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Axioms of Incidence

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Summary. This article is based on *"Foundations of Geometry"* by Karol Borsuk and Wanda Szmielew ([1]). The fourth axiom of incidence is weakened. In [1] it has the form *for any plane there exist three non-collinear points in the plane* and in the article *for any plane there exists one point in the plane.* The original axiom is proved. The article includes: theorems concerning collinearity of points and coplanarity of points and lines, basic theorems concerning lines and planes, fundamental existence theorems, theorems concerning intersection of lines and planes.

The articles [3], [2], and [4] provide the terminology and notation for this paper. We consider structures IncStruct, which are systems

$$\langle\!\langle \mathrm{Points}, \mathrm{Lines}, \mathrm{Planes}, \mathrm{Inc}_1, \mathrm{Inc}_2, \mathrm{Inc}_3 \rangle\!\rangle$$

where Points, Lines, Planes have the type DOMAIN, Inc_1 has the type Relation **of the** Points, **the** Lines, Inc_2 has the type Relation **of the** Points, **the** Planes, and Inc_3 has the type Relation **of the** Lines, **the** Planes. We now define three new modes. Let S have the type IncStruct.

POINT **of** S stands for Element **of the** Points **of** S.

LINE **of** S stands for Element **of the** Lines **of** S.

PLANE **of** S stands for Element **of the** Planes **of** S.

In the sequel S will have the type IncStruct; A will have the type Element **of the** Points **of** S; L will have the type Element **of the** Lines **of** S; P will have the type Element **of the** Planes **of** S. The following propositions are true:

(1) A **is** POINT **of** S,

(2) L **is** LINE **of** S,

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(3) $$P \textbf{ is } \mathrm{PLANE} \textbf{ of } S.$$

For simplicity we adopt the following convention: A, B, C, D will denote objects of the type POINT **of** S; L will denote an object of the type LINE **of** S; P will denote an object of the type PLANE **of** S; F, G will denote objects of the type Subset **of the** Points **of** S. The arguments of the notions defined below are the following: S which is an object of the type reserved above; A which is an object of the type POINT **of** S; L which is an object of the type LINE **of** S. The predicate

$$A \text{ on } L \qquad \text{is defined by} \qquad \langle A, L\rangle \in \textbf{the}\, \mathrm{Inc}_1 \textbf{ of } S.$$

The arguments of the notions defined below are the following: S which is an object of the type reserved above; A which is an object of the type POINT **of** S; P which is an object of the type PLANE **of** S. The predicate

$$A \text{ on } P \qquad \text{is defined by} \qquad \langle A, P\rangle \in \textbf{the}\, \mathrm{Inc}_2 \textbf{ of } S.$$

The arguments of the notions defined below are the following: S which is an object of the type reserved above; L which is an object of the type LINE **of** S; P which is an object of the type PLANE **of** S. The predicate

$$L \text{ on } P \qquad \text{is defined by} \qquad \langle L, P\rangle \in \textbf{the}\, \mathrm{Inc}_3 \textbf{ of } S.$$

The arguments of the notions defined below are the following: S which is an object of the type reserved above; F which is an object of the type **set of** POINT **of** S; L which is an object of the type LINE **of** S. The predicate

$$F \text{ on } L \qquad \text{is defined by} \qquad \textbf{for } A \textbf{ being } \mathrm{POINT} \textbf{ of } S \textbf{ st } A \in F \textbf{ holds } A \text{ on } L.$$

The arguments of the notions defined below are the following: S which is an object of the type reserved above; F which is an object of the type **set of** POINT **of** S; P which is an object of the type PLANE **of** S. The predicate

$$F \text{ on } P \qquad \text{is defined by} \qquad \textbf{for } A \textbf{ st } A \in F \textbf{ holds } A \text{ on } P.$$

