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Families of Sets

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Summary. The article contains definitions of the following concepts: family of sets, family of subsets of a set, the intersection of a family of sets. Functors $\cup$, $\cap$, and $\setminus$ are redefined for families of subsets of a set. Some properties of these notions are presented.

The terminology and notation used in this paper are introduced in the following papers: [1], [3], and [2]. For simplicity we adopt the following convention: X, Y, Z, $Z1$, D will denote objects of the type set; x, y will denote objects of the type Any. Let us consider X. The functor

$$\bigcap X,$$

with values of the type set, is defined by

$$\begin{cases} \textbf{for}\, x \,\textbf{holds}\, x \in \textbf{it iff for}\, Y \,\textbf{holds}\, Y \in X \,\textbf{implies}\, x \in Y, & \textbf{if} \quad X \neq \emptyset, \\ \textbf{it} = \emptyset, & \textbf{otherwise}. \end{cases}$$

The following propositions are true:

$$X \neq \emptyset \,\textbf{implies for}\, x \,\textbf{holds}\, x \in \bigcap X \,\textbf{iff for}\, Y \,\textbf{st}\, Y \in X \,\textbf{holds}\, x \in Y, \tag{1}$$

$$\bigcap \emptyset = \emptyset, \tag{2}$$

$$\bigcap X \subseteq \bigcup X, \tag{3}$$

$$Z \in X \,\textbf{implies}\, \bigcap X \subseteq Z, \tag{4}$$

$$\emptyset \in X \,\textbf{implies}\, \bigcap X = \emptyset, \tag{5}$$

$$X \neq \emptyset \,\&\, (\textbf{for}\, Z1 \,\textbf{st}\, Z1 \in X \,\textbf{holds}\, Z \subseteq Z1) \,\textbf{implies}\, Z \subseteq \bigcap X, \tag{6}$$

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(7) $$X \neq \emptyset \ \& \ X \subseteq Y \textbf{ implies } \bigcap Y \subseteq \bigcap X,$$

(8) $$X \in Y \ \& \ X \subseteq Z \textbf{ implies } \bigcap Y \subseteq Z,$$

(9) $$X \in Y \ \& \ X \cap Z = \emptyset \textbf{ implies } \bigcap Y \cap Z = \emptyset,$$

(10) $$X \neq \emptyset \ \& \ Y \neq \emptyset \textbf{ implies } \bigcap (X \cup Y) = \bigcap X \cap \bigcap Y,$$

(11) $$\bigcap \{x\} = x,$$

(12) $$\bigcap \{X, Y\} = X \cap Y.$$

Set-Family stands for set .

In the sequel SFX, SFY, SFZ will have the type Set-Family. One can prove the following two propositions:

(13) $$x \textbf{ is } \text{Set-Family},$$

(14) $$SFX = SFY \textbf{ iff for } X \textbf{ holds } X \in SFX \textbf{ iff } X \in SFY.$$

We now define two new predicates. Let us consider SFX, SFY. The predicate

$$SFX \text{ is\_finer\_than } SFY$$

is defined by

$$\textbf{for } X \textbf{ st } X \in SFX \textbf{ ex } Y \textbf{ st } Y \in SFY \ \& \ X \subseteq Y.$$

The predicate

$$SFX \text{ is\_coarser\_than } SFY$$

is defined by

$$\textbf{for } Y \textbf{ st } Y \in SFY \textbf{ ex } X \textbf{ st } X \in SFX \ \& \ X \subseteq Y.$$

Next we state several propositions:

(15) $$SFX \text{ is\_finer\_than } SFY \textbf{ iff for } X \textbf{ st } X \in SFX \textbf{ ex } Y \textbf{ st } Y \in SFY \ \& \ X \subseteq Y,$$

(16) $$SFX \text{ is\_coarser\_than } SFY$$
$$\textbf{iff for } Y \textbf{ st } Y \in SFY \textbf{ ex } X \textbf{ st } X \in SFX \ \& \ X \subseteq Y,$$

(17) $$SFX \subseteq SFY \textbf{ implies } SFX \text{ is\_finer\_than } SFY,$$

(18) $$SFX \text{ is\_finer\_than } SFY \textbf{ implies } \bigcup SFX \subseteq \bigcup SFY,$$

