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Properties of Subsets

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Summary. The text includes theorems concerning properties of subsets, and some operations on sets. The functions yielding improper subsets of a set, i.e. the empty set and the set itself are introduced. Functions and predicates introduced for sets are redefined. Some theorems about enumerated sets are proved.

The articles [2], [3], and [1] provide the terminology and notation for this paper. In the sequel E, X denote objects of the type set; x denotes an object of the type Any. One can prove the following propositions:

(1) $E \neq \emptyset$ **implies** (x **is** Element **of** E **iff** $x \in E$),

(2) $x \in E$ **implies** x **is** Element **of** E,

(3) X **is** Subset **of** E **iff** $X \subseteq E$.

We now define two new functors. Let us consider E. The functor

$$\emptyset E,$$

yields the type Subset **of** E and is defined by

$$\textbf{it} = \emptyset.$$

The functor

$$\Omega\, E,$$

with values of the type Subset **of** E, is defined by

$$\textbf{it} = E.$$

We now state two propositions:

(4) $\emptyset$ **is** Subset **of** X,

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$$X \textbf{ is } \text{Subset } \textbf{of } X. \tag{5}$$

In the sequel A, B, C denote objects of the type Subset **of** E. Next we state several propositions:

$$x \in A \textbf{ implies } x \textbf{ is } \text{Element } \textbf{of } E, \tag{6}$$

$$(\textbf{for } x \textbf{ being } \text{Element } \textbf{of } E \textbf{ holds } x \in A \textbf{ implies } x \in B) \textbf{ implies } A \subseteq B, \tag{7}$$

$$(\textbf{for } x \textbf{ being } \text{Element } \textbf{of } E \textbf{ holds } x \in A \textbf{ iff } x \in B) \textbf{ implies } A = B, \tag{8}$$

$$x \in A \textbf{ implies } x \in E, \tag{9}$$

$$A \neq \emptyset \textbf{ iff } \textbf{ex } x \textbf{ being } \text{Element } \textbf{of } E \textbf{ st } x \in A. \tag{10}$$

Let us consider E, A. The functor

$$A^{\mathrm{c}},$$

yields the type Subset **of** E and is defined by

$$\textbf{it} = E \setminus A.$$

Let us consider B. Let us note that it makes sense to consider the following functors on restricted areas. Then

$A \cup B$	is	Subset **of** E,
$A \cap B$	is	Subset **of** E,
$A \setminus B$	is	Subset **of** E,
$A \dot{-} B$	is	Subset **of** E.

One can prove the following propositions:

$$x \in A \cap B \textbf{ implies } x \textbf{ is } \text{Element } \textbf{of } A \;\&\; x \textbf{ is } \text{Element } \textbf{of } B, \tag{11}$$

$$x \in A \cup B \textbf{ implies } x \textbf{ is } \text{Element } \textbf{of } A \textbf{ or } x \textbf{ is } \text{Element } \textbf{of } B, \tag{12}$$

$$x \in A \setminus B \textbf{ implies } x \textbf{ is } \text{Element } \textbf{of } A, \tag{13}$$

$$x \in A \dot{-} B \textbf{ implies } x \textbf{ is } \text{Element } \textbf{of } A \textbf{ or } x \textbf{ is } \text{Element } \textbf{of } B, \tag{14}$$

$$(\textbf{for } x \textbf{ being } \text{Element } \textbf{of } E \textbf{ holds } x \in A \textbf{ iff } x \in B \textbf{ or } x \in C) \\ \textbf{implies } A = B \cup C, \tag{15}$$

$$(\textbf{for } x \textbf{ being } \text{Element } \textbf{of } E \textbf{ holds } x \in A \textbf{ iff } x \in B \;\&\; x \in C) \\ \textbf{implies } A = B \cap C, \tag{16}$$

$$(\textbf{for } x \textbf{ being } \text{Element } \textbf{of } E \textbf{ holds } x \in A \textbf{ iff } x \in B \;\&\; \textbf{not } x \in C) \\ \textbf{implies } A = B \setminus C, \tag{17}$$

$$(18)\quad (\textbf{for } x \textbf{ being } \text{Element } \textbf{of } E \textbf{ holds } x \in A \textbf{ iff not } (x \in B \textbf{ iff } x \in C)) \textbf{ implies } A = B \mathbin{\dot{-}} C,$$