The arguments of the notions defined below are the following: S which is an object of the type reserved above; F which is an object of the type **set of** POINT **of** S. The predicate

$$F \text{ is\_linear} \qquad \text{is defined by} \qquad \textbf{ex } L \textbf{ st } F \text{ on } L.$$

The arguments of the notions defined below are the following: S which is an object of the type reserved above; F which is an object of the type **set of** POINT **of** S. The predicate

$$F \text{ is\_planar} \qquad \text{is defined by} \qquad \textbf{ex } P \textbf{ st } F \text{ on } P.$$

Next we state a number of propositions:

(4) $$A \text{ on } L \textbf{ iff } \langle A, L\rangle \in \textbf{the}\, \mathrm{Inc}_1 \textbf{ of } S,$$

$$A \text{ on } P \textbf{ iff } \langle A, P\rangle \in \textbf{the}\,\mathrm{Inc}_2 \textbf{ of } S, \tag{5}$$

$$L \text{ on } P \textbf{ iff } \langle L, P\rangle \in \textbf{the}\,\mathrm{Inc}_3 \textbf{ of } S, \tag{6}$$

$$F \text{ on } L \textbf{ iff for } A \textbf{ st } A \in F \textbf{ holds } A \text{ on } L, \tag{7}$$

$$F \text{ on } P \textbf{ iff for } A \textbf{ st } A \in F \textbf{ holds } A \text{ on } P, \tag{8}$$

$$F \text{ is\_linear } \textbf{iff ex } L \textbf{ st } F \text{ on } L, \tag{9}$$

$$F \text{ is\_planar } \textbf{iff ex } P \textbf{ st } F \text{ on } P, \tag{10}$$

$$\{A, B\} \text{ on } L \textbf{ iff } A \text{ on } L \ \&\ B \text{ on } L, \tag{11}$$

$$\{A, B, C\} \text{ on } L \textbf{ iff } A \text{ on } L \ \&\ B \text{ on } L \ \&\ C \text{ on } L, \tag{12}$$

$$\{A, B\} \text{ on } P \textbf{ iff } A \text{ on } P \ \&\ B \text{ on } P, \tag{13}$$

$$\{A, B, C\} \text{ on } P \textbf{ iff } A \text{ on } P \ \&\ B \text{ on } P \ \&\ C \text{ on } P, \tag{14}$$

$$\{A, B, C, D\} \text{ on } P \textbf{ iff } A \text{ on } P \ \&\ B \text{ on } P \ \&\ C \text{ on } P \ \&\ D \text{ on } P, \tag{15}$$

$$G \subseteq F \ \&\ F \text{ on } L \textbf{ implies } G \text{ on } L, \tag{16}$$

$$G \subseteq F \ \&\ F \text{ on } P \textbf{ implies } G \text{ on } P, \tag{17}$$

$$F \text{ on } L \ \&\ A \text{ on } L \textbf{ iff } F \cup \{A\} \text{ on } L, \tag{18}$$

$$F \text{ on } P \ \&\ A \text{ on } P \textbf{ iff } F \cup \{A\} \text{ on } P, \tag{19}$$

$$F \cup G \text{ on } L \textbf{ iff } F \text{ on } L \ \&\ G \text{ on } L, \tag{20}$$

$$F \cup G \text{ on } P \textbf{ iff } F \text{ on } P \ \&\ G \text{ on } P, \tag{21}$$

$$G \subseteq F \ \&\ F \text{ is\_linear } \textbf{implies } G \text{ is\_linear}, \tag{22}$$

$$G \subseteq F \ \&\ F \text{ is\_planar } \textbf{implies } G \text{ is\_planar}. \tag{23}$$

