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Graphs of Functions.

Czesław Byliński[1]
Warsaw University
Białystok

Summary. The graph of a function is defined in [1]. In this paper the graph of a function is redefined as a Relation. Operations on functions are interpreted as the corresponding operations on relations. Some theorems about graphs of functions are proved.

The terminology and notation used in this paper have been introduced in the following papers: [2], [3], [1], and [4]. For simplicity we adopt the following convention: X, $X1$, $X2$, Y, $Y1$, $Y2$ denote objects of the type set; x, $x1$, $x2$, y, $y1$, $y2$, z denote objects of the type Any; f, $f1$, $f2$, g, $g1$, $g2$, h, $h1$ denote objects of the type Function. Let us consider f. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\mathrm{graph}\, f \quad \text{is} \quad \text{Relation}.$$

Next we state a number of propositions:

(1) $$\textbf{for}\ R\ \textbf{being}\ \text{Relation}\ \textbf{st}\ \textbf{for}\ x,y1,y2\ \textbf{st}\ \langle x, y1\rangle \in R\ \&\ \langle x, y2\rangle \in R\ \textbf{holds}\ y1 = y2\ \textbf{ex}\ f\ \textbf{st}\ \mathrm{graph}\, f = R,$$

(2) $$y \in \mathrm{rng}\, f\ \textbf{iff}\ \textbf{ex}\ x\ \textbf{st}\ \langle x, y\rangle \in \mathrm{graph}\, f,$$

(3) $$\mathrm{dom}\,\mathrm{graph}\, f = \mathrm{dom}\, f\ \&\ \mathrm{rng}\,\mathrm{graph}\, f = \mathrm{rng}\, f,$$

(4) $$\mathrm{graph}\, f \subseteq [\!:\mathrm{dom}\, f, \mathrm{rng}\, f:\!],$$

(5) $$(\textbf{for}\ x,y\ \textbf{holds}\ \langle x, y\rangle \in \mathrm{graph}\, f1\ \textbf{iff}\ \langle x, y\rangle \in \mathrm{graph}\, f2)\ \textbf{implies}\ f1 = f2,$$

(6) $$\textbf{for}\ G\ \textbf{being}\ \text{set}\ \textbf{st}\ G \subseteq \mathrm{graph}\, f\ \textbf{ex}\ g\ \textbf{st}\ \mathrm{graph}\, g = G,$$

(7) $$\mathrm{graph}\, f \subseteq \mathrm{graph}\, g\ \textbf{implies}\ \mathrm{dom}\, f \subseteq \mathrm{dom}\, g\ \&\ \mathrm{rng}\, f \subseteq \mathrm{rng}\, g,$$

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$$\text{graph}\, f \subseteq \text{graph}\, g \ \textbf{iff}\ \text{dom}\, f \subseteq \text{dom}\, g \ \&\ \textbf{for}\ x\ \textbf{st}\ x \in \text{dom}\, f\ \textbf{holds}\ f\,.x = g\,.x, \tag{8}$$

$$\text{dom}\, f = \text{dom}\, g \ \&\ \text{graph}\, f \subseteq \text{graph}\, g\ \textbf{implies}\ f = g, \tag{9}$$

$$\langle x, z\rangle \in \text{graph}\,(g \cdot f)\ \textbf{iff}\ \textbf{ex}\ y\ \textbf{st}\ \langle x, y\rangle \in \text{graph}\, f \ \&\ \langle y, z\rangle \in \text{graph}\, g, \tag{10}$$

$$(\text{graph}\, f) \cdot (\text{graph}\, g) = \text{graph}\,(g \cdot f), \tag{11}$$

$$\langle x, z\rangle \in \text{graph}\,(g \cdot f)\ \textbf{implies}\ \langle x, f\,.x\rangle \in \text{graph}\, f \ \&\ \langle f\,.x, z\rangle \in \text{graph}\, g, \tag{12}$$

$$\text{graph}\, h \subseteq \text{graph}\, f \quad \textbf{implies}\ \text{graph}\,(g \cdot h) \subseteq \text{graph}\,(g \cdot f) \ \&\ \text{graph}\,(h \cdot g) \subseteq \text{graph}\,(f \cdot g), \tag{13}$$

$$\text{graph}\, g2 \subseteq \text{graph}\, g1 \ \&\ \text{graph}\, f2 \subseteq \text{graph}\, f1 \quad \textbf{implies}\ \text{graph}\,(g2 \cdot f2) \subseteq \text{graph}\,(g1 \cdot f1), \tag{14}$$

$$\textbf{ex}\ f\ \textbf{st}\ \text{graph}\, f = \{\langle x, y\rangle\}, \tag{15}$$

$$\text{graph}\, f = \{\langle x, y\rangle\}\ \textbf{implies}\ f\,.x = y, \tag{16}$$

