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The Ordinal Numbers

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Summary. In the beginning of article we show some consequences of the regularity axiom. In the second part we introduce the successor of a set and the notions of transitivity and connectedness wrt membership relation. Then we define ordinal numbers as transitive and connected sets, and we prove some theorems of them and of their sets. Lastly we introduce the concept of a transfinite sequence and we show transfinite induction and schemes of defining by transfinite induction.

The notation and terminology used in this paper have been introduced in the following articles: [2], [3], and [1]. For simplicity we adopt the following convention: X, Y, Z, A, B, $X1$, $X2$, $X3$, $X4$, $X5$, $X6$ will denote objects of the type set; x will denote an object of the type Any. Next we state several propositions:

(1) $$\textbf{not}\, X \in X,$$

(2) $$\textbf{not}\, (X \in Y \;\&\; Y \in X),$$

(3) $$\textbf{not}\, (X \in Y \;\&\; Y \in Z \;\&\; Z \in X),$$

(4) $$\textbf{not}\, (X1 \in X2 \;\&\; X2 \in X3 \;\&\; X3 \in X4 \;\&\; X4 \in X1),$$

(5) $$\textbf{not}\, (X1 \in X2 \;\&\; X2 \in X3 \;\&\; X3 \in X4 \;\&\; X4 \in X5 \;\&\; X5 \in X1),$$

(6) $$\textbf{not}\, (X1 \in X2 \;\&\; X2 \in X3 \;\&\; X3 \in X4 \;\&\; X4 \in X5 \;\&\; X5 \in X6 \;\&\; X6 \in X1),$$

(7) $$Y \in X \;\textbf{implies not}\; X \subseteq Y.$$

The scheme *Comprehension* deals with a constant $\mathcal{A}$ that has the type set and a unary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex}\, B \;\textbf{st for}\; Z \;\textbf{being}\; \text{set} \;\textbf{holds}\; Z \in B \;\textbf{iff}\; Z \in \mathcal{A} \;\&\; \mathcal{P}[Z]$$

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for all values of the parameters.

One can prove the following proposition

$$(\textbf{for}\, X \;\textbf{holds}\; X \in A \;\textbf{iff}\; X \in B) \;\textbf{implies}\; A = B. \tag{8}$$

Let us consider X. The functor

$$\mathrm{succ}\, X,$$

with values of the type set, is defined by

$$\textbf{it} = X \cup \{X\}.$$

Next we state several propositions:

$$\mathrm{succ}\, X = X \cup \{X\}, \tag{9}$$

$$X \in \mathrm{succ}\, X, \tag{10}$$

$$\mathrm{succ}\, X \neq \emptyset, \tag{11}$$

$$\mathrm{succ}\, X = \mathrm{succ}\, Y \;\textbf{implies}\; X = Y, \tag{12}$$

$$x \in \mathrm{succ}\, X \;\textbf{iff}\; x \in X \;\textbf{or}\; x = X, \tag{13}$$

$$X \neq \mathrm{succ}\, X. \tag{14}$$

For simplicity we adopt the following convention: a has the type Any; X, Y, Z, x, y have the type set. We now define two new predicates. Let us consider X. The predicate

$$X \text{ is\_}\!\in\!\text{-transitive} \qquad \text{is defined by} \qquad \textbf{for}\, x \;\textbf{st}\; x \in X \;\textbf{holds}\; x \subseteq X.$$

The predicate

$$X \text{ is\_}\!\in\!\text{-connected}$$

is defined by

$$\textbf{for}\, x{,}y \;\textbf{st}\; x \in X \;\&\; y \in X \;\textbf{holds}\; x \in y \;\textbf{or}\; x = y \;\textbf{or}\; y \in x.$$

One can prove the following two propositions:

$$X \text{ is\_}\!\in\!\text{-transitive} \;\textbf{iff}\; \textbf{for}\, x \;\textbf{st}\; x \in X \;\textbf{holds}\; x \subseteq X, \tag{15}$$

$$X \text{ is\_}\!\in\!\text{-connected} \;\textbf{iff}\; \textbf{for}\, x{,}y \;\textbf{st}\; x \in X \;\&\; y \in X \;\textbf{holds}\; x \in y \;\textbf{or}\; x = y \;\textbf{or}\; y \in x. \tag{16}$$

The mode

$$\mathrm{Ordinal},$$

which widens to the type set, is defined by

$$\textbf{it} \text{ is\_}\!\in\!\text{-transitive} \;\&\; \textbf{it} \text{ is\_}\!\in\!\text{-connected}.$$

In the sequel A, B, C will have the type Ordinal. The following propositions are true:

(17) $$X \textbf{ is } \text{Ordinal} \textbf{ iff } X \text{ is\_}{\in}\text{-transitive} \ \& \ X \text{ is\_}{\in}\text{-connected}\,,$$

(18) $$x \in A \textbf{ implies } x \subseteq A,$$

(19) $$A \in B \ \& \ B \in C \textbf{ implies } A \in C,$$

(20) $$x \in A \ \& \ y \in A \textbf{ implies } x \in y \textbf{ or } x = y \textbf{ or } y \in x,$$

(21) $$\textbf{for } x{,}A \textbf{ being } \text{Ordinal} \textbf{ st } x \subseteq A \ \& \ x \neq A \textbf{ holds } x \in A,$$

(22) $$A \subseteq B \ \& \ B \in C \textbf{ implies } A \in C,$$

(23) $$a \in A \textbf{ implies } a \textbf{ is } \text{Ordinal}\,,$$

(24) $$A \in B \textbf{ or } A = B \textbf{ or } B \in A,$$

(25) $$A \subseteq B \textbf{ or } B \subseteq A,$$

(26) $$A \subseteq B \textbf{ or } B \in A,$$

(27) $$\emptyset \textbf{ is } \text{Ordinal}\,.$$

The constant $\mathbf{0}$ has the type Ordinal, and is defined by

$$\textbf{it} = \emptyset.$$

Next we state three propositions:

(28) $$\mathbf{0} = \emptyset,$$

(29) $$x \textbf{ is } \text{Ordinal} \textbf{ implies } \text{succ}\, x \textbf{ is } \text{Ordinal}\,,$$

(30) $$x \textbf{ is } \text{Ordinal} \textbf{ implies } \bigcup x \textbf{ is } \text{Ordinal}\,.$$

Let us consider A. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$\text{succ}\, A \qquad \text{is} \qquad \text{Ordinal}\,,$$

$$\bigcup A \qquad \text{is} \qquad \text{Ordinal}\,.$$

One can prove the following propositions:

(31) $$(\textbf{for } x \textbf{ st } x \in X \textbf{ holds } x \textbf{ is } \text{Ordinal} \ \& \ x \subseteq X) \textbf{ implies } X \textbf{ is } \text{Ordinal}\,,$$

(32) $$X \subseteq A \ \& \ X \neq \emptyset \textbf{ implies ex } C \textbf{ st } C \in X \ \& \textbf{ for } B \textbf{ st } B \in X \textbf{ holds } C \subseteq B,$$

(33) $$A \in B \textbf{ iff } \text{succ}\, A \subseteq B,$$

(34) $$A \in \operatorname{succ} C \textbf{ iff } A \subseteq C.$$

Now we present two schemes. The scheme *Ordinal_Min* concerns a unary predicate $\mathcal{P}$ states that the following holds

$$\textbf{ex } A \textbf{ st } \mathcal{P}[A] \ \& \ \textbf{for } B \textbf{ st } \mathcal{P}[B] \textbf{ holds } A \subseteq B$$

provided the parameter satisfies the following condition:

- $$\textbf{ex } A \textbf{ st } \mathcal{P}[A].$$

The scheme *Transfinite_Ind* concerns a unary predicate $\mathcal{P}$ states that the following holds

$$\textbf{for } A \textbf{ holds } \mathcal{P}[A]$$

provided the parameter satisfies the following condition:

- $$\textbf{for } A \textbf{ st for } C \textbf{ st } C \in A \textbf{ holds } \mathcal{P}[C] \textbf{ holds } \mathcal{P}[A].$$

One can prove the following propositions:

(35) $$\textbf{for } X \textbf{ st for } a \textbf{ st } a \in X \textbf{ holds } a \textbf{ is } \text{Ordinal} \textbf{ holds } \bigcup X \textbf{ is } \text{Ordinal},$$

(36) $$\textbf{for } X \textbf{ st for } a \textbf{ st } a \in X \textbf{ holds } a \textbf{ is } \text{Ordinal} \textbf{ ex } A \textbf{ st } X \subseteq A,$$

(37) $$\textbf{not ex } X \textbf{ st for } x \textbf{ holds } x \in X \textbf{ iff } x \textbf{ is } \text{Ordinal},$$

(38) $$\textbf{not ex } X \textbf{ st for } A \textbf{ holds } A \in X,$$

(39) $$\textbf{for } X \textbf{ ex } A \textbf{ st not } A \in X \ \& \ \textbf{for } B \textbf{ st not } B \in X \textbf{ holds } A \subseteq B.$$

Let us consider A. The predicate

$$A \text{ is\_limit\_ordinal} \qquad \text{is defined by} \qquad A = \bigcup A.$$

One can prove the following three propositions:

(40) $$A \text{ is\_limit\_ordinal} \textbf{ iff } A = \bigcup A,$$

(41) $$\textbf{for } A \textbf{ holds } A \text{ is\_limit\_ordinal} \textbf{ iff for } C \textbf{ st } C \in A \textbf{ holds } \operatorname{succ} C \in A,$$

(42) $$\textbf{not } A \text{ is\_limit\_ordinal} \textbf{ iff ex } B \textbf{ st } A = \operatorname{succ} B.$$

In the sequel F denotes an object of the type Function. The mode

$$\text{Transfinite-Sequence},$$

which widens to the type Function, is defined by

$$\textbf{ex } A \textbf{ st } \operatorname{dom} \textbf{it} = A.$$

Let us consider Z. The mode

$$\text{Transfinite-Sequence } \mathbf{of}\ Z,$$

which widens to the type Transfinite-Sequence, is defined by

$$\text{rng}\,\mathbf{it} \subseteq Z.$$

The following propositions are true:

(43) F **is** Transfinite-Sequence **iff ex** A **st** dom $F = A$,

(44) F **is** Transfinite-Sequence **of** Z **iff** F **is** Transfinite-Sequence & rng $F \subseteq Z$,

(45) $\emptyset$ **is** Transfinite-Sequence **of** Z.

In the sequel L, $L1$, $L2$ will have the type Transfinite-Sequence. The following proposition is true

(46) dom F **is** Ordinal **implies** F **is** Transfinite-Sequence **of** rng F.

Let us consider L. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\text{dom}\,L \quad \text{is} \quad \text{Ordinal}.$$

We now state a proposition

(47) $X \subseteq Y$ **implies**
for L **being** Transfinite-Sequence **of** X **holds** L **is** Transfinite-Sequence **of** Y.

Let us consider L, A. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$L \mid A \quad \text{is} \quad \text{Transfinite-Sequence } \mathbf{of}\ \text{rng}\,L.$$

The following two propositions are true:

(48) **for** L **being** Transfinite-Sequence **of** X
for A **holds** $L \mid A$ **is** Transfinite-Sequence **of** X,

(49) (**for** a **st** $a \in X$ **holds** a **is** Transfinite-Sequence) & (**for** $L1$,$L2$
st $L1 \in X$ & $L2 \in X$ **holds** graph $L1 \subseteq$ graph $L2$ **or** graph $L2 \subseteq$ graph $L1$)
implies $\bigcup X$ **is** Transfinite-Sequence.

Now we present three schemes. The scheme *TS_Uniq* deals with a constant $\mathcal{A}$ that has the type Ordinal, a unary functor $\mathcal{F}$, a constant $\mathcal{B}$ that has the type Transfinite-Sequence and a constant $\mathcal{C}$ that has the type Transfinite-Sequence, and states that the following holds

$$\mathcal{B} = \mathcal{C}$$

provided the parameters satisfy the following conditions:

- $\operatorname{dom}\mathcal{B} = \mathcal{A}$ & **for** B,L **st** $B \in \mathcal{A}$ & $L = \mathcal{B} \mid B$ **holds** $\mathcal{B}.B = \mathcal{F}(L)$,
- $\operatorname{dom}\mathcal{C} = \mathcal{A}$ & **for** B,L **st** $B \in \mathcal{A}$ & $L = \mathcal{C} \mid B$ **holds** $\mathcal{C}.B = \mathcal{F}(L)$.

The scheme *TS_Exist* deals with a constant $\mathcal{A}$ that has the type Ordinal and a unary functor $\mathcal{F}$ and states that the following holds

ex L **st** $\operatorname{dom} L = \mathcal{A}$ & **for** B,$L1$ **st** $B \in \mathcal{A}$ & $L1 = L \mid B$ **holds** $L.B = \mathcal{F}(L1)$

for all values of the parameters.

The scheme *Func_TS* concerns a constant $\mathcal{A}$ that has the type Transfinite-Sequence, a unary functor $\mathcal{F}$ and a unary functor $\mathcal{G}$ and states that the following holds

for B **st** $B \in \operatorname{dom}\mathcal{A}$ **holds** $\mathcal{A}.B = \mathcal{G}(\mathcal{A} \mid B)$

provided the parameters satisfy the following conditions:

- **for** A,a **holds** $a = \mathcal{F}(A)$
  iff ex L **st** $a = \mathcal{G}(L)$ & $\operatorname{dom} L = A$ & **for** B **st** $B \in A$ **holds** $L.B = \mathcal{G}(L \mid B)$,
- **for** A **st** $A \in \operatorname{dom}\mathcal{A}$ **holds** $\mathcal{A}.A = \mathcal{F}(A)$.

References

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