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Cardinal Numbers

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Summary. We present the choice function rule in the beginning of the article. In the main part of the article we formalize the base of cardinal theory. In the first section we introduce the concept of cardinal numbers and order relations between them. We present here Cantor-Bernstein theorem and other properties of order relation of cardinals. In the second section we show that every set has cardinal number equipotence to it. We introduce notion of alephs and we deal with the concept of finite set. At the end of the article we show two schemes of cardinal induction. Some definitions are based on [9] and [11].

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The papers [12], [10], [1], [13], [7], [4], [2], [3], [5], [6], and [8] provide the notation and terminology for this paper. For simplicity we follow the rules: A, B will be ordinal numbers, X, X_1, Y, Y_1, Z will be sets, R will be a relation, f will be a function, x, y will be arbitrary, m, n will be natural numbers, and M will be a non-empty family of sets. We now state a proposition

(1) If for every X such that $X \in M$ holds $X \neq \emptyset$, then there exists $Choice$ being a function such that $\mathrm{dom}\, Choice = M$ and for every X such that $X \in M$ holds $Choice(X) \in X$.

The mode cardinal number, which widens to the type a set, is defined by:
there exists B such that it $= B$ and for every A such that $A \approx B$ holds $B \subseteq A$.

One can prove the following proposition

(2) X is a cardinal number if and only if there exists A such that $X = A$ and for every B such that $B \approx A$ holds $A \subseteq B$.

Let M be a cardinal number. The functor $\overline{M}$ yielding an ordinal number, is defined by:
$\overline{M} = M$.

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In the sequel K, M, N will be cardinal numbers. We now state three propositions:

(3) $\overline{M} = M$.

(4) For every X there exists A such that $X \approx A$.

(5) M is an ordinal number.

We now define two new predicates. Let us consider M, N. The predicate $M \leq N$ is defined by:

$M \subseteq N$.

The predicate $M < N$ is defined by:

$M \in N$.

Next we state a number of propositions:

(6) $M \leq N$ if and only if $M \subseteq N$.

(7) $M < N$ if and only if $M \in N$.

(8) $M = N$ if and only if $M \approx N$.

(9) $M \leq M$.

(10) If $M \leq N$ and $N \leq M$, then $M = N$.

(11) If $M \leq N$ and $N \leq K$, then $M \leq K$.

(12) $M \leq N$ or $N \leq M$.

(13) $M < N$ if and only if $M \leq N$ and $M \neq N$.

(14) $M < N$ if and only if $N \nleq M$.

(15) If $M < N$, then $N \nless M$.

(16) $M < N$ or $M = N$ or $N < M$.

(17) If $M < N$ and $N < K$, then $M < K$.

(18) If $M < N$ and $N \leq K$ or $M \leq N$ and $N < K$, then $M < K$.

Let us consider X. The functor $\overline{\overline{X}}$ yields a cardinal number and is defined by:

$X \approx \overline{\overline{X}}$.

Next we state a number of propositions:

(19) $M = \overline{\overline{X}}$ if and only if $X \approx M$.

(20) $\overline{\overline{M}} = M$.

(21) $X \approx Y$ if and only if $\overline{\overline{X}} = \overline{\overline{Y}}$.

(22) If R is well ordering relation, then field $R \approx \overline{R}$.

(23) If $X \subseteq M$, then $\overline{\overline{X}} \leq M$.

(24) $\overline{\overline{A}} \subseteq A$.

(25) If $X \in M$, then $\overline{\overline{X}} < M$.

(26) $\overline{\overline{X}} \leq \overline{\overline{Y}}$ if and only if there exists f such that f is one-to-one and $\operatorname{dom} f = X$ and $\operatorname{rng} f \subseteq Y$.

(27) If $X \subseteq Y$, then $\overline{\overline{X}} \leq \overline{\overline{Y}}$.

(28) $\overline{\overline{X}} \leq \overline{\overline{Y}}$ if and only if there exists f such that $\operatorname{dom} f = Y$ and $X \subseteq \operatorname{rng} f$.

(29) $X \not\approx 2^X$.

(30) $\overline{\overline{X}} < \overline{\overline{2^X}}$.

Let us consider X. The functor X^+ yielding a cardinal number, is defined by: $\overline{\overline{X}} < X^+$ and for every M such that $\overline{\overline{X}} < M$ holds $X^+ \leq M$.

We now state several propositions:

(31) $M = X^+$ if and only if $\overline{\overline{X}} < M$ and for every N such that $\overline{\overline{X}} < N$ holds $M \leq N$.

