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# Relations and Their Basic Properties

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**Summary.** We define here: mode Relation as a set of pairs, the domain, the codomain, and the field of relation, the empty and the identity relations, the composition of relations, the image and the inverse image of a set under a relation. Two predicates = and ⊆ , and three functions ∪, ∩ and \ are redefined. Basic facts about the above mentioned notions are presented.

The terminology and notation used in this paper have been introduced in the articles [1] and [2]. For simplicity we adopt the following convention: $A$, $B$, $X$, $Y$, $Y1$, $Y2$ denote objects of the type set; $a$, $b$, $c$, $d$, $x$, $y$, $z$ denote objects of the type Any. The mode

Relation,

which widens to the type set, is defined by

$$x \in \textbf{it implies ex}\, y{,}z \,\textbf{st}\, x = \langle y, z\rangle.$$

One can prove the following proposition

(1) $\textbf{for}\ R\ \textbf{being}\ \text{set}\ \textbf{st}\ \textbf{for}\ x\ \textbf{st}\ x \in R\ \textbf{ex}\ y{,}z\ \textbf{st}\ x = \langle y, z\rangle\ \textbf{holds}\ R\ \textbf{is}\ \text{Relation}.$

In the sequel $P$, $P1$, $P2$, $Q$, $R$, $S$ will have the type Relation. Next we state several propositions:

(2) $x \in R\ \textbf{implies ex}\ y{,}z\ \textbf{st}\ x = \langle y, z\rangle,$

(3) $A \subseteq R\ \textbf{implies}\ A\ \textbf{is}\ \text{Relation},$

(4) $\{\langle x, y\rangle\}\ \textbf{is}\ \text{Relation},$

(5) $\{\langle a, b\rangle, \langle c, d\rangle\}\ \textbf{is}\ \text{Relation},$

(6) $[:X, Y:]\ \textbf{is}\ \text{Relation}.$

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The scheme *Rel_Existence* deals with a constant $\mathcal{A}$ that has the type set, a constant $\mathcal{B}$ that has the type set and a binary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex}\, R \,\textbf{being}\, \text{Relation} \,\textbf{st for}\, x{,}y \,\textbf{holds}\, \langle x,y\rangle \in R \,\textbf{iff}\, x \in \mathcal{A} \,\&\, y \in \mathcal{B} \,\&\, \mathcal{P}[x,y]$$

for all values of the parameters.

Let us consider $P$, $R$. Let us note that one can characterize the predicate

$$P = R$$

by the following (equivalent) condition:

$$\textbf{for}\, a{,}b \,\textbf{holds}\, \langle a,b\rangle \in P \,\textbf{iff}\, \langle a,b\rangle \in R.$$

The following proposition is true

$$P = R \,\textbf{iff for}\, a{,}b \,\textbf{holds}\, \langle a,b\rangle \in P \,\textbf{iff}\, \langle a,b\rangle \in R. \tag{7}$$

For convenience we may adopt another formulas defining notions considered in the paper. From now on we shall treat them as new definitions.

Let us consider $P$, $R$. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$P \cap R \quad \text{is} \quad \text{Relation}\,,$$

$$P \cup R \quad \text{is} \quad \text{Relation}\,,$$

$$P \setminus R \quad \text{is} \quad \text{Relation}\,.$$

Let us note that one can characterize the predicate

$$P \subseteq R$$

by the following (equivalent) condition:

$$\textbf{for}\, a{,}b \,\textbf{holds}\, \langle a,b\rangle \in P \,\textbf{implies}\, \langle a,b\rangle \in R.$$

