mathbf{st}\ x \in X\ \&\ y \in Y\ \mathbf{holds}\ \mathbf{it}.\langle x, y\rangle = y.$$

Next we state several propositions:

(57) $$\mathbf{for}\ f\ \mathbf{being}\ \mathrm{Function}\ \mathbf{holds}\ f = \pi_1\,(X, Y)\ \mathbf{iff}\ \mathrm{dom}\, f = [:X, Y:]\ \&\ \mathbf{for}\ x,y\ \mathbf{st}\ x \in X\ \&\ y \in Y\ \mathbf{holds}\ f.\langle x, y\rangle = x,$$

(58) $$\mathbf{for}\ f\ \mathbf{being}\ \mathrm{Function}\ \mathbf{holds}\ f = \pi_2\,(X, Y)\ \mathbf{iff}\ \mathrm{dom}\, f = [:X, Y:]\ \&\ \mathbf{for}\ x,y\ \mathbf{st}\ x \in X\ \&\ y \in Y\ \mathbf{holds}\ f.\langle x, y\rangle = y,$$

(59) $$\mathrm{rng}\,\pi_1\,(X, Y) \subseteq X,$$

(60) $$Y \neq \emptyset\ \mathbf{implies}\ \mathrm{rng}\,\pi_1\,(X, Y) = X,$$

(61) $$\mathrm{rng}\,\pi_2\,(X, Y) \subseteq Y,$$

(62) $$X \neq \emptyset\ \mathbf{implies}\ \mathrm{rng}\,\pi_2\,(X, Y) = Y.$$

Let us consider X, Y. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$\pi_1\,(X, Y)\quad \text{is}\quad \mathrm{Function}\ \mathbf{of}\ [:X, Y:], X,$$

$$\pi_2\,(X, Y)\quad \text{is}\quad \mathrm{Function}\ \mathbf{of}\ [:X, Y:], Y.$$

We now state two propositions:

(63) $$\mathbf{for}\ d1\ \mathbf{being}\ \mathrm{Element}\ \mathbf{of}\ D1\ \mathbf{for}\ d2\ \mathbf{being}\ \mathrm{Element}\ \mathbf{of}\ D2\ \mathbf{holds}\ \pi_1\,(D1,D2).\langle d1,d2\rangle = d1,$$

(64) $$\mathbf{for}\ d1\ \mathbf{being}\ \mathrm{Element}\ \mathbf{of}\ D1\ \mathbf{for}\ d2\ \mathbf{being}\ \mathrm{Element}\ \mathbf{of}\ D2\ \mathbf{holds}\ \pi_2\,(D1,D2).\langle d1,d2\rangle = d2.$$

Let us consider X. The functor

$$\delta\, X,$$

with values of the type Function, is defined by

$$\mathrm{dom}\,\mathbf{it} = X\ \&\ \mathbf{for}\ x\ \mathbf{st}\ x \in X\ \mathbf{holds}\ \mathbf{it}.x = \langle x, x\rangle.$$

The following two propositions are true:

(65) $$\textbf{for } f \textbf{ being } \mathrm{Function} \textbf{ holds } f = \delta X \textbf{ iff } \mathrm{dom}\, f = X \ \&\ \textbf{for } x \textbf{ st } x \in X \textbf{ holds } f.x = \langle x, x \rangle,$$

(66) $$\mathrm{rng}\, \delta X \subseteq [:X, X:].$$

Let us consider X. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\delta X \quad \text{is} \quad \mathrm{Function}\ \textbf{of}\ X, [:X, X:].$$

Let f, g have the type Function. The functor

$$[\!\langle f, g \rangle\!],$$

with values of the type Function, is defined by

$$\mathrm{dom}\, \textbf{it} = \mathrm{dom}\, f \cap \mathrm{dom}\, g \ \&\ \textbf{for } x \textbf{ st } x \in \mathrm{dom}\, \textbf{it} \textbf{ holds } \textbf{it}.x = \langle f.x, g.x \rangle.$$

We now state a number of propositions:

(67) $$\textbf{for } f,g,fg \textbf{ being } \mathrm{Function} \textbf{ holds } fg = [\!\langle f, g \rangle\!] \textbf{ iff } \mathrm{dom}\, fg = \mathrm{dom}\, f \cap \mathrm{dom}\, g \ \&\ \textbf{for } x \textbf{ st } x \in \mathrm{dom}\, fg \textbf{ holds } fg.x = \langle f.x, g.x \rangle,$$

(68) $$\textbf{for } f,g \textbf{ being } \mathrm{Function} \textbf{ st } x \in \mathrm{dom}\, f \cap \mathrm{dom}\, g \textbf{ holds } [\!\langle f, g \rangle\!].x = \langle f.x, g.x \rangle,$$

(69) $$\textbf{for } f,g \textbf{ being } \mathrm{Function} \textbf{ st } \mathrm{dom}\, f = X \ \&\ \mathrm{dom}\, g = X \ \&\ x \in X \textbf{ holds } [\!\langle f, g \rangle\!].x = \langle f.x, g.x \rangle,$$

(70) $$\textbf{for } f,g \textbf{ being } \mathrm{Function} \textbf{ st } \mathrm{dom}\, f = X \ \&\ \mathrm{dom}\, g = X \textbf{ holds } \mathrm{dom}\, [\!\langle f, g \rangle\!] = X,$$

(71) $$\textbf{for } f,g \textbf{ being } \mathrm{Function} \textbf{ holds } \mathrm{rng}\, [\!\langle f, g \rangle\!] \subseteq [:\mathrm{rng}\, f, \mathrm{rng}\, g:],$$

(72) $$\textbf{for } f,g \textbf{ being } \mathrm{Function} \textbf{ st } \mathrm{dom}\, f = \mathrm{dom}\, g \ \&\ \mathrm{rng}\, f \subseteq Y \ \&\ \mathrm{rng}\, g \subseteq Z \textbf{ holds } \pi_1(Y, Z) \cdot [\!\langle f, g \rangle\!] = f \ \&\ \pi_2(Y, Z) \cdot [\!\langle f, g \rangle\!] = g,$$

(73) $$[\!\langle \pi_1(X, Y), \pi_2(X, Y) \rangle\!] = \mathrm{id}\, [:X, Y:],$$

(74) $$\textbf{for } f,g,h,k \textbf{ being } \mathrm{Function} \textbf{ st } \mathrm{dom}\, f = \mathrm{dom}\, g \ \&\ \mathrm{dom}\, k = \mathrm{dom}\, h \ \&\ [\!\langle f, g \rangle\!] = [\!\langle k, h \rangle\!] \textbf{ holds } f = k \ \&\ g = h,$$

(75) $$\textbf{for } f,g,h \textbf{ being } \mathrm{Function} \textbf{ holds } [\!\langle f \cdot h, g \cdot h \rangle\!] = [\!\langle f, g \rangle\!] \cdot h,$$

(76) $$\textbf{for } f,g \textbf{ being } \mathrm{Function} \textbf{ holds } [\!\langle f, g \rangle\!] \circ A \subseteq [:f \circ A, g \circ A:],$$

(77) $$\textbf{for } f,g \textbf{ being } \mathrm{Function} \textbf{ holds } [\!\langle f, g \rangle\!]^{-1} [:B, C:] = f^{-1} B \cap g^{-1} C,$$

(78) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ for } g \textbf{ being } \text{Function } \textbf{of } X, Z \textbf{ st } (Y = \emptyset \textbf{ implies } X = \emptyset) \;\&\; (Z = \emptyset \textbf{ implies } X = \emptyset) \textbf{ holds } [\![f, g]\!] \textbf{ is } \text{Function } \textbf{of } X, [\![: Y, Z :]\!].$$

The arguments of the notions defined below are the following: X, $D1$, $D2$ which are objects of the type reserved above; $f1$ which is an object of the type Function **of** X, $D1$; $f2$ which is an object of the type Function **of** X, $D2$. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$[\![f1, f2]\!] \quad \text{is} \quad \text{Function } \textbf{of } X, [: D1, D2 :].$$

We now state several propositions:

(79) $$\textbf{for } f1 \textbf{ being } \text{Function } \textbf{of } C, D1 \textbf{ for } f2 \textbf{ being } \text{Function } \textbf{of } C, D2 \textbf{ for } c \textbf{ being } \text{Element } \textbf{of } C \textbf{ holds } [\![f1, f2]\!].c = \langle f1.c, f2.c \rangle,$$