$$(19)\quad \emptyset E = \emptyset,$$

$$(20)\quad \Omega\, E = E,$$

$$(21)\quad \emptyset E = (\Omega\, E)^{\,\mathrm{c}},$$

$$(22)\quad \Omega\, E = (\emptyset E)^{\,\mathrm{c}},$$

$$(23)\quad A^{\,\mathrm{c}} = E \setminus A,$$

$$(24)\quad A^{\,\mathrm{c}\,\mathrm{c}} = A,$$

$$(25)\quad A \cup A^{\,\mathrm{c}} = \Omega\, E \;\&\; A^{\,\mathrm{c}} \cup A = \Omega\, E,$$

$$(26)\quad A \cap A^{\,\mathrm{c}} = \emptyset E \;\&\; A^{\,\mathrm{c}} \cap A = \emptyset E,$$

$$(27)\quad A \cap \emptyset E = \emptyset E \;\&\; \emptyset E \cap A = \emptyset E,$$

$$(28)\quad A \cup \Omega\, E = \Omega\, E \;\&\; \Omega\, E \cup A = \Omega\, E,$$

$$(29)\quad (A \cup B)^{\,\mathrm{c}} = A^{\,\mathrm{c}} \cap B^{\,\mathrm{c}},$$

$$(30)\quad (A \cap B)^{\,\mathrm{c}} = A^{\,\mathrm{c}} \cup B^{\,\mathrm{c}},$$

$$(31)\quad A \subseteq B \textbf{ iff } B^{\,\mathrm{c}} \subseteq A^{\,\mathrm{c}},$$

$$(32)\quad A \setminus B = A \cap B^{\,\mathrm{c}},$$

$$(33)\quad (A \setminus B)^{\,\mathrm{c}} = A^{\,\mathrm{c}} \cup B,$$

$$(34)\quad (A \mathbin{\dot{-}} B)^{\,\mathrm{c}} = A \cap B \cup A^{\,\mathrm{c}} \cap B^{\,\mathrm{c}},$$

$$(35)\quad A \subseteq B^{\,\mathrm{c}} \textbf{ implies } B \subseteq A^{\,\mathrm{c}},$$

$$(36)\quad A^{\,\mathrm{c}} \subseteq B \textbf{ implies } B^{\,\mathrm{c}} \subseteq A,$$

$$(37)\quad \emptyset E \subseteq E,$$

$$(38)\quad A \subseteq A^{\,\mathrm{c}} \textbf{ iff } A = \emptyset E,$$

$$(39)\quad A^{\,\mathrm{c}} \subseteq A \textbf{ iff } A = \Omega\, E,$$

$$(40)\quad X \subseteq A \;\&\; X \subseteq A^{\,\mathrm{c}} \textbf{ implies } X = \emptyset,$$

$$(41)\quad (A \cup B)^{\,\mathrm{c}} \subseteq A^{\,\mathrm{c}} \;\&\; (A \cup B)^{\,\mathrm{c}} \subseteq B^{\,\mathrm{c}},$$

$$A^{c} \subseteq (A \cap B)^{c} \;\&\; B^{c} \subseteq (A \cap B)^{c}, \tag{42}$$

$$A \text{ misses } B \textbf{ iff } A \subseteq B^{c}, \tag{43}$$

$$A \text{ misses } B^{c} \textbf{ iff } A \subseteq B, \tag{44}$$

$$A \text{ misses } A^{c}, \tag{45}$$

$$A \text{ misses } B \;\&\; A^{c} \text{ misses } B^{c} \textbf{ implies } A = B^{c}, \tag{46}$$

$$A \subseteq B \;\&\; C \text{ misses } B \textbf{ implies } A \subseteq C^{c}, \tag{47}$$

$$(\textbf{for } a \textbf{ being } \text{Element } \textbf{of } A \textbf{ holds } a \in B) \textbf{ implies } A \subseteq B, \tag{48}$$

$$(\textbf{for } x \textbf{ being } \text{Element } \textbf{of } E \textbf{ holds } x \in A) \textbf{ implies } E = A, \tag{49}$$

$$E \neq \emptyset \textbf{ implies for } A,B \tag{50}$$
$$\textbf{holds } A = B^{c} \textbf{ iff for } x \textbf{ being } \text{Element } \textbf{of } E \textbf{ holds } x \in A \textbf{ iff not } x \in B,$$