The following three propositions are true:

$$P \subseteq R \,\textbf{iff for}\, a{,}b \,\textbf{holds}\, \langle a,b\rangle \in P \,\textbf{implies}\, \langle a,b\rangle \in R, \tag{8}$$

$$X \cap R \,\textbf{is}\, \text{Relation} \,\&\, R \cap X \,\textbf{is}\, \text{Relation}\,, \tag{9}$$

$$R \setminus X \,\textbf{is}\, \text{Relation}\,. \tag{10}$$

Let us consider $R$. The functor

$$\operatorname{dom} R,$$

with values of the type set, is defined by

$$x \in \textbf{it iff ex}\, y \,\textbf{st}\, \langle x,y\rangle \in R.$$

We now state several propositions:

$$X = \operatorname{dom} R \textbf{ iff for } x \textbf{ holds } x \in X \textbf{ iff ex } y \textbf{ st } \langle x, y\rangle \in R, \tag{11}$$

$$x \in \operatorname{dom} R \textbf{ iff ex } y \textbf{ st } \langle x, y\rangle \in R, \tag{12}$$

$$\operatorname{dom}(P \cup R) = \operatorname{dom} P \cup \operatorname{dom} R, \tag{13}$$

$$\operatorname{dom}(P \cap R) \subseteq \operatorname{dom} P \cap \operatorname{dom} R, \tag{14}$$

$$\operatorname{dom} P \setminus \operatorname{dom} R \subseteq \operatorname{dom}(P \setminus R). \tag{15}$$

Let us consider $R$. The functor

$$\operatorname{rng} R,$$

yields the type set and is defined by

$$y \in \textbf{it iff ex } x \textbf{ st } \langle x, y\rangle \in R.$$

One can prove the following propositions:

$$X = \operatorname{rng} R \textbf{ iff for } x \textbf{ holds } x \in X \textbf{ iff ex } y \textbf{ st } \langle y, x\rangle \in R, \tag{16}$$

$$x \in \operatorname{rng} R \textbf{ iff ex } y \textbf{ st } \langle y, x\rangle \in R, \tag{17}$$

$$x \in \operatorname{dom} R \textbf{ implies ex } y \textbf{ st } y \in \operatorname{rng} R, \tag{18}$$

$$y \in \operatorname{rng} R \textbf{ implies ex } x \textbf{ st } x \in \operatorname{dom} R, \tag{19}$$

$$\langle x, y\rangle \in R \textbf{ implies } x \in \operatorname{dom} R \;\&\; y \in \operatorname{rng} R, \tag{20}$$

$$R \subseteq [\!:\operatorname{dom} R, \operatorname{rng} R:\!], \tag{21}$$

$$R \cap [\!:\operatorname{dom} R, \operatorname{rng} R:\!] = R, \tag{22}$$

$$R = \{\langle x, y\rangle\} \textbf{ implies } \operatorname{dom} R = \{x\} \;\&\; \operatorname{rng} R = \{y\}, \tag{23}$$

$$R = \{\langle a, b\rangle, \langle x, y\rangle\} \textbf{ implies } \operatorname{dom} R = \{a, x\} \;\&\; \operatorname{rng} R = \{b, y\}, \tag{24}$$

$$P \subseteq R \textbf{ implies } \operatorname{dom} P \subseteq \operatorname{dom} R \;\&\; \operatorname{rng} P \subseteq \operatorname{rng} R, \tag{25}$$

$$\operatorname{rng}(P \cup R) = \operatorname{rng} P \cup \operatorname{rng} R, \tag{26}$$

$$\operatorname{rng}(P \cap R) \subseteq \operatorname{rng} P \cap \operatorname{rng} R, \tag{27}$$

$$\operatorname{rng} P \setminus \operatorname{rng} R \subseteq \operatorname{rng}(P \setminus R). \tag{28}$$

Let us consider $R$. The functor

$$\operatorname{field} R,$$

yields the type set and is defined by

$$\mathbf{it} = \operatorname{dom} R \cup \operatorname{rng} R.$$

We now state several propositions:

$$\text{field}\, R = \operatorname{dom} R \cup \operatorname{rng} R, \tag{29}$$

$$\langle a, b\rangle \in R \;\mathbf{implies}\; a \in \text{field}\, R \;\&\; b \in \text{field}\, R, \tag{30}$$

$$P \subseteq R \;\mathbf{implies}\; \text{field}\, P \subseteq \text{field}\, R, \tag{31}$$

$$R = \{\langle x, y\rangle\} \;\mathbf{implies}\; \text{field}\, R = \{x, y\}, \tag{32}$$

$$\text{field}\,(P \cup R) = \text{field}\, P \cup \text{field}\, R, \tag{33}$$

$$\text{field}\,(P \cap R) \subseteq \text{field}\, P \cap \text{field}\, R. \tag{34}$$