(32) $M < M^+$.

(33) $\overline{\overline{\mathbf{0}}} < X^+$.

(34) If $\overline{\overline{X}} = \overline{\overline{Y}}$, then $X^+ = Y^+$.

(35) If $X \approx Y$, then $X^+ = Y^+$.

(36) $A \in A^+$.

In the sequel L, L_1 will be transfinite sequences. Let us consider M. The predicate M is a limit cardinal number is defined by:

for no N holds $M = N^+$.

One can prove the following proposition

(37) M is a limit cardinal number if and only if for no N holds $M = N^+$.

Let us consider A. The functor $\aleph_A$ yielding any, is defined by:

there exists L such that $\aleph_A = \mathrm{last}\, L$ and $\mathrm{dom}\, L = \mathrm{succ}\, A$ and $L(\mathbf{0}) = \overline{\overline{\mathbb{N}}}$ and for all B, y such that $\mathrm{succ}\, B \in \mathrm{succ}\, A$ and $y = L(B)$ holds $L(\mathrm{succ}\, B) = (\bigcup\{y\})^+$ and for all B, L_1 such that $B \in \mathrm{succ}\, A$ and $B \neq \mathbf{0}$ and B is a limit ordinal number and $L_1 = L \restriction B$ holds $L(B) = \overline{\overline{\sup L_1}}$.

Let us consider A. Then $\aleph_A$ is a cardinal number.

The following propositions are true:

(38) $\aleph_{\mathbf{0}} = \overline{\overline{\mathbb{N}}}$.

(39) $\aleph_{\mathrm{succ}\, A} = {\aleph_A}^+$.

(40) If $A \neq \mathbf{0}$ and A is a limit ordinal number, then for every L such that $\mathrm{dom}\, L = A$ and for every B such that $B \in A$ holds $L(B) = \aleph_B$ holds $\aleph_A = \overline{\overline{\sup L}}$.

(41) $A \in B$ if and only if $\aleph_A < \aleph_B$.

(42) If $\aleph_A = \aleph_B$, then $A = B$.

(43) $A \subseteq B$ if and only if $\aleph_A \leq \aleph_B$.

(44) If $X \subseteq Y$ and $Y \subseteq Z$ and $X \approx Z$, then $X \approx Y$ and $Y \approx Z$.

(45) If $2^Y \subseteq X$, then $\overline{\overline{Y}} < \overline{\overline{X}}$ and $Y \not\approx X$.

(46) $X \approx \emptyset$ if and only if $X = \emptyset$.

(47) $\overline{\overline{\emptyset}} = \mathbf{0}$.

(48) $X \approx \{x\}$ if and only if there exists x such that $X = \{x\}$.

(49) $\overline{\overline{X}} = \overline{\overline{\{x\}}}$ if and only if there exists x such that $X = \{x\}$.

(50) $\overline{\overline{\{x\}}} = \mathbf{1}$.

Let us consider n. The functor $\overline{n}$ yielding an ordinal number, is defined by:

there exists f such that $\overline{n} = f(n)$ and $\operatorname{dom} f = \mathbb{N}$ and $f(0) = \mathbf{0}$ and for every element n of $\mathbb{N}$ for every x such that $x = f(n)$ holds $f(n+1) = \operatorname{succ}(\bigcup\{x\})$.

We now state a number of propositions:

(51) $\overline{0} = \mathbf{0}$.

(52) $\overline{n+1} = \operatorname{succ}(\overline{n})$.

(53) $\overline{n} \in \omega$.

(54) If A is natural, then there exists n such that $\overline{n} = A$.

(55) If $\overline{n} = \overline{m}$, then $n = m$.

(56) $n \leq m$ if and only if $\overline{n} \subseteq \overline{m}$.

(57) $\mathbb{N} \approx \omega$.

(58) If $X \cap X_1 = \emptyset$ and $Y \cap Y_1 = \emptyset$ and $X \approx Y$ and $X_1 \approx Y_1$, then $X \cup X_1 \approx Y \cup Y_1$.

(59) If $x \in X$ and $y \in X$, then $X \setminus \{x\} \approx X \setminus \{y\}$.

(60) If $X \subseteq \operatorname{dom} f$ and f is one-to-one, then $X \approx f \circ X$.

(61) If $X \approx Y$ and $x \in X$ and $y \in Y$, then $X \setminus \{x\} \approx Y \setminus \{y\}$.