The mode

IncSpace,

which widens to the type IncStruct, is defined by

$$\begin{gathered}(\textbf{for } L \textbf{ being } \mathrm{LINE} \textbf{ of it ex } A{,}B \textbf{ being } \mathrm{POINT} \textbf{ of it st } A \neq B \ \&\ \{A, B\} \text{ on } L)\ \&\\ (\textbf{for } A{,}B \textbf{ being } \mathrm{POINT} \textbf{ of it ex } L \textbf{ being } \mathrm{LINE} \textbf{ of it st } \{A, B\} \text{ on } L)\ \&\\ (\textbf{for } A{,}B \textbf{ being } \mathrm{POINT} \textbf{ of it}, K{,}L \textbf{ being } \mathrm{LINE} \textbf{ of it}\\ \textbf{st } A \neq B \ \&\ \{A, B\} \text{ on } K \ \&\ \{A, B\} \text{ on } L \textbf{ holds } K = L)\\ \&\ (\textbf{for } P \textbf{ being } \mathrm{PLANE} \textbf{ of it ex } A \textbf{ being } \mathrm{POINT} \textbf{ of it st } A \text{ on } P)\ \&\end{gathered}$$

(**for** A,B,C **being** POINT **of it ex** P **being** PLANE **of it st** $\{A, B, C\}$ on P) &
(**for** A,B,C **being** POINT **of it**, P,Q **being** PLANE **of it**
st not $\{A, B, C\}$ is_linear & $\{A, B, C\}$ on P & $\{A, B, C\}$ on Q **holds** $P = Q$)
&
(**for** L **being** LINE **of it**, P **being** PLANE **of it**
st ex A,B **being** POINT **of it st** $A \neq B$ & $\{A, B\}$ on L & $\{A, B\}$ on P **holds** L on P)
&
(**for** A **being** POINT **of it**, P,Q **being** PLANE **of it**
st A on P & A on Q **ex** B **being** POINT **of it st** $A \neq B$ & B on P & B on Q)
& (**ex** A,B,C,D **being** POINT **of it st not** $\{A, B, C, D\}$ is_planar) &
for A **being** POINT **of it**, L **being** LINE **of it**, P **being** PLANE **of it**
st A on L & L on P **holds** A on P.

The following proposition is true

(24) (**for** L **being** LINE **of** S **ex** A,B **being** POINT **of** S **st** $A \neq B$ & $\{A, B\}$ on L)
& (**for** A,B **being** POINT **of** S **ex** L **being** LINE **of** S **st** $\{A, B\}$ on L) &
(**for** A,B **being** POINT **of** S, K,L **being** LINE **of** S
st $A \neq B$ & $\{A, B\}$ on K & $\{A, B\}$ on L **holds** $K = L$)
& (**for** P **being** PLANE **of** S **ex** A **being** POINT **of** S **st** A on P) &
(**for** A,B,C **being** POINT **of** S **ex** P **being** PLANE **of** S **st** $\{A, B, C\}$ on P)
&
(**for** A,B,C **being** POINT **of** S, P,Q **being** PLANE **of** S
st not $\{A, B, C\}$ is_linear & $\{A, B, C\}$ on P & $\{A, B, C\}$ on Q **holds** $P = Q$)
&
(**for** L **being** LINE **of** S, P **being** PLANE **of** S **st**
ex A,B **being** POINT **of** S **st** $A \neq B$ & $\{A, B\}$ on L & $\{A, B\}$ on P
holds L on P)
&
(**for** A **being** POINT **of** S, P,Q **being** PLANE **of** S
st A on P & A on Q **ex** B **being** POINT **of** S **st** $A \neq B$ & B on P & B on Q)
& (**ex** A,B,C,D **being** POINT **of** S **st not** $\{A, B, C, D\}$ is_planar) & (
for A **being** POINT **of** S, L **being** LINE **of** S, P **being** PLANE **of** S
st A on L & L on P **holds** A on P)
implies S **is** IncSpace.