Let us consider $R$. The functor

$$R\,\check{},$$

yields the type Relation and is defined by

$$\langle x, y\rangle \in \mathbf{it} \;\mathbf{iff}\; \langle y, x\rangle \in R.$$

One can prove the following propositions:

$$R = P\,\check{} \;\mathbf{iff}\; \mathbf{for}\; x{,}y \;\mathbf{holds}\; \langle x, y\rangle \in R \;\mathbf{iff}\; \langle y, x\rangle \in P, \tag{35}$$

$$\langle x, y\rangle \in P\,\check{} \;\mathbf{iff}\; \langle y, x\rangle \in P, \tag{36}$$

$$(R\,\check{})\,\check{} = R, \tag{37}$$

$$\text{field}\, R = \text{field}\,(R\,\check{}), \tag{38}$$

$$(P \cap R)\,\check{} = P\,\check{} \cap R\,\check{}, \tag{39}$$

$$(P \cup R)\,\check{} = P\,\check{} \cup R\,\check{}, \tag{40}$$

$$(P \setminus R)\,\check{} = P\,\check{} \setminus R\,\check{}. \tag{41}$$

Let us consider $P$, $R$. The functor

$$P \cdot R,$$

with values of the type Relation, is defined by

$$\langle x, y\rangle \in \mathbf{it} \;\mathbf{iff}\; \mathbf{ex}\; z \;\mathbf{st}\; \langle x, z\rangle \in P \;\&\; \langle z, y\rangle \in R.$$

We now state a number of propositions:

$$Q = P \cdot R \;\mathbf{iff}\; \mathbf{for}\; x{,}y \;\mathbf{holds}\; \langle x, y\rangle \in Q \;\mathbf{iff}\; \mathbf{ex}\; z \;\mathbf{st}\; \langle x, z\rangle \in P \;\&\; \langle z, y\rangle \in R, \tag{42}$$

(43) $$\langle x, y\rangle \in P \cdot R \textbf{ iff ex } z \textbf{ st } \langle x, z\rangle \in P \;\&\; \langle z, y\rangle \in R,$$

(44) $$\operatorname{dom}(P \cdot R) \subseteq \operatorname{dom} P,$$

(45) $$\operatorname{rng}(P \cdot R) \subseteq \operatorname{rng} R,$$

(46) $$\operatorname{rng} R \subseteq \operatorname{dom} P \textbf{ implies } \operatorname{dom}(R \cdot P) = \operatorname{dom} R,$$

(47) $$\operatorname{dom} P \subseteq \operatorname{rng} R \textbf{ implies } \operatorname{rng}(R \cdot P) = \operatorname{rng} P,$$

(48) $$P \subseteq R \textbf{ implies } Q \cdot P \subseteq Q \cdot R,$$

(49) $$P \subseteq Q \textbf{ implies } P \cdot R \subseteq Q \cdot R,$$

(50) $$P \subseteq R \;\&\; Q \subseteq S \textbf{ implies } P \cdot Q \subseteq R \cdot S,$$

(51) $$P \cdot (R \cup Q) = (P \cdot R) \cup (P \cdot Q),$$

(52) $$P \cdot (R \cap Q) \subseteq (P \cdot R) \cap (P \cdot Q),$$

(53) $$(P \cdot R) \setminus (P \cdot Q) \subseteq P \cdot (R \setminus Q),$$

(54) $$(P \cdot R)\breve{} = R\breve{} \cdot P\breve{},$$

(55) $$(P \cdot R) \cdot Q = P \cdot (R \cdot Q).$$

The constant ∅ has the type Relation, and is defined by

$$\textbf{not } \langle x, y\rangle \in \textbf{it}\,.$$

One can prove the following propositions:

(56) $$R = \varnothing \textbf{ iff for } x, y \textbf{ holds not } \langle x, y\rangle \in R,$$