(19) $$SFY \neq \emptyset \ \& \ SFX \text{ is\_coarser\_than } SFY \textbf{ implies } \bigcap SFX \subseteq \bigcap SFY.$$

Let us note that it makes sense to consider the following constant. Then $\emptyset$ is Set-Family. Let us consider x. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\{x\} \quad \text{is} \quad \text{Set-Family}.$$

Let us consider y. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\{x, y\} \quad \text{is} \quad \text{Set-Family}.$$

One can prove the following propositions:

(20) $$\emptyset \text{ is\_finer\_than } SFX,$$

(21) $$SFX \text{ is\_finer\_than } \emptyset \textbf{ implies } SFX = \emptyset,$$

(22) $$SFX \text{ is\_finer\_than } SFX,$$

(23) $$SFX \text{ is\_finer\_than } SFY \;\&\; SFY \text{ is\_finer\_than } SFZ \textbf{ implies } SFX \text{ is\_finer\_than } SFZ,$$

(24) $$SFX \text{ is\_finer\_than } \{Y\} \textbf{ implies for } X \textbf{ st } X \in SFX \textbf{ holds } X \subseteq Y,$$

(25) $$SFX \text{ is\_finer\_than } \{X, Y\} \textbf{ implies for } Z \textbf{ st } Z \in SFX \textbf{ holds } Z \subseteq X \textbf{ or } Z \subseteq Y.$$

We now define three new functors. Let us consider SFX, SFY. The functor

$$\text{UNION}\,(SFX,SFY),$$

yields the type Set-Family and is defined by

$$Z \in \textbf{it iff ex } X,Y \textbf{ st } X \in SFX \;\&\; Y \in SFY \;\&\; Z = X \cup Y.$$

The functor

$$\text{INTERSECTION}\,(SFX,SFY),$$

with values of the type Set-Family, is defined by

$$Z \in \textbf{it iff ex } X,Y \textbf{ st } X \in SFX \;\&\; Y \in SFY \;\&\; Z = X \cap Y.$$

The functor

$$\text{DIFFERENCE}\,(SFX,SFY),$$

with values of the type Set-Family, is defined by

$$Z \in \textbf{it iff ex } X,Y \textbf{ st } X \in SFX \;\&\; Y \in SFY \;\&\; Z = X \setminus Y.$$

One can prove the following propositions:

(26) $Z \in \text{UNION}\,(SFX,SFY)$ **iff ex** X,Y **st** $X \in SFX \;\&\; Y \in SFY \;\&\; Z = X \cup Y$,

(27) $$Z \in \mathrm{INTERSECTION}\,(SFX,SFY) \\ \textbf{iff ex}\,X,Y\ \textbf{st}\ X \in SFX\ \&\ Y \in SFY\ \&\ Z = X \cap Y,$$

(28) $$Z \in \mathrm{DIFFERENCE}\,(SFX,SFY) \\ \textbf{iff ex}\,X,Y\ \textbf{st}\ X \in SFX\ \&\ Y \in SFY\ \&\ Z = X \setminus Y,$$

(29) $$SFX\ \text{is\_finer\_than}\ \mathrm{UNION}\,(SFX,SFX),$$

(30) $$\mathrm{INTERSECTION}\,(SFX,SFX)\ \text{is\_finer\_than}\ SFX,$$

(31) $$\mathrm{DIFFERENCE}\,(SFX,SFX)\ \text{is\_finer\_than}\ SFX,$$

(32) $$\mathrm{UNION}\,(SFX,SFY) = \mathrm{UNION}\,(SFY,SFX),$$

(33) $$\mathrm{INTERSECTION}\,(SFX,SFY) = \mathrm{INTERSECTION}\,(SFY,SFX),$$

(34) $$SFX \cap SFY \neq \emptyset \\ \textbf{implies} \bigcap SFX \cap \bigcap SFY = \bigcap \mathrm{INTERSECTION}\,(SFX,SFY),$$

(35) $$SFY \neq \emptyset\ \textbf{implies}\ X \cup \bigcap SFY = \bigcap \mathrm{UNION}\,(\{X\},SFY),$$