(80) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ for } g \textbf{ being } \text{Function } \textbf{of } X, Z \textbf{ st } (Y = \emptyset \textbf{ implies } X = \emptyset) \;\&\; (Z = \emptyset \textbf{ implies } X = \emptyset) \textbf{ holds } \operatorname{rng} [\![f, g]\!] \subseteq [: Y, Z :],$$

(81) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ for } g \textbf{ being } \text{Function } \textbf{of } X, Z \textbf{ st } (Y = \emptyset \textbf{ implies } X = \emptyset) \;\&\; (Z = \emptyset \textbf{ implies } X = \emptyset) \textbf{ holds } \pi_1(Y, Z) \cdot [\![f, g]\!] = f \;\&\; \pi_2(Y, Z) \cdot [\![f, g]\!] = g,$$

(82) $$\textbf{for } f \textbf{ being } \text{Function } \textbf{of } X, D1 \textbf{ for } g \textbf{ being } \text{Function } \textbf{of } X, D2 \textbf{ holds } \pi_1(D1, D2) \cdot [\![f, g]\!] = f \;\&\; \pi_2(D1, D2) \cdot [\![f, g]\!] = g,$$

(83) $$\textbf{for } f1, f2 \textbf{ being } \text{Function } \textbf{of } X, Y \textbf{ for } g1, g2 \textbf{ being } \text{Function } \textbf{of } X, Z \textbf{ st } (Y = \emptyset \textbf{ implies } X = \emptyset) \;\&\; (Z = \emptyset \textbf{ implies } X = \emptyset) \;\&\; [\![f1, g1]\!] = [\![f2, g2]\!] \textbf{ holds } f1 = f2 \;\&\; g1 = g2,$$

(84) $$\textbf{for } f1, f2 \textbf{ being } \text{Function } \textbf{of } X, D1 \textbf{ for } g1, g2 \textbf{ being } \text{Function } \textbf{of } X, D2 \textbf{ st } [\![f1, g1]\!] = [\![f2, g2]\!] \textbf{ holds } f1 = f2 \;\&\; g1 = g2.$$

Let f, g have the type Function. The functor

$$[: f, g :],$$

yields the type Function and is defined by

$$\operatorname{dom} \textbf{it} = [: \operatorname{dom} f, \operatorname{dom} g :] \;\&\; \textbf{for } x, y \textbf{ st } x \in \operatorname{dom} f \;\&\; y \in \operatorname{dom} g \textbf{ holds } \textbf{it}.\langle x, y \rangle = \langle f.x, g.y \rangle.$$

The following propositions are true:

(85) $$\textbf{for } f, g, fg \textbf{ being } \text{Function } \textbf{holds } fg = [: f, g :] \textbf{ iff } \operatorname{dom} fg = [: \operatorname{dom} f, \operatorname{dom} g :] \;\&\; \textbf{for } x, y \textbf{ st } x \in \operatorname{dom} f \;\&\; y \in \operatorname{dom} g \textbf{ holds } fg.\langle x, y \rangle = \langle f.x, g.y \rangle,$$

(86) $$\textbf{for } f,g \textbf{ being } \text{Function}, x,y \textbf{ st } \langle x,y\rangle \in [:\operatorname{dom} f,\operatorname{dom} g:] \textbf{ holds } [:f,g:].\langle x,y\rangle = \langle f.x,g.y\rangle,$$

(87) $$\textbf{for } f,g \textbf{ being } \text{Function} \textbf{ holds } [:f,g:] = [\![f \cdot \pi_1(\operatorname{dom} f,\operatorname{dom} g), g\cdot \pi_2(\operatorname{dom} f,\operatorname{dom} g)]\!],$$

(88) $$\textbf{for } f,g \textbf{ being } \text{Function} \textbf{ holds } \operatorname{rng}[:f,g:] = [:\operatorname{rng} f,\operatorname{rng} g:],$$

(89) $$\textbf{for } f,g \textbf{ being } \text{Function} \textbf{ st } \operatorname{dom} f = X \ \&\ \operatorname{dom} g = X \textbf{ holds } [\![f,g]\!] = [:f,g:]\cdot(\delta X),$$