(57) $$\textbf{not } \langle x, y\rangle \in \varnothing\,,$$

(58) $$\varnothing \subseteq [:A, B:],$$

(59) $$\varnothing \subseteq R,$$

(60) $$\operatorname{dom} \varnothing = \emptyset \;\&\; \operatorname{rng} \varnothing = \emptyset,$$

(61) $$\varnothing \cap R = \varnothing \;\&\; \varnothing \cup R = R,$$

(62) $$\varnothing \cdot R = \varnothing \;\&\; R \cdot \varnothing = \varnothing\,,$$

(63) $$R \cdot \varnothing = \varnothing \cdot R,$$

(64) $$\operatorname{dom} R = \emptyset \textbf{ or } \operatorname{rng} R = \emptyset \textbf{ implies } R = \varnothing\,,$$

$$\text{dom}\, R = \emptyset \textbf{ iff } \text{rng}\, R = \emptyset, \tag{65}$$

$$\varnothing\breve{} = \varnothing\,, \tag{66}$$

$$\text{rng}\, R \cap \text{dom}\, P = \emptyset \textbf{ implies } R \cdot P = \varnothing\,. \tag{67}$$

Let us consider $X$. The functor

$$\triangle\, X,$$

with values of the type Relation, is defined by

$$\langle x, y\rangle \in \textbf{it iff } x \in X \ \&\ x = y.$$

The following propositions are true:

$$P = \triangle\, X \textbf{ iff for } x,y \textbf{ holds } \langle x, y\rangle \in P \textbf{ iff } x \in X \ \&\ x = y, \tag{68}$$

$$\langle x, y\rangle \in \triangle\, X \textbf{ iff } x \in X \ \&\ x = y, \tag{69}$$

$$x \in X \textbf{ iff } \langle x, x\rangle \in \triangle\, X, \tag{70}$$

$$\text{dom} \triangle\, X = X \ \&\ \text{rng} \triangle\, X = X, \tag{71}$$

$$(\triangle\, X)\breve{} = \triangle\, X, \tag{72}$$

$$(\textbf{for } x \textbf{ st } x \in X \textbf{ holds } \langle x, x\rangle \in R) \textbf{ implies } \triangle\, X \subseteq R, \tag{73}$$

$$\langle x, y\rangle \in (\triangle\, X) \cdot R \textbf{ iff } x \in X \ \&\ \langle x, y\rangle \in R, \tag{74}$$

$$\langle x, y\rangle \in R \cdot \triangle\, Y \textbf{ iff } y \in Y \ \&\ \langle x, y\rangle \in R, \tag{75}$$

$$R \cdot (\triangle\, X) \subseteq R \ \&\ (\triangle\, X) \cdot R \subseteq R, \tag{76}$$

$$\text{dom}\, R \subseteq X \textbf{ implies } (\triangle\, X) \cdot R = R, \tag{77}$$

$$(\triangle\, \text{dom}\, R) \cdot R = R, \tag{78}$$

$$\text{rng}\, R \subseteq Y \textbf{ implies } R \cdot (\triangle\, Y) = R, \tag{79}$$

$$R \cdot (\triangle\, \text{rng}\, R) = R, \tag{80}$$

$$\triangle\, \emptyset = \varnothing\,, \tag{81}$$

$$\text{dom}\, R = X \ \&\ \text{rng}\, P2 \subseteq X \ \&\ P2 \cdot R = \triangle\, (\text{dom}\, P1) \ \&\ R \cdot P1 = \triangle\, X \textbf{ implies } P1 = P2, \tag{82}$$

$$\text{dom}\, R = X \ \&\ \text{rng}\, P2 = X \ \&\ P2 \cdot R = \triangle\, (\text{dom}\, P1) \ \&\ R \cdot P1 = \triangle\, X \textbf{ implies } P1 = P2. \tag{83}$$

Let us consider $R$, $X$. The functor

$$R \mid X,$$

with values of the type Relation, is defined by

$$\langle x, y\rangle \in \textbf{it} \textbf{ iff } x \in X \ \& \ \langle x, y\rangle \in R.$$

We now state a number of propositions:

(84) $$P = R \mid X \textbf{ iff for } x{,}y \textbf{ holds } \langle x, y\rangle \in P \textbf{ iff } x \in X \ \& \ \langle x, y\rangle \in R,$$

(85) $$\langle x, y\rangle \in R \mid X \textbf{ iff } x \in X \ \& \ \langle x, y\rangle \in R,$$

(86) $$x \in \operatorname{dom}(R \mid X) \textbf{ iff } x \in X \ \& \ x \in \operatorname{dom} R,$$

(87) $$\operatorname{dom}(R \mid X) \subseteq X,$$

(88) $$R \mid X \subseteq R,$$

(89) $$\operatorname{dom}(R \mid X) \subseteq \operatorname{dom} R,$$

(90) $$\operatorname{dom}(R \mid X) = \operatorname{dom} R \cap X,$$

(91) $$X \subseteq \operatorname{dom} R \textbf{ implies } \operatorname{dom}(R \mid X) = X,$$

(92) $$(R \mid X) \cdot P \subseteq R \cdot P,$$

(93) $$P \cdot (R \mid X) \subseteq P \cdot R,$$

(94) $$R \mid X = (\triangle\, X) \cdot R,$$

(95) $$R \mid X = \emptyset \textbf{ iff } (\operatorname{dom} R) \cap X = \emptyset,$$

(96) $$R \mid X = R \cap [\!:X, \operatorname{rng} R:\!],$$

(97) $$\operatorname{dom} R \subseteq X \textbf{ implies } R \mid X = R,$$

(98) $$R \mid \operatorname{dom} R = R,$$

(99) $$\operatorname{rng}(R \mid X) \subseteq \operatorname{rng} R,$$

(100) $$(R \mid X) \mid Y = R \mid (X \cap Y),$$

(101) $$(R \mid X) \mid X = R \mid X,$$

(102) $$X \subseteq Y \textbf{ implies } (R \mid X) \mid Y = R \mid X,$$

(103) $$Y \subseteq X \textbf{ implies } (R \mid X) \mid Y = R \mid Y,$$

(104) $$X \subseteq Y \textbf{ implies } R \mid X \subseteq R \mid Y,$$

(105) $$P \subseteq R \textbf{ implies } P \mid X \subseteq R \mid X,$$

(106) $$P \subseteq R \,\&\, X \subseteq Y \textbf{ implies } P \mid X \subseteq R \mid Y,$$

(107) $$R \mid (X \cup Y) = (R \mid X) \cup (R \mid Y),$$

(108) $$R \mid (X \cap Y) = (R \mid X) \cap (R \mid Y),$$

(109) $$R \mid (X \setminus Y) = R \mid X \setminus R \mid Y,$$

(110) $$R \mid \emptyset = \varnothing\,,$$

(111) $$\varnothing \mid X = \varnothing\,,$$

(112) $$(P \cdot R) \mid X = (P \mid X) \cdot R.$$

Let us consider $Y$, $R$. The functor

$$Y \mid R,$$

yields the type Relation and is defined by

$$\langle x, y\rangle \in \textbf{it iff } y \in Y \;\&\; \langle x, y\rangle \in R.$$

The following propositions are true:

(113) $$P = Y \mid R \textbf{ iff for } x{,}y \textbf{ holds } \langle x, y\rangle \in P \textbf{ iff } y \in Y \;\&\; \langle x, y\rangle \in R,$$

(114) $$\langle x, y\rangle \in Y \mid R \textbf{ iff } y \in Y \;\&\; \langle x, y\rangle \in R,$$

(115) $$y \in \operatorname{rng}(Y \mid R) \textbf{ iff } y \in Y \;\&\; y \in \operatorname{rng} R,$$

(116) $$\operatorname{rng}(Y \mid R) \subseteq Y,$$

(117) $$Y \mid R \subseteq R,$$

(118) $$\operatorname{rng}(Y \mid R) \subseteq \operatorname{rng} R,$$

(119) $$\operatorname{rng}(Y \mid R) = \operatorname{rng} R \cap Y,$$

(120) $$Y \subseteq \operatorname{rng} R \textbf{ implies } \operatorname{rng}(Y \mid R) = Y,$$