For simplicity we adopt the following convention: S will denote an object of the type IncSpace; A, B, C, D will denote objects of the type POINT **of** S; K, L, $L1$, $L2$ will denote objects of the type LINE **of** S; P, Q will denote objects of the type PLANE **of** S; F will denote an object of the type Subset **of the** Points **of** S. The following propositions are true:

(25) $$\mathbf{ex}\, A,B\ \mathbf{st}\ A \neq B\ \&\ \{A, B\} \text{ on } L,$$

(26) $$\mathbf{ex}\, L\ \mathbf{st}\ \{A, B\} \text{ on } L,$$

(27) $$A \neq B\ \&\ \{A, B\} \text{ on } K\ \&\ \{A, B\} \text{ on } L\ \mathbf{implies}\ K = L,$$

(28) $$\mathbf{ex}\, A\ \mathbf{st}\ A \text{ on } P,$$

(29) $$\mathbf{ex}\, P\ \mathbf{st}\ \{A, B, C\} \text{ on } P,$$

(30) $$\mathbf{not}\, \{A, B, C\} \text{ is\_linear}\ \&\ \{A, B, C\} \text{ on } P\ \&\ \{A, B, C\} \text{ on } Q\ \mathbf{implies}\ P = Q,$$

(31) $$(\mathbf{ex}\, A,B\ \mathbf{st}\ A \neq B\ \&\ \{A, B\} \text{ on } L\ \&\ \{A, B\} \text{ on } P)\ \mathbf{implies}\ L \text{ on } P,$$

(32) $$A \text{ on } P\ \&\ A \text{ on } Q\ \mathbf{implies}\ \mathbf{ex}\, B\ \mathbf{st}\ A \neq B\ \&\ B \text{ on } P\ \&\ B \text{ on } Q,$$

(33) $$\mathbf{ex}\, A,B,C,D\ \mathbf{st}\ \mathbf{not}\, \{A, B, C, D\} \text{ is\_planar},$$

(34) $$A \text{ on } L\ \&\ L \text{ on } P\ \mathbf{implies}\ A \text{ on } P,$$

(35) $$F \text{ on } L\ \&\ L \text{ on } P\ \mathbf{implies}\ F \text{ on } P,$$

(36) $$\{A, A, B\} \text{ is\_linear},$$

(37) $$\{A, A, B, C\} \text{ is\_planar},$$

(38) $$\{A, B, C\} \text{ is\_linear}\ \mathbf{implies}\ \{A, B, C, D\} \text{ is\_planar},$$

(39) $$A \neq B\ \&\ \{A, B\} \text{ on } L\ \&\ \mathbf{not}\, C \text{ on } L\ \mathbf{implies}\ \mathbf{not}\, \{A, B, C\} \text{ is\_linear},$$

(40) $$\mathbf{not}\, \{A, B, C\} \text{ is\_linear}\ \&\ \{A, B, C\} \text{ on } P\ \&\ \mathbf{not}\, D \text{ on } P$$
$$\mathbf{implies}\ \mathbf{not}\, \{A, B, C, D\} \text{ is\_planar},$$

(41) $$\mathbf{not}\, (\mathbf{ex}\, P\ \mathbf{st}\ K \text{ on } P\ \&\ L \text{ on } P)\ \mathbf{implies}\ K \neq L,$$

(42) $$\mathbf{not}\, (\mathbf{ex}\, P\ \mathbf{st}\ L \text{ on } P\ \&\ L1 \text{ on } P\ \&\ L2 \text{ on } P)$$
$$\&\ (\mathbf{ex}\, A\ \mathbf{st}\ A \text{ on } L\ \&\ A \text{ on } L1\ \&\ A \text{ on } L2)$$
$$\mathbf{implies}\ L \neq L1,$$