$$\text{graph}\, f = \{\langle x, y\rangle\}\ \textbf{implies}\ \text{dom}\, f = \{x\} \ \&\ \text{rng}\, f = \{y\}, \tag{17}$$

$$\text{dom}\, f = \{x\}\ \textbf{implies}\ \text{graph}\, f = \{\langle x, f\,.x\rangle\}, \tag{18}$$

$$(\textbf{ex}\ f\ \textbf{st}\ \text{graph}\, f = \{\langle x1, y1\rangle, \langle x2, y2\rangle\})\ \textbf{iff}\ (x1 = x2\ \textbf{implies}\ y1 = y2), \tag{19}$$

$$\textbf{ex}\ f\ \textbf{st}\ \text{graph}\, f = \emptyset, \tag{20}$$

$$\text{graph}\, f = \emptyset\ \textbf{implies}\ \text{dom}\, f = \emptyset \ \&\ \text{rng}\, f = \emptyset, \tag{21}$$

$$\text{rng}\, f = \emptyset\ \textbf{or}\ \text{dom}\, f = \emptyset\ \textbf{implies}\ \text{graph}\, f = \emptyset, \tag{22}$$

$$\text{rng}\, f \cap \text{dom}\, g = \emptyset\ \textbf{implies}\ \text{graph}\,(g \cdot f) = \emptyset, \tag{23}$$

$$\text{graph}\, g = \emptyset\ \textbf{implies}\ \text{graph}\,(g \cdot f) = \emptyset \ \&\ \text{graph}\,(f \cdot g) = \emptyset, \tag{24}$$

$$f\ \text{is\_one-to-one} \quad \textbf{iff}\ \textbf{for}\ x1, x2, y\ \textbf{st}\ \langle x1, y\rangle \in \text{graph}\, f \ \&\ \langle x2, y\rangle \in \text{graph}\, f\ \textbf{holds}\ x1 = x2, \tag{25}$$

$$\text{graph}\, g \subseteq \text{graph}\, f \ \&\ f\ \text{is\_one-to-one}\ \textbf{implies}\ g\ \text{is\_one-to-one}, \tag{26}$$

$$(\textbf{ex}\ g\ \textbf{st}\ \text{graph}\, g = \text{graph}\, f \cap X) \ \&\ \textbf{ex}\ g\ \textbf{st}\ \text{graph}\, g = X \cap \text{graph}\, f, \tag{27}$$

$$\text{graph}\, h = \text{graph}\, f \cap \text{graph}\, g \quad \textbf{implies}\ \text{dom}\, h \subseteq \text{dom}\, f \cap \text{dom}\, g \ \&\ \text{rng}\, h \subseteq \text{rng}\, f \cap \text{rng}\, g, \tag{28}$$

$$\text{graph}\, h = \text{graph}\, f \cap \text{graph}\, g \ \&\ x \in \text{dom}\, h\ \textbf{implies}\ h\,.x = f\,.x \ \&\ h\,.x = g\,.x, \tag{29}$$

(30) $(f \text{ is\_one-to-one } \mathbf{or}\ g \text{ is\_one-to-one}) \ \&\ \text{graph}\, h = \text{graph}\, f \cap \text{graph}\, g$ $\mathbf{implies}\ h \text{ is\_one-to-one}$,

(31) $\text{dom}\, f \cap \text{dom}\, g = \emptyset\ \mathbf{implies\ ex}\ h\ \mathbf{st}\ \text{graph}\, h = \text{graph}\, f \cup \text{graph}\, g$,

(32) $\text{graph}\, f \subseteq \text{graph}\, h \ \&\ \text{graph}\, g \subseteq \text{graph}\, h$ $\mathbf{implies\ ex}\ h1\ \mathbf{st}\ \text{graph}\, h1 = \text{graph}\, f \cup \text{graph}\, g$,

(33) $\text{graph}\, h = \text{graph}\,(f) \cup \text{graph}\,(g)$ $\mathbf{implies}\ \text{dom}\, h = \text{dom}\, f \cup \text{dom}\, g \ \&\ \text{rng}\, h = \text{rng}\, f \cup \text{rng}\, g$,

(34) $x \in \text{dom}\, f \ \&\ \text{graph}\, h = \text{graph}\, f \cup \text{graph}\, g\ \mathbf{implies}\ h.x = f.x$,

(35) $x \in \text{dom}\, g \ \&\ \text{graph}\, h = \text{graph}\, f \cup \text{graph}\, g\ \mathbf{implies}\ h.x = g.x$,

(36) $x \in \text{dom}\, h \ \&\ \text{graph}\, h = \text{graph}\, f \cup \text{graph}\, g\ \mathbf{implies}\ h.x = f.x\ \mathbf{or}\ h.x = g.x$,