(121) $$(Y \mid R) \cdot P \subseteq R \cdot P,$$

(122) $$P \cdot (Y \mid R) \subseteq P \cdot R,$$

(123) $$Y \mid R = R \cdot (\triangle\, Y),$$

(124) $$Y \mid R = R \cap [\!:\operatorname{dom} R, Y:\!],$$

(125) $$\operatorname{rng} R \subseteq Y \textbf{ implies } Y \mid R = R,$$

$$\operatorname{rng} R \mid R = R, \tag{126}$$

$$Y \mid (X \mid R) = (Y \cap X) \mid R, \tag{127}$$

$$Y \mid (Y \mid R) = Y \mid R, \tag{128}$$

$$X \subseteq Y \textbf{ implies } Y \mid (X \mid R) = X \mid R, \tag{129}$$

$$Y \subseteq X \textbf{ implies } Y \mid (X \mid R) = Y \mid R, \tag{130}$$

$$X \subseteq Y \textbf{ implies } X \mid R \subseteq Y \mid R, \tag{131}$$

$$P1 \subseteq P2 \textbf{ implies } Y \mid P1 \subseteq Y \mid P2, \tag{132}$$

$$P1 \subseteq P2 \mathbin{\&} Y1 \subseteq Y2 \textbf{ implies } Y1 \mid P1 \subseteq Y2 \mid P2, \tag{133}$$

$$(X \cup Y) \mid R = (X \mid R) \cup (Y \mid R), \tag{134}$$

$$(X \cap Y) \mid R = X \mid R \cap Y \mid R, \tag{135}$$

$$(X \setminus Y) \mid R = X \mid R \setminus Y \mid R, \tag{136}$$

$$\emptyset \mid R = \emptyset\,, \tag{137}$$

$$Y \mid \emptyset = \emptyset\,, \tag{138}$$

$$Y \mid (P \cdot R) = P \cdot (Y \mid R), \tag{139}$$

$$(Y \mid R) \mid X = Y \mid (R \mid X). \tag{140}$$

Let us consider $R$, $X$. The functor

$$R^{\circ}\, X,$$

yields the type set and is defined by

$$y \in \textbf{it iff ex } x \textbf{ st } \langle x, y\rangle \in R \mathbin{\&} x \in X.$$

One can prove the following propositions:

$$Y = R^{\circ}\, X \textbf{ iff for } y \textbf{ holds } y \in Y \textbf{ iff ex } x \textbf{ st } \langle x, y\rangle \in R \mathbin{\&} x \in X, \tag{141}$$

$$y \in R^{\circ}\, X \textbf{ iff ex } x \textbf{ st } \langle x, y\rangle \in R \mathbin{\&} x \in X, \tag{142}$$

$$y \in R^{\circ}\, X \textbf{ iff ex } x \textbf{ st } x \in \operatorname{dom} R \mathbin{\&} \langle x, y\rangle \in R \mathbin{\&} x \in X, \tag{143}$$

$$R^{\circ}\, X \subseteq \operatorname{rng} R, \tag{144}$$

$$R^{\circ}\, X = R^{\circ}\, (\operatorname{dom} R \cap X), \tag{145}$$

$$R^{\circ} \operatorname{dom} R = \operatorname{rng} R, \tag{146}$$

(147) $$R^{\circ} X \subseteq R^{\circ} (\operatorname{dom} R),$$

(148) $$\operatorname{rng}(R \mid X) = R^{\circ} X,$$

(149) $$R^{\circ} \emptyset = \emptyset,$$

(150) $$\varnothing^{\circ} X = \emptyset,$$

(151) $$R^{\circ} X = \emptyset \textbf{ iff } \operatorname{dom} R \cap X = \emptyset,$$

(152) $$X \neq \emptyset \;\&\; X \subseteq \operatorname{dom} R \textbf{ implies } R^{\circ} X \neq \emptyset,$$