(43) $$L1 \text{ on } P\ \&\ L2 \text{ on } P\ \&\ \mathbf{not}\, L \text{ on } P\ \&\ L1 \neq L2$$
$$\mathbf{implies}\ \mathbf{not}\ \mathbf{ex}\, Q\ \mathbf{st}\ L \text{ on } Q\ \&\ L1 \text{ on } Q\ \&\ L2 \text{ on } Q,$$

$$\textbf{ex}\, P\ \textbf{st}\ A \text{ on } P\ \&\ L \text{ on } P, \tag{44}$$

$$(\textbf{ex}\, A\ \textbf{st}\ A \text{ on } K\ \&\ A \text{ on } L)\ \textbf{implies ex}\, P\ \textbf{st}\ K \text{ on } P\ \&\ L \text{ on } P, \tag{45}$$

$$A \neq B\ \textbf{implies ex}\, L\ \textbf{st for}\, K\ \textbf{holds}\ \{A, B\} \text{ on } K\ \textbf{iff}\ K = L, \tag{46}$$

$$\begin{gathered}\textbf{not}\, \{A, B, C\}\, \text{is\_linear} \\ \textbf{implies ex}\, P\ \textbf{st for}\, Q\ \textbf{holds}\ \{A, B, C\} \text{ on } Q\ \textbf{iff}\ P = Q,\end{gathered} \tag{47}$$

$$\textbf{not}\, A \text{ on } L\ \textbf{implies ex}\, P\ \textbf{st for}\, Q\ \textbf{holds}\ A \text{ on } Q\ \&\ L \text{ on } Q\ \textbf{iff}\ P = Q, \tag{48}$$

$$\begin{gathered}K \neq L\ \&\ (\textbf{ex}\, A\ \textbf{st}\ A \text{ on } K\ \&\ A \text{ on } L) \\ \textbf{implies ex}\, P\ \textbf{st for}\, Q\ \textbf{holds}\ K \text{ on } Q\ \&\ L \text{ on } Q\ \textbf{iff}\ P = Q.\end{gathered} \tag{49}$$

Let us consider S, A, B. Assume that the following holds

$$A \neq B.$$

The functor

$$\text{Line}\,(A, B),$$

with values of the type LINE **of** S, is defined by

$$\{A, B\} \text{ on } \textbf{it}\,.$$

Let us consider S, A, B, C. Assume that the following holds

$$\textbf{not}\, \{A, B, C\}\, \text{is\_linear}\,.$$

The functor

$$\text{Plane}\,(A, B, C),$$

yields the type PLANE **of** S and is defined by

$$\{A, B, C\} \text{ on } \textbf{it}\,.$$

Let us consider S, A, L. Assume that the following holds

$$\textbf{not}\, A \text{ on } L.$$

The functor

$$\text{Plane}\,(A, L),$$

with values of the type PLANE **of** S, is defined by

$$A \text{ on } \textbf{it}\ \&\ L \text{ on } \textbf{it}\,.$$

Let us consider S, K, L. Assume that the following holds

$$K \neq L.$$

Moreover we assume that

$$\textbf{ex}\, A \;\textbf{st}\; A \text{ on } K \;\&\; A \text{ on } L.$$

The functor

$$\text{Plane}\,(K, L),$$

with values of the type PLANE **of** S, is defined by

$$K \text{ on } \textbf{it} \;\&\; L \text{ on } \textbf{it}\,.$$

Next we state a number of propositions:

(50) $$A \neq B \;\textbf{implies}\; \{A, B\} \text{ on Line}\,(A, B),$$

(51) $$A \neq B \;\&\; \{A, B\} \text{ on } K \;\textbf{implies}\; K = \text{Line}\,(A, B),$$

(52) $$\textbf{not}\, \{A, B, C\} \text{ is\_linear } \textbf{implies}\; \{A, B, C\} \text{ on Plane}\,(A, B, C),$$

(53) $$\textbf{not}\, \{A, B, C\} \text{ is\_linear } \& \; \{A, B, C\} \text{ on } Q \;\textbf{implies}\; Q = \text{Plane}\,(A, B, C),$$

(54) $$\textbf{not}\, A \text{ on } L \;\textbf{implies}\; A \text{ on Plane}\,(A, L) \;\&\; L \text{ on Plane}\,(A, L),$$

