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Topological Spaces and Continuous Functions

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Summary. The paper contains a definition of topological space. The following notions are defined: point of topological space, subset of topological space, subspace of topological space, and continuous function.

The articles [5], [7], [6], [1], [4], [2], and [3] provide the terminology and notation for this paper. We consider structures TopStruct, which are systems

$$\langle\!\langle \text{carrier}\,, \text{topology} \rangle\!\rangle$$

where carrier has the type DOMAIN, and topology has the type Subset-Family **of the** carrier. In the sequel T has the type TopStruct. The mode

$$\text{TopSpace}\,,$$

which widens to the type TopStruct, is defined by

$$\textbf{the}\,\text{carrier}\ \textbf{of it} \in \textbf{the}\,\text{topology}\ \textbf{of it}\ \&$$
$$(\textbf{for}\ a\ \textbf{being}\ \text{Subset-Family}\ \textbf{of the}\,\text{carrier}\ \textbf{of it}$$
$$\textbf{st}\ a \subseteq \textbf{the}\,\text{topology}\ \textbf{of it holds} \bigcup a \in \textbf{the}\,\text{topology}\ \textbf{of it})$$
$$\&\ \textbf{for}\ a,b\ \textbf{being}\ \text{Subset}\ \textbf{of the}\,\text{carrier}\ \textbf{of it}$$
$$\textbf{st}\ a \in \textbf{the}\,\text{topology}\ \textbf{of it}\ \&\ b \in \textbf{the}\,\text{topology}\ \textbf{of it holds}\ a \cap b \in \textbf{the}\,\text{topology}\ \textbf{of it}\,.$$

We now state a proposition

(1) $$\textbf{the}\,\text{carrier}\ \textbf{of}\ T \in \textbf{the}\,\text{topology}\ \textbf{of}\ T\ \&$$
$$(\textbf{for}\ a\ \textbf{being}\ \text{Subset-Family}\ \textbf{of the}\,\text{carrier}\ \textbf{of}\ T$$

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$$\textbf{st } a \subseteq \textbf{the} \text{ topology } \textbf{of } T \textbf{ holds} \bigcup a \in \textbf{the} \text{ topology } \textbf{of } T)$$
$$\& \ (\textbf{for } p,q \textbf{ being} \text{ Subset } \textbf{of the} \text{ carrier } \textbf{of } T \textbf{ st}$$
$$p \in \textbf{the} \text{ topology } \textbf{of } T \ \& \ q \in \textbf{the} \text{ topology } \textbf{of } T$$
$$\textbf{holds } p \cap q \in \textbf{the} \text{ topology } \textbf{of } T)$$
$$\textbf{implies } T \textbf{ is} \text{ TopSpace}.$$

In the sequel T, S, GX will have the type TopSpace. Let us consider T.

Point **of** T stands for Element **of the** carrier **of** T.

The following proposition is true

(2) **for** x **being** Element **of the** carrier **of** T **holds** x **is** Point **of** T.

Let us consider T.

Subset **of** T stands for **set of** Point **of** T.

We now state a proposition

(3) **for** P **being** Subset **of the** carrier **of** T **holds** P **is** Subset **of** T.

In the sequel P, Q will have the type Subset **of** T; p will have the type Point **of** T. Let us consider T.

Subset-Family **of** T stands for Subset-Family **of the** carrier **of** T.

Next we state a proposition

(4) **for** F **being** Subset-Family **of the** carrier **of** T
holds F **is** Subset-Family **of** T.

In the sequel F will denote an object of the type Subset-Family **of** T. The scheme *SubFamEx1* concerns a constant $\mathcal{A}$ that has the type TopSpace and a unary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex } F \textbf{ being} \text{ Subset-Family } \textbf{of } \mathcal{A} \textbf{ st for } B \textbf{ being} \text{ Subset } \textbf{of } \mathcal{A} \textbf{ holds } B \in F \textbf{ iff } \mathcal{P}[B]$$

for all values of the parameters.