(153) $$R^{\circ} (X \cup Y) = R^{\circ} X \cup R^{\circ} Y,$$

(154) $$R^{\circ} (X \cap Y) \subseteq R^{\circ} X \cap R^{\circ} Y,$$

(155) $$R^{\circ} X \setminus R^{\circ} Y \subseteq R^{\circ} (X \setminus Y),$$

(156) $$X \subseteq Y \textbf{ implies } R^{\circ} X \subseteq R^{\circ} Y,$$

(157) $$P \subseteq R \textbf{ implies } P^{\circ} X \subseteq R^{\circ} X,$$

(158) $$P \subseteq R \;\&\; X \subseteq Y \textbf{ implies } P^{\circ} X \subseteq R^{\circ} Y,$$

(159) $$(P \cdot R)^{\circ} X = R^{\circ} (P^{\circ} X),$$

(160) $$\operatorname{rng}(P \cdot R) = R^{\circ} (\operatorname{rng} P),$$

(161) $$(R \mid X)^{\circ} Y \subseteq R^{\circ} Y,$$

(162) $$R \mid X = \varnothing \textbf{ iff } (\operatorname{dom} R) \cap X = \emptyset,$$

(163) $$(\operatorname{dom} R) \cap X \subseteq (R\check{})^{\circ} (R^{\circ} X).$$

Let us consider $R$, $Y$. The functor

$$R^{-1}\, Y,$$

with values of the type set, is defined by

$$x \in \textbf{it iff ex}\, y \textbf{ st } \langle x, y\rangle \in R \;\&\; y \in Y.$$

Next we state a number of propositions:

(164) $$X = R^{-1}\, Y \textbf{ iff for } x \textbf{ holds } x \in X \textbf{ iff ex}\, y \textbf{ st } \langle x, y\rangle \in R \;\&\; y \in Y,$$

(165) $$x \in R^{-1}\, Y \textbf{ iff ex}\, y \textbf{ st } \langle x, y\rangle \in R \;\&\; y \in Y,$$

(166) $$x \in R^{-1}\, Y \textbf{ iff ex}\, y \textbf{ st } y \in \operatorname{rng} R \;\&\; \langle x, y\rangle \in R \;\&\; y \in Y,$$

(167) $$R^{-1}\, Y \subseteq \operatorname{dom} R,$$

(168) $$R^{-1}\,Y = R^{-1}\,(\operatorname{rng} R \cap Y),$$

(169) $$R^{-1}\operatorname{rng} R = \operatorname{dom} R,$$

(170) $$R^{-1}\,Y \subseteq R^{-1}\operatorname{rng} R,$$

(171) $$R^{-1}\,\emptyset = \emptyset,$$

(172) $$\emptyset^{-1}\,Y = \emptyset,$$

(173) $$R^{-1}\,Y = \emptyset \textbf{ iff } \operatorname{rng} R \cap Y = \emptyset,$$

(174) $$Y \neq \emptyset \,\&\, Y \subseteq \operatorname{rng} R \textbf{ implies } R^{-1}\,Y \neq \emptyset,$$

(175) $$R^{-1}\,(X \cup Y) = R^{-1}\,X \cup R^{-1}\,Y,$$

(176) $$R^{-1}\,(X \cap Y) \subseteq R^{-1}\,Y \cap R^{-1}\,Y,$$

(177) $$R^{-1}\,X \setminus R^{-1}\,Y \subseteq R^{-1}\,(X \setminus Y),$$

(178) $$X \subseteq Y \textbf{ implies } R^{-1}\,X \subseteq R^{-1}\,Y,$$

(179) $$P \subseteq R \textbf{ implies } P^{-1}\,Y \subseteq R^{-1}\,Y,$$

(180) $$P \subseteq R \,\&\, X \subseteq Y \textbf{ implies } P^{-1}\,X \subseteq R^{-1}\,Y,$$

(181) $$(P \cdot R)^{-1}\,Y = P^{-1}\,(R^{-1}\,Y),$$

(182) $$\operatorname{dom}(P \cdot R) = P^{-1}\,(\operatorname{dom} R),$$

(183) $$(\operatorname{rng} R) \cap Y \subseteq (R\breve{})^{-1}\,(R^{-1}\,Y).$$

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