(55) $$\textbf{not}\, A \text{ on } L \;\&\; A \text{ on } Q \;\&\; L \text{ on } Q \;\textbf{implies}\; Q = \text{Plane}\,(A, L),$$

(56) $$K \neq L \;\&\; (\textbf{ex}\, A \;\textbf{st}\; A \text{ on } K \;\&\; A \text{ on } L) \;\textbf{implies}\; K \text{ on Plane}\,(K, L) \;\&\; L \text{ on Plane}\,(K, L),$$

(57) $$A \neq B \;\textbf{implies}\; \text{Line}\,(A, B) = \text{Line}\,(B, A),$$

(58) $$\textbf{not}\, \{A, B, C\} \text{ is\_linear } \textbf{implies}\; \text{Plane}\,(A, B, C) = \text{Plane}\,(A, C, B),$$

(59) $$\textbf{not}\, \{A, B, C\} \text{ is\_linear } \textbf{implies}\; \text{Plane}\,(A, B, C) = \text{Plane}\,(B, A, C),$$

(60) $$\textbf{not}\, \{A, B, C\} \text{ is\_linear } \textbf{implies}\; \text{Plane}\,(A, B, C) = \text{Plane}\,(B, C, A),$$

(61) $$\textbf{not}\, \{A, B, C\} \text{ is\_linear } \textbf{implies}\; \text{Plane}\,(A, B, C) = \text{Plane}\,(C, A, B),$$

(62) $$\textbf{not}\, \{A, B, C\} \text{ is\_linear } \textbf{implies}\; \text{Plane}\,(A, B, C) = \text{Plane}\,(C, B, A),$$

(63) $$K \neq L \;\&\; (\textbf{ex}\, A \;\textbf{st}\; A \text{ on } K \;\&\; A \text{ on } L) \;\&\; K \text{ on } Q \;\&\; L \text{ on } Q \;\textbf{implies}\; Q = \text{Plane}\,(K, L),$$

(64) $$K \neq L \;\&\; (\textbf{ex}\, A \;\textbf{st}\; A \text{ on } K \;\&\; A \text{ on } L) \;\textbf{implies}\; \text{Plane}\,(K, L) = \text{Plane}\,(L, K),$$

(65) $$A \neq B \;\&\; C \text{ on Line}\,(A, B) \;\textbf{implies}\; \{A, B, C\} \text{ is\_linear}\,,$$

(66) $$A \neq B \;\&\; A \neq C \;\&\; \{A, B, C\} \text{ is\_linear } \textbf{implies}\; \text{Line}\,(A, B) = \text{Line}\,(A, C),$$

(67) $$\textbf{not}\, \{A, B, C\} \text{ is\_linear } \textbf{implies}\; \text{Plane}\,(A, B, C) = \text{Plane}\,(C, \text{Line}\,(A, B)),$$

$$\textbf{not}\ \{A, B, C\}\ \text{is\_linear}\ \&\ D\ \text{on Plane}\,(A, B, C)\ \textbf{implies}\ \{A, B, C, D\}\ \text{is\_planar}\,, \tag{68}$$

$$\textbf{not}\ C\ \text{on}\ L\ \&\ \{A, B\}\ \text{on}\ L\ \&\ A \neq B\ \textbf{implies}\ \text{Plane}\,(C, L) = \text{Plane}\,(A, B, C), \tag{69}$$

$$\textbf{not}\ \{A, B, C\}\ \text{is\_linear}\ \textbf{implies}\ \text{Plane}\,(A, B, C) = \text{Plane}\,(\text{Line}\,(A, B), \text{Line}\,(A, C)), \tag{70}$$

$$\textbf{ex}\ A, B, C\ \textbf{st}\ \{A, B, C\}\ \text{on}\ P\ \&\ \textbf{not}\ \{A, B, C\}\ \text{is\_linear}\,, \tag{71}$$

$$\textbf{ex}\ A, B, C, D\ \textbf{st}\ A\ \text{on}\ P\ \&\ \textbf{not}\ \{A, B, C, D\}\ \text{is\_planar}\,, \tag{72}$$