(36) $$X \cap \bigcup SFY = \bigcup \mathrm{INTERSECTION}\,(\{X\},SFY),$$

(37) $$SFY \neq \emptyset\ \textbf{implies}\ X \setminus \bigcup SFY = \bigcap \mathrm{DIFFERENCE}\,(\{X\},SFY),$$

(38) $$SFY \neq \emptyset\ \textbf{implies}\ X \setminus \bigcap SFY = \bigcup \mathrm{DIFFERENCE}\,(\{X\},SFY),$$

(39) $$\bigcup \mathrm{INTERSECTION}\,(SFX,SFY) \subseteq \bigcup SFX \cap \bigcup SFY,$$

(40) $$SFX \neq \emptyset\ \&\ SFY \neq \emptyset\ \textbf{implies} \bigcap SFX \cup \bigcap SFY \subseteq \bigcap \mathrm{UNION}\,(SFX,SFY),$$

(41) $$SFX \neq \emptyset\ \&\ SFY \neq \emptyset \\ \textbf{implies} \bigcap \mathrm{DIFFERENCE}\,(SFX,SFY) \subseteq \bigcap SFX \setminus \bigcap SFY.$$

Let D have the type set.

Subset-Family **of** D stands for Subset **of** bool D.

We now state a proposition

(42) **for** F **being** Subset **of** bool D **holds** F **is** Subset-Family **of** D.

In the sequel F, G have the type Subset-Family **of** D; P has the type Subset **of** D. Let us consider D, F, G. Let us note that it makes sense to consider the following functors on restricted areas. Then

$F \cup G$ is Subset-Family **of** D,

$$F \cap G \quad \text{is} \quad \text{Subset-Family } \mathbf{of}\ D,$$

$$F \setminus G \quad \text{is} \quad \text{Subset-Family } \mathbf{of}\ D.$$

Next we state a proposition

$$X \in F \ \mathbf{implies}\ X\ \mathbf{is}\ \text{Subset}\ \mathbf{of}\ D. \tag{43}$$

Let us consider D, F. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\bigcup F \quad \text{is} \quad \text{Subset } \mathbf{of}\ D.$$

Let us consider D, F. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\bigcap F \quad \text{is} \quad \text{Subset } \mathbf{of}\ D.$$

The following proposition is true

$$F = G\ \mathbf{iff}\ \mathbf{for}\ P\ \mathbf{holds}\ P \in F\ \mathbf{iff}\ P \in G. \tag{44}$$

The scheme *SubFamEx* deals with a constant $\mathcal{A}$ that has the type set and a unary predicate $\mathcal{P}$ and states that the following holds

$$\mathbf{ex}\ F\ \mathbf{being}\ \text{Subset-Family}\ \mathbf{of}\ \mathcal{A}\ \mathbf{st}\ \mathbf{for}\ B\ \mathbf{being}\ \text{Subset}\ \mathbf{of}\ \mathcal{A}\ \mathbf{holds}\ B \in F\ \mathbf{iff}\ \mathcal{P}[B]$$

for all values of the parameters.

Let us consider D, F. The functor

$$F^{\mathrm{c}},$$

yields the type Subset-Family **of** D and is defined by

$$\mathbf{for}\ P\ \mathbf{being}\ \text{Subset}\ \mathbf{of}\ D\ \mathbf{holds}\ P \in \mathbf{it}\ \mathbf{iff}\ P^{\mathrm{c}} \in F.$$

Next we state four propositions:

$$\mathbf{for}\ P\ \mathbf{holds}\ P \in F^{\mathrm{c}}\ \mathbf{iff}\ P^{\mathrm{c}} \in F, \tag{45}$$

$$F \neq \emptyset\ \mathbf{implies}\ F^{\mathrm{c}} \neq \emptyset, \tag{46}$$

$$F \neq \emptyset\ \mathbf{implies}\ \Omega\, D \setminus \bigcup F = \bigcap (F^{\mathrm{c}}), \tag{47}$$

$$F \neq \emptyset\ \mathbf{implies}\ \bigcup F^{\mathrm{c}} = \Omega\, D \setminus \bigcap F. \tag{48}$$

References

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