(62) If $\operatorname{Seg} n \approx \operatorname{Seg} m$, then $n = m$.

(63) $\operatorname{Seg} n \approx \overline{n}$.

(64) If $\overline{n} \approx \overline{m}$, then $n = m$.

(65) If $A \in \omega$, then A is a cardinal number.

(66) $\overline{n} = \overline{\overline{\overline{n}}}$.

Let us consider n. The functor $\overline{\overline{n}}$ yielding a cardinal number, is defined by:

$\overline{\overline{n}} = \overline{n}$.

One can prove the following propositions:

(67) $\overline{\overline{n}} = \overline{n}$.

(68) If $X \approx Y$ or $Y \approx X$ but X is finite, then Y is finite.

(69) $\overline{n}$ is finite and $\overline{\overline{n}}$ is finite.

(70) $\overline{\overline{\operatorname{Seg} n}} = \overline{\overline{n}}$.

(71) If $\overline{\overline{n}} = \overline{\overline{m}}$, then $n = m$.

(72) $\overline{\overline{n}} \leq \overline{\overline{m}}$ if and only if $n \leq m$.

(73) $\overline{\overline{n}} < \overline{\overline{m}}$ if and only if $n < m$.

(74) If X is finite, then there exists n such that $X \approx \overline{n}$.

(75) If X is finite, then there exists n such that $X \approx \operatorname{Seg} n$.

(76) $\overline{\overline{n}}^{+} = \overline{\overline{n+1}}$.

Let us consider X. Let us assume that X is finite. The functor $\operatorname{card} X$ yields a natural number and is defined by:

$\overline{\overline{\operatorname{card} X}} = \overline{\overline{X}}$.

We now state several propositions:

(77) If X is finite, then $\operatorname{card} X = n$ if and only if $\overline{\overline{n}} = \overline{\overline{X}}$.

(78) $\operatorname{card} \emptyset = 0$.

(79) $\operatorname{card}\{x\} = 1$.

(80) If Y is finite and $X \subseteq Y$, then $\operatorname{card} X \leq \operatorname{card} Y$.

(81) If X is finite or Y is finite but $X \approx Y$, then $\operatorname{card} X = \operatorname{card} Y$.

(82) If X is finite, then X^+ is finite.

In the article we present several logical schemes. The scheme *Cardinal_Ind* concerns a unary predicate $\mathcal{P}$ and states that:

for every M holds $\mathcal{P}[M]$

provided the parameter satisfies the following conditions:

- $\mathcal{P}[\mathbf{0}]$,
- for every M such that $\mathcal{P}[M]$ holds $\mathcal{P}[M^+]$,
- for every M such that $M \neq \mathbf{0}$ and M is a limit cardinal number and for every N such that $N < M$ holds $\mathcal{P}[N]$ holds $\mathcal{P}[M]$.

The scheme *Cardinal_CompInd* concerns a unary predicate $\mathcal{P}$ and states that:

for every M holds $\mathcal{P}[M]$

provided the parameter satisfies the following condition:

- for every M such that for every N such that $N < M$ holds $\mathcal{P}[N]$ holds $\mathcal{P}[M]$.

Next we state several propositions:

(83) $\aleph_{\mathbf{0}} = \omega$.

(84) $\overline{\overline{\omega}} = \omega$ and $\overline{\overline{\mathbb{N}}} = \omega$.

(85) $\overline{\overline{\omega}}$ is a limit cardinal number.

(86) If M is finite, then there exists n such that $M = \overline{\overline{n}}$.

(87) $\operatorname{card}(\operatorname{Seg} n) = n$ and $\operatorname{card}(\overline{n}) = n$ and $\operatorname{card} \overline{\overline{n}} = n$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[3] Grzegorz Bancerek. Sequences of ordinal numbers. *Formalized Mathematics*, 1(**2**):281–290, 1990.

[4] Grzegorz Bancerek. The well ordering relations. *Formalized Mathematics*, 1(**1**):123–129, 1990.

[5] Grzegorz Bancerek. Zermelo theorem and axiom of choice. *Formalized Mathematics*, 1(**2**):265–267, 1990.

[6] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[8] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[9] Wojciech Guzicki and Paweł Zbierski. *Podstawy teorii mnogości.* PWN, Warszawa, 1978.

[10] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[11] Kazimierz Kuratowski and Andrzej Mostowski. *Teoria mnogości.* PTM, Wrocław, 1952.

[12] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[13] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

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