$$E \neq \emptyset \textbf{ implies for } A,B \tag{51}$$
$$\textbf{holds } A = B^{c} \textbf{ iff for } x \textbf{ being } \text{Element } \textbf{of } E \textbf{ holds not } x \in A \textbf{ iff } x \in B,$$

$$E \neq \emptyset \textbf{ implies for } A,B \tag{52}$$
$$\textbf{holds } A = B^{c} \textbf{ iff for } x \textbf{ being } \text{Element } \textbf{of } E \textbf{ holds not } (x \in A \textbf{ iff } x \in B),$$

$$x \in A^{c} \textbf{ implies not } x \in A. \tag{53}$$

In the sequel $x1$, $x2$, $x3$, $x4$, $x5$, $x6$, $x7$, $x8$ will have the type Element **of** X. One can prove the following propositions:

$$X \neq \emptyset \textbf{ implies } \{x1\} \textbf{ is } \text{Subset } \textbf{of } X, \tag{54}$$

$$X \neq \emptyset \textbf{ implies } \{x1,x2\} \textbf{ is } \text{Subset } \textbf{of } X, \tag{55}$$

$$X \neq \emptyset \textbf{ implies } \{x1,x2,x3\} \textbf{ is } \text{Subset } \textbf{of } X, \tag{56}$$

$$X \neq \emptyset \textbf{ implies } \{x1,x2,x3,x4\} \textbf{ is } \text{Subset } \textbf{of } X, \tag{57}$$

$$X \neq \emptyset \textbf{ implies } \{x1,x2,x3,x4,x5\} \textbf{ is } \text{Subset } \textbf{of } X, \tag{58}$$

$$X \neq \emptyset \textbf{ implies } \{x1,x2,x3,x4,x5,x6\} \textbf{ is } \text{Subset } \textbf{of } X, \tag{59}$$

$$X \neq \emptyset \textbf{ implies } \{x1,x2,x3,x4,x5,x6,x7\} \textbf{ is } \text{Subset } \textbf{of } X, \tag{60}$$

$$X \neq \emptyset \textbf{ implies } \{x1,x2,x3,x4,x5,x6,x7,x8\} \textbf{ is } \text{Subset } \textbf{of } X. \tag{61}$$

In the sequel $x1$, $x2$, $x3$, $x4$, $x5$, $x6$, $x7$, $x8$ denote objects of the type Any. We now state several propositions:

$$x1 \in X \textbf{ implies } \{x1\} \textbf{ is } \text{Subset } \textbf{of } X, \tag{62}$$

(63) $x1 \in X$ & $x2 \in X$ **implies** $\{x1,x2\}$ **is** Subset **of** X,

(64) $x1 \in X$ & $x2 \in X$ & $x3 \in X$ **implies** $\{x1,x2,x3\}$ **is** Subset **of** X,

(65) $x1 \in X$ & $x2 \in X$ & $x3 \in X$ & $x4 \in X$ **implies** $\{x1,x2,x3,x4\}$ **is** Subset **of** X,

(66) $x1 \in X$ & $x2 \in X$ & $x3 \in X$ & $x4 \in X$ & $x5 \in X$
implies $\{x1,x2,x3,x4,x5\}$ **is** Subset **of** X,

(67) $x1 \in X$ & $x2 \in X$ & $x3 \in X$ & $x4 \in X$ & $x5 \in X$ & $x6 \in X$
implies $\{x1,x2,x3,x4,x5,x6\}$ **is** Subset **of** X,

(68) $x1 \in X$ & $x2 \in X$ & $x3 \in X$ & $x4 \in X$ & $x5 \in X$ & $x6 \in X$ & $x7 \in X$
implies $\{x1,x2,x3,x4,x5,x6,x7\}$ **is** Subset **of** X,

(69) $x1 \in X$
& $x2 \in X$ & $x3 \in X$ & $x4 \in X$ & $x5 \in X$ & $x6 \in X$ & $x7 \in X$ & $x8 \in X$
implies $\{x1,x2,x3,x4,x5,x6,x7,x8\}$ **is** Subset **of** X.

The scheme *Subset_Ex* concerns a constant $\mathcal{A}$ that has the type set and a unary predicate $\mathcal{P}$ and states that the following holds

ex X **being** Subset **of** $\mathcal{A}$ **st for** x **holds** $x \in X$ **iff** $x \in \mathcal{A}$ & $\mathcal{P}[x]$

for all values of the parameters.

References

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