$$\textbf{ex}\ B\ \textbf{st}\ A \neq B\ \&\ B\ \text{on}\ L, \tag{73}$$

$$A \neq B\ \textbf{implies ex}\ C\ \textbf{st}\ C\ \text{on}\ P\ \&\ \textbf{not}\ \{A, B, C\}\ \text{is\_linear}\,, \tag{74}$$

$$\textbf{not}\ \{A, B, C\}\ \text{is\_linear}\ \textbf{implies ex}\ D\ \textbf{st not}\ \{A, B, C, D\}\ \text{is\_planar}\,, \tag{75}$$

$$\textbf{ex}\ B, C\ \textbf{st}\ \{B, C\}\ \text{on}\ P\ \&\ \textbf{not}\ \{A, B, C\}\ \text{is\_linear}\,, \tag{76}$$

$$A \neq B\ \textbf{implies ex}\ C, D\ \textbf{st not}\ \{A, B, C, D\}\ \text{is\_planar}\,, \tag{77}$$

$$\textbf{ex}\ B, C, D\ \textbf{st not}\ \{A, B, C, D\}\ \text{is\_planar}\,, \tag{78}$$

$$\textbf{ex}\ L\ \textbf{st not}\ A\ \text{on}\ L\ \&\ L\ \text{on}\ P, \tag{79}$$

$$A\ \text{on}\ P\ \textbf{implies ex}\ L, L1, L2\ \textbf{st}\ L1 \neq L2\ \&\ L1\ \text{on}\ P\ \&\ L2\ \text{on}\ P\ \&\ \textbf{not}\ L\ \text{on}\ P\ \&\ A\ \text{on}\ L\ \&\ A\ \text{on}\ L1\ \&\ A\ \text{on}\ L2, \tag{80}$$

$$\textbf{ex}\ L, L1, L2\ \textbf{st}\ A\ \text{on}\ L\ \&\ A\ \text{on}\ L1\ \&\ A\ \text{on}\ L2\ \&\ \textbf{not ex}\ P\ \textbf{st}\ L\ \text{on}\ P\ \&\ L1\ \text{on}\ P\ \&\ L2\ \text{on}\ P, \tag{81}$$

$$\textbf{ex}\ P\ \textbf{st}\ A\ \text{on}\ P\ \&\ \textbf{not}\ L\ \text{on}\ P, \tag{82}$$

$$\textbf{ex}\ A\ \textbf{st}\ A\ \text{on}\ P\ \&\ \textbf{not}\ A\ \text{on}\ L, \tag{83}$$

$$\textbf{ex}\ K\ \textbf{st not ex}\ P\ \textbf{st}\ L\ \text{on}\ P\ \&\ K\ \text{on}\ P, \tag{84}$$

$$\textbf{ex}\ P, Q\ \textbf{st}\ P \neq Q\ \&\ L\ \text{on}\ P\ \&\ L\ \text{on}\ Q, \tag{85}$$

$$K \neq L\ \&\ \{A, B\}\ \text{on}\ K\ \&\ \{A, B\}\ \text{on}\ L\ \textbf{implies}\ A = B, \tag{86}$$

$$\textbf{not}\ L\ \text{on}\ P\ \&\ \{A, B\}\ \text{on}\ L\ \&\ \{A, B\}\ \text{on}\ P\ \textbf{implies}\ A = B, \tag{87}$$

$$P \neq Q\ \textbf{implies not}\ (\textbf{ex}\ A\ \textbf{st}\ A\ \text{on}\ P\ \&\ A\ \text{on}\ Q)\ \textbf{or ex}\ L\ \textbf{st for}\ B\ \textbf{holds}\ B\ \text{on}\ P\ \&\ B\ \text{on}\ Q\ \textbf{iff}\ B\ \text{on}\ L. \tag{88}$$

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