(37) $f \text{ is\_one-to-one}$ $\&\ g \text{ is\_one-to-one} \ \&\ \text{graph}\, h = \text{graph}\, f \cup \text{graph}\, g \ \&\ \text{rng}\, f \cap \text{rng}\, g = \emptyset$ $\mathbf{implies}\ h \text{ is\_one-to-one}$,

(38) $\mathbf{ex}\ g\ \mathbf{st}\ \text{graph}\, g = \text{graph}\,(f) \setminus X$,

(39) $\langle x, y \rangle \in \text{graph}\, \text{id}\,(X)\ \mathbf{iff}\ x \in X \ \&\ x = y$,

(40) $\text{graph}\, \text{id}\, X = \triangle\, X$,

(41) $x \in X\ \mathbf{iff}\ \langle x, x \rangle \in \text{graph}\, \text{id}\,(X)$,

(42) $\langle x, y \rangle \in \text{graph}\,(f \cdot \text{id}\,(X))\ \mathbf{iff}\ x \in X \ \&\ \langle x, y \rangle \in \text{graph}\, f$,

(43) $\langle x, y \rangle \in \text{graph}\,(\text{id}\,(Y) \cdot f)\ \mathbf{iff}\ \langle x, y \rangle \in \text{graph}\, f \ \&\ y \in Y$,

(44) $\text{graph}\,(f \cdot \text{id}\,(X)) \subseteq \text{graph}\, f \ \&\ \text{graph}\,(\text{id}\,(X) \cdot f) \subseteq \text{graph}\,(f)$,

(45) $\text{graph}\, \text{id}\, \emptyset = \emptyset$,

(46) $\text{graph}\, f = \emptyset\ \mathbf{implies}\ f \text{ is\_one-to-one}$,

(47) $f \text{ is\_one-to-one}\ \mathbf{implies\ for}\ x,y\ \mathbf{holds}\ \langle y, x \rangle \in \text{graph}\,(f^{-1})\ \mathbf{iff}\ \langle x, y \rangle \in \text{graph}\, f$,

(48) $f \text{ is\_one-to-one}\ \mathbf{implies}\ \text{graph}\,(f^{-1}) = (\text{graph}\, f)\tilde{}$,

(49) $\text{graph}\, f = \emptyset\ \mathbf{implies}\ \text{graph}\,(f^{-1}) = \emptyset$,

(50) $\langle x, y \rangle \in \text{graph}\,(f \mid X)\ \mathbf{iff}\ x \in X \ \&\ \langle x, y \rangle \in \text{graph}\, f$,

$$\text{graph}\,(f \mid X) = (\text{graph}\,f) \mid X, \tag{51}$$

$$x \in \text{dom}\,f \;\&\; x \in X \textbf{ iff } \langle x, f\,.x\rangle \in \text{graph}\,(f \mid X), \tag{52}$$

$$\text{graph}\,(f \mid X) \subseteq \text{graph}\,f, \tag{53}$$

$$\text{graph}\,((f \mid X) \cdot h) \subseteq \text{graph}\,(f \cdot h) \;\&\; \text{graph}\,(g \cdot (f \mid X)) \subseteq \text{graph}\,(g \cdot f), \tag{54}$$

$$\text{graph}\,(f \mid X) = \text{graph}\,(f) \cap [\!:X, \text{rng}\,f:\!], \tag{55}$$

$$X \subseteq Y \textbf{ implies } \text{graph}\,(f \mid X) \subseteq \text{graph}\,(f \mid Y), \tag{56}$$

$$\text{graph}\,f1 \subseteq \text{graph}\,f2 \textbf{ implies } \text{graph}\,(f1 \mid X) \subseteq \text{graph}\,(f2 \mid X), \tag{57}$$

$$\text{graph}\,f1 \subseteq \text{graph}\,f2 \;\&\; X1 \subseteq X2 \textbf{ implies } \text{graph}\,(f1 \mid X1) \subseteq \text{graph}\,(f2 \mid X2), \tag{58}$$

$$\text{graph}\,(f \mid (X \cup Y)) = \text{graph}\,(f \mid X) \cup \text{graph}\,(f \mid Y), \tag{59}$$

$$\text{graph}\,(f \mid (X \cap Y)) = \text{graph}\,(f \mid X) \cap \text{graph}\,(f \mid Y), \tag{60}$$

$$\text{graph}\,(f \mid (X \setminus Y)) = \text{graph}\,(f \mid X) \setminus \text{graph}\,(f \mid Y), \tag{61}$$

$$\text{graph}\,(f \mid \emptyset) = \emptyset, \tag{62}$$

$$\text{graph}\,f = \emptyset \textbf{ implies } \text{graph}\,(f \mid X) = \emptyset, \tag{63}$$