One can prove the following propositions:

(5) $\emptyset \in$ **the** topology **of** T,

(6) **the** carrier **of** $T \in$ **the** topology **of** T,

(7) **for** a **being** Subset-Family **of** T
st $a \subseteq$ **the** topology **of** T **holds** $\bigcup a \in$ **the** topology **of** T,

(8) $$P \in \textbf{the}\,\text{topology}\,\textbf{of}\,T \;\&\; Q \in \textbf{the}\,\text{topology}\,\textbf{of}\,T \;\textbf{implies}\; P \cap Q \in \textbf{the}\,\text{topology}\,\textbf{of}\,T.$$

We now define two new functors. Let us consider T. The functor

$$\emptyset T,$$

with values of the type Subset **of** T, is defined by

$$\textbf{it} = \emptyset\,\textbf{the}\,\text{carrier}\,\textbf{of}\,T.$$

The functor

$$\Omega\,T,$$

with values of the type Subset **of** T, is defined by

$$\textbf{it} = \Omega\,\textbf{the}\,\text{carrier}\,\textbf{of}\,T.$$

One can prove the following four propositions:

(9) $$\emptyset T = \emptyset\,\textbf{the}\,\text{carrier}\,\textbf{of}\,T,$$

(10) $$\Omega\,T = \Omega\,\textbf{the}\,\text{carrier}\,\textbf{of}\,T,$$

(11) $$\emptyset(T) = \emptyset,$$

(12) $$\Omega\,(T) = \textbf{the}\,\text{carrier}\,\textbf{of}\,T.$$

Let us consider T, P. The functor

$$P^{\,\mathrm{c}},$$

yields the type Subset **of** T and is defined by

$$\textbf{it} = P^{\,\mathrm{c}}.$$

Let us consider T, P, Q. Let us note that it makes sense to consider the following functors on restricted areas. Then

$P \cup Q$	is	Subset **of** T,
$P \cap Q$	is	Subset **of** T,
$P \setminus Q$	is	Subset **of** T,
$P \dot{-} Q$	is	Subset **of** T.

The following propositions are true:

(13) $$p \in \Omega\,(T),$$

(14) $$P \subseteq \Omega\,(T),$$

(15) $$P \cap \Omega(T) = P,$$

(16) $$\textbf{for } A \textbf{ being } \text{set } \textbf{holds } A \subseteq \Omega(T) \textbf{ implies } A \textbf{ is } \text{Subset } \textbf{of } T,$$

(17) $$P^{\mathrm{c}} = \Omega(T) \setminus P,$$

(18) $$P \cup P^{\mathrm{c}} = \Omega(T),$$

(19) $$P \subseteq Q \textbf{ iff } Q^{\mathrm{c}} \subseteq P^{\mathrm{c}},$$

(20) $$P = P^{\mathrm{c}\,\mathrm{c}},$$

(21) $$P \subseteq Q^{\mathrm{c}} \textbf{ iff } P \cap Q = \emptyset,$$

(22) $$\Omega(T) \setminus (\Omega(T) \setminus P) = P,$$

(23) $$P \neq \Omega(T) \textbf{ iff } \Omega(T) \setminus P \neq \emptyset,$$

(24) $$\Omega(T) \setminus P = Q \textbf{ implies } \Omega(T) = P \cup Q,$$

(25) $$\Omega(T) = P \cup Q \;\&\; P \cap Q = \emptyset \textbf{ implies } Q = \Omega(T) \setminus P,$$

(26) $$P \cap P^{\mathrm{c}} = \emptyset(T),$$

(27) $$\Omega(T) = (\emptyset T)^{\mathrm{c}},$$

(28) $$P \setminus Q = P \cap Q^{\mathrm{c}},$$

(29) $$P = Q \textbf{ implies } \Omega(T) \setminus P = \Omega(T) \setminus Q.$$

Let us consider T, P. The predicate

$$P \text{ is\_open} \qquad \text{is defined by} \qquad P \in \textbf{the } \text{topology } \textbf{of } T.$$

One can prove the following proposition

(30) $$P \text{ is\_open } \textbf{iff } P \in \textbf{the } \text{topology } \textbf{of } T.$$

Let us consider T, P. The predicate

$$P \text{ is\_closed} \qquad \text{is defined by} \qquad \Omega(T) \setminus P \text{ is\_open}.$$

One can prove the following proposition

(31) $$P \text{ is\_closed } \textbf{iff } \Omega(T) \setminus P \text{ is\_open}.$$

Let us consider T, P. The predicate

$$P \text{ is\_open\_closed} \qquad \text{is defined by} \qquad P \text{ is\_open } \& \; P \text{ is\_closed}.$$

We now state a proposition

(32) P is_open_closed **iff** P is_open & P is_closed .