(90) $$[:\operatorname{id} X,\operatorname{id} Y:] = \operatorname{id}[:X,Y:],$$

(91) $$\textbf{for } f,g,h,k \textbf{ being } \text{Function} \textbf{ holds } [:f,h:]\cdot[\![g,k]\!] = [\![f\cdot g,h\cdot k]\!],$$

(92) $$\textbf{for } f,g,h,k \textbf{ being } \text{Function} \textbf{ holds } [:f,h:]\cdot[:g,k:] = [:f\cdot g,h\cdot k:],$$

(93) $$\textbf{for } f,g \textbf{ being } \text{Function} \textbf{ holds } [:f,g:]^{\circ}[:B,C:] = [:f^{\circ}B,g^{\circ}C:],$$

(94) $$\textbf{for } f,g \textbf{ being } \text{Function} \textbf{ holds } [:f,g:]^{-1}[:B,C:] = [:f^{-1}B,g^{-1}C:],$$

(95) $$\textbf{for } f \textbf{ being } \text{Function} \textbf{ of } X,Y \textbf{ for } g \textbf{ being } \text{Function} \textbf{ of } V,Z \textbf{ st } (Y=\emptyset \textbf{ implies } X=\emptyset)\ \&\ (Z=\emptyset \textbf{ implies } V=\emptyset) \textbf{ holds } [:f,g:] \textbf{ is } \text{Function} \textbf{ of } [:X,V:],[:Y,Z:].$$

The arguments of the notions defined below are the following: $X1$, $X2$, $D1$, $D2$ which are objects of the type reserved above; $f1$ which is an object of the type Function **of** $X1$, $D1$; $f2$ which is an object of the type Function **of** $X2$, $D2$. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$[:f1,f2:] \quad \text{is} \quad \text{Function } \textbf{of } [:X1,X2:],[:D1,D2:].$$

One can prove the following propositions:

(96) $$\textbf{for } f1 \textbf{ being } \text{Function} \textbf{ of } C1,D1 \textbf{ for } f2 \textbf{ being } \text{Function} \textbf{ of } C2,D2 \textbf{ for } c1 \textbf{ being } \text{Element} \textbf{ of } C1 \textbf{ for } c2 \textbf{ being } \text{Element} \textbf{ of } C2 \textbf{ holds } [:f1,f2:].\langle c1,c2\rangle = \langle f1.c1,f2.c2\rangle,$$

(97) $$\textbf{for } f1 \textbf{ being } \text{Function} \textbf{ of } X1,Y1 \textbf{ for } f2 \textbf{ being } \text{Function} \textbf{ of } X2,Y2 \textbf{ st } (Y1=\emptyset \textbf{ implies } X1=\emptyset)\ \&\ (Y2=\emptyset \textbf{ implies } X2=\emptyset) \textbf{ holds } [:f1,f2:] = [\![f1\cdot\pi_1(X1,X2),f2\cdot\pi_2(X1,X2)]\!],$$

(98) $$\textbf{for } f1 \textbf{ being } \text{Function} \textbf{ of } X1,D1 \textbf{ for } f2 \textbf{ being } \text{Function} \textbf{ of } X2,D2 \textbf{ holds } [:f1,f2:] = [\![f1\cdot\pi_1(X1,X2),f2\cdot\pi_2(X1,X2)]\!],$$

(99) $$\textbf{for } f1 \textbf{ being } \text{Function } \textbf{of } X, Y1 \textbf{ for } f2 \textbf{ being } \text{Function } \textbf{of } X, Y2 \textbf{ st } (Y1 = \emptyset \textbf{ implies } X = \emptyset) \ \& \ (Y2 = \emptyset \textbf{ implies } X = \emptyset) \textbf{ holds } [\![f1, f2]\!] = [\!: f1, f2 :\!] \cdot (\delta X),$$

(100) $$\textbf{for } f1 \textbf{ being } \text{Function } \textbf{of } X, D1 \textbf{ for } f2 \textbf{ being } \text{Function } \textbf{of } X, D2 \textbf{ holds } [\![f1, f2]\!] = [\!: f1, f2 :\!] \cdot (\delta X).$$

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