$$\text{graph}\,g \subseteq \text{graph}\,f \textbf{ implies } f \mid \text{dom}\,g = g, \tag{64}$$

$$\langle x, y\rangle \in \text{graph}\,(Y \mid f) \textbf{ iff } y \in Y \;\&\; \langle x, y\rangle \in \text{graph}\,f, \tag{65}$$

$$\text{graph}\,(Y \mid f) = Y \mid (\text{graph}\,f), \tag{66}$$

$$x \in \text{dom}\,f \;\&\; f\,.x \in Y \textbf{ iff } \langle x, f\,.x\rangle \in \text{graph}\,(Y \mid f), \tag{67}$$

$$\text{graph}\,(Y \mid f) \subseteq \text{graph}\,(f), \tag{68}$$

$$\text{graph}\,((Y \mid f) \cdot h) \subseteq \text{graph}\,(f \cdot h) \;\&\; \text{graph}\,(g \cdot (Y \mid f)) \subseteq \text{graph}\,(g \cdot f), \tag{69}$$

$$\text{graph}\,(Y \mid f) = \text{graph}\,(f) \cap [\!:\text{dom}\,f, Y:\!], \tag{70}$$

$$X \subseteq Y \textbf{ implies } \text{graph}\,(X \mid f) \subseteq \text{graph}\,(Y \mid f), \tag{71}$$

$$\text{graph}\,f1 \subseteq \text{graph}\,f2 \textbf{ implies } \text{graph}\,(Y \mid f1) \subseteq \text{graph}\,(Y \mid f2), \tag{72}$$

$$\text{graph}\,f1 \subseteq \text{graph}\,f2 \;\&\; Y1 \subseteq Y2 \textbf{ implies } \text{graph}\,(Y1 \mid f1) \subseteq \text{graph}\,(Y2 \mid f2), \tag{73}$$

$$\text{graph}\,((X \cup Y) \mid f) = \text{graph}\,(X \mid f) \cup \text{graph}\,(Y \mid f), \tag{74}$$

$$\text{graph}\,((X \cap Y) \mid f) = \text{graph}\,(X \mid f) \cap \text{graph}\,(Y \mid f), \tag{75}$$

$$\operatorname{graph}((X \setminus Y) \mid f) = \operatorname{graph}(X \mid f) \setminus \operatorname{graph}(Y \mid f), \tag{76}$$

$$\operatorname{graph}(\emptyset \mid f) = \emptyset, \tag{77}$$

$$\operatorname{graph} f = \emptyset \textbf{ implies } \operatorname{graph}(Y \mid f) = \emptyset, \tag{78}$$

$$\operatorname{graph} g \subseteq \operatorname{graph} f \ \&\ f \text{ is\_one-to-one } \textbf{implies } \operatorname{rng} g \mid f = g, \tag{79}$$

$$y \in f^{\circ} X \textbf{ iff ex } x \textbf{ st } \langle x, y\rangle \in \operatorname{graph} f \ \&\ x \in X, \tag{80}$$

$$f^{\circ} X = (\operatorname{graph} f)^{\circ} X, \tag{81}$$

$$\operatorname{graph} f = \emptyset \textbf{ implies } f^{\circ} X = \emptyset, \tag{82}$$

$$\operatorname{graph} f1 \subseteq \operatorname{graph} f2 \textbf{ implies } f1^{\circ} X \subseteq f2^{\circ} X, \tag{83}$$

$$\operatorname{graph} f1 \subseteq \operatorname{graph} f2 \ \&\ X1 \subseteq X2 \textbf{ implies } f1^{\circ} X1 \subseteq f2^{\circ} X2, \tag{84}$$

$$x \in f^{-1} Y \textbf{ iff ex } y \textbf{ st } \langle x, y\rangle \in \operatorname{graph} f \ \&\ y \in Y, \tag{85}$$

$$f^{-1} Y = (\operatorname{graph} f)^{-1} Y, \tag{86}$$

$$x \in f^{-1} Y \textbf{ iff } \langle x, f.x\rangle \in \operatorname{graph} f \ \&\ f.x \in Y, \tag{87}$$

$$\operatorname{graph} f = \emptyset \textbf{ implies } f^{-1} Y = \emptyset, \tag{88}$$

$$\operatorname{graph} f1 \subseteq \operatorname{graph} f2 \textbf{ implies } f1^{-1} Y \subseteq f2^{-1} Y, \tag{89}$$

$$\operatorname{graph} f1 \subseteq \operatorname{graph} f2 \ \&\ Y1 \subseteq Y2 \textbf{ implies } f1^{-1} Y1 \subseteq f2^{-1} Y2. \tag{90}$$

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