Let us consider T, F. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\bigcup F \quad \text{is} \quad \text{Subset } \mathbf{of}\, T.$$

Let us consider T, F. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\bigcap F \quad \text{is} \quad \text{Subset } \mathbf{of}\, T.$$

Let us consider T, F. The predicate

$$F \text{ is\_a\_cover\_of } T \qquad \text{is defined by} \qquad \Omega\,(T) = \bigcup F.$$

The following proposition is true

(33) F is_a_cover_of T **iff** $\Omega\,(T) = \bigcup F$.

Let us consider T. The mode

SubSpace **of** T,

which widens to the type TopSpace, is defined by

$\Omega\,(\mathbf{it}) \subseteq \Omega\,(T)$ & **for** P **being** Subset **of it holds** $P \in$ **the** topology **of it**
iff ex Q **being** Subset **of** T **st** $Q \in$ **the** topology **of** T & $P = Q \cap \Omega\,(\mathbf{it})$.

Next we state two propositions:

(34) $\Omega\,(S) \subseteq \Omega\,(T)$ & (**for** P **being** Subset **of** S **holds** $P \in$ **the** topology **of** S
iff ex Q **being** Subset **of** T **st** $Q \in$ **the** topology **of** T & $P = Q \cap \Omega\,(S)$)
implies S **is** SubSpace **of** T,

(35) **for** V **being** SubSpace **of** T **holds** $\Omega\,(V) \subseteq \Omega\,(T)$ & **for** P **being** Subset **of** V
holds $P \in$ **the** topology **of** V
iff ex Q **being** Subset **of** T **st** $Q \in$ **the** topology **of** T & $P = Q \cap \Omega\,(V)$.

Let us consider T, P. Assume that the following holds

$$P \neq \emptyset(T).$$

The functor

$$T \mid P,$$

with values of the type SubSpace **of** T, is defined by

$$\Omega\,(\mathbf{it}) = P.$$

One can prove the following proposition

(36) $P \neq \emptyset(T)$ **implies for** S **being** SubSpace **of** T **holds** $S = T \mid P$ **iff** $\Omega(S) = P$.

Let us consider T, S.

map **of** T, S stands for Function **of** (**the** carrier **of** T),(**the** carrier **of** S).

Next we state a proposition

(37) **for** f **being** Function **of the** carrier **of** T,**the** carrier **of** S **holds** f **is** map **of** T, S.

In the sequel f has the type map **of** T, S; $P1$ has the type Subset **of** S. Let us consider T, S, f, P. Let us note that it makes sense to consider the following functor on a restricted area. Then

$f\,^\circ\,P$ is Subset **of** S.

Let us consider T, S, f, $P1$. Let us note that it makes sense to consider the following functor on a restricted area. Then

$f\,^{-1}\,P1$ is Subset **of** T.

Let us consider T, S, f. The predicate

f is_continuous

is defined by

for $P1$ **holds** $P1$ is_closed **implies** $f\,^{-1}\,P1$ is_closed .

The following proposition is true

(38) f is_continuous **iff for** $P1$ **holds** $P1$ is_closed **implies** $f\,^{-1}\,P1$ is_closed .

The scheme *TopAbstr* concerns a constant $\mathcal{A}$ that has the type TopSpace and a unary predicate $\mathcal{P}$ and states that the following holds

ex P **being** Subset **of** $\mathcal{A}$ **st for** x **being** Point **of** $\mathcal{A}$ **holds** $x \in P$ **iff** $\mathcal{P}[x]$

for all values of the parameters.

One can prove the following propositions:

(39) **for** X' **being** SubSpace **of** GX **for** A **being** Subset **of** X' **holds** A **is** Subset **of** GX,

(40) **for** A **being** Subset **of** GX, x **being** Any **st** $x \in A$ **holds** x **is** Point **of** GX,

(41) **for** A **being** Subset **of** GX **st** $A \neq \emptyset(GX)$ **ex** x **being** Point **of** GX **st** $x \in A$,

(42) $\Omega(GX)$ is_closed ,

(43) $$\textbf{for } X' \textbf{ being } \mathrm{SubSpace} \textbf{ of } GX, B \textbf{ being } \mathrm{Subset} \textbf{ of } X' \textbf{ holds}$$
$$B \,\mathrm{is\_closed} \textbf{ iff ex } C \textbf{ being } \mathrm{Subset} \textbf{ of } GX \textbf{ st } C \,\mathrm{is\_closed} \,\&\, C \cap (\Omega\,(X')) = B,$$

(44) $$\textbf{for } F \textbf{ being } \mathrm{Subset\text{-}Family} \textbf{ of } GX \textbf{ st}$$
$$F \neq \emptyset \,\&\, \textbf{for } A \textbf{ being } \mathrm{Subset} \textbf{ of } GX \textbf{ st } A \in F \textbf{ holds } A \,\mathrm{is\_closed}$$
$$\textbf{holds} \bigcap F \,\mathrm{is\_closed}\,.$$

The arguments of the notions defined below are the following: GX which is an object of the type TopSpace; A which is an object of the type Subset **of** GX. The functor

$$\mathrm{Cl}\, A,$$

yields the type Subset **of** GX and is defined by

$$\textbf{for } p \textbf{ being } \mathrm{Point} \textbf{ of } GX \textbf{ holds } p \in \textbf{it}$$
$$\textbf{iff for } G \textbf{ being } \mathrm{Subset} \textbf{ of } GX \textbf{ st } G \,\mathrm{is\_open} \textbf{ holds } p \in G \textbf{ implies } A \cap G \neq \emptyset(GX).$$

We now state a number of propositions:

(45) $$\textbf{for } A \textbf{ being } \mathrm{Subset} \textbf{ of } GX, p \textbf{ being } \mathrm{Point} \textbf{ of } GX \textbf{ holds } p \in \mathrm{Cl}\, A$$
$$\textbf{iff for } C \textbf{ being } \mathrm{Subset} \textbf{ of } GX \textbf{ st } C \,\mathrm{is\_closed} \textbf{ holds } A \subseteq C \textbf{ implies } p \in C,$$

(46) $$\textbf{for } A \textbf{ being } \mathrm{Subset} \textbf{ of } GX \textbf{ ex } F \textbf{ being } \mathrm{Subset\text{-}Family} \textbf{ of } GX \textbf{ st}$$
$$(\textbf{for } C \textbf{ being } \mathrm{Subset} \textbf{ of } GX \textbf{ holds } C \in F \textbf{ iff } C \,\mathrm{is\_closed} \,\&\, A \subseteq C)$$
$$\&\ \mathrm{Cl}\, A = \bigcap F,$$

(47) $$\textbf{for}$$
$$X' \textbf{ being } \mathrm{SubSpace} \textbf{ of } GX, A \textbf{ being } \mathrm{Subset} \textbf{ of } GX, A1 \textbf{ being } \mathrm{Subset} \textbf{ of } X'$$
$$\textbf{st } A = A1 \textbf{ holds } \mathrm{Cl}\, A1 = (\mathrm{Cl}\, A) \cap (\Omega\,(X')),$$

(48) $$\textbf{for } A \textbf{ being } \mathrm{Subset} \textbf{ of } GX \textbf{ holds } A \subseteq \mathrm{Cl}\, A,$$

(49) $$\textbf{for } A,B \textbf{ being } \mathrm{Subset} \textbf{ of } GX \textbf{ st } A \subseteq B \textbf{ holds } \mathrm{Cl}\, A \subseteq \mathrm{Cl}\, B,$$

(50) $$\textbf{for } A,B \textbf{ being } \mathrm{Subset} \textbf{ of } GX \textbf{ holds } \mathrm{Cl}\,(A \cup B) = \mathrm{Cl}\, A \cup \mathrm{Cl}\, B,$$

(51) $$\textbf{for } A,B \textbf{ being } \mathrm{Subset} \textbf{ of } GX \textbf{ holds } \mathrm{Cl}\,(A \cap B) \subseteq (\mathrm{Cl}\, A) \cap \mathrm{Cl}\, B,$$

(52) $$\textbf{for } A \textbf{ being } \mathrm{Subset} \textbf{ of } GX \textbf{ holds } A \,\mathrm{is\_closed} \textbf{ iff } \mathrm{Cl}\, A = A,$$

(53) $$\textbf{for } A \textbf{ being } \mathrm{Subset} \textbf{ of } GX$$
$$\textbf{holds } A \,\mathrm{is\_open} \textbf{ iff } \mathrm{Cl}\,(\Omega\,(GX) \setminus A) = \Omega\,(GX) \setminus A,$$

(54) $$\textbf{for } A \textbf{ being } \mathrm{Subset} \textbf{ of } GX, p \textbf{ being } \mathrm{Point} \textbf{ of } GX \textbf{ holds } p \in \mathrm{Cl}\, A \textbf{ iff}$$
$$\textbf{for } G \textbf{ being } \mathrm{Subset} \textbf{ of } GX$$
$$\textbf{st } G \,\mathrm{is\_open} \textbf{ holds } p \in G \textbf{ implies } A \cap G \neq \emptyset(GX).$$

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