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Kuratowski - Zorn Lemma [1]

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Summary. The goal of this article is to prove Kuratowski - Zorn lemma. We prove it in a number of forms (theorems and schemes). We introduce the following notions: a relation is a quasi (or partial, or linear) order, a relation quasi (or partially, or lineary) orders a set, minimal and maximal element in a relation, inferior and superior element of a relation, a set has lower (or upper) Zorn property w.r.t. a relation. We prove basic theorems concerning those notions and theorems that relate them to the notions introduced in [6]. At the end of the article we prove some theorems that belong rather to [7], [9] or [2].

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The notation and terminology used here are introduced in the following articles: [5], [3], [7], [9], [8], [2], [4], [6], and [1]. For simplicity we follow a convention: R, P are relations, X, X_1, X_2, Y, Z are sets, O is an order in X, D, D_1 are non-empty sets, x, y are arbitrary, A is a poset, C is a chain of A, S is a subset of A, and a, b are elements of A. In the article we present several logical schemes. The scheme *RelOnDomEx* deals with a constant $\mathcal{A}$ that is a non-empty set, a constant $\mathcal{B}$ that is a non-empty set and a binary predicate $\mathcal{P}$ and states that:

there exists R being a relation between $\mathcal{A}$ and $\mathcal{B}$ such that for every element a of $\mathcal{A}$ for every element b of $\mathcal{B}$ holds $\langle a, b\rangle \in R$ if and only if $\mathcal{P}[a, b]$

for all values of the parameters.

The scheme *RelOnDomEx1* deals with a constant $\mathcal{A}$ that is a non-empty set and a binary predicate $\mathcal{P}$ and states that:

there exists R being a relation on $\mathcal{A}$ such that for all elements a, b of $\mathcal{A}$ holds $\langle a, b\rangle \in R$ if and only if $\mathcal{P}[a, b]$

for all values of the parameters.

One can prove the following propositions:

(1) $\operatorname{dom} O = X$ and $\operatorname{rng} O = X$.

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(2) field $O = X$.

We now define three new predicates. Let us consider R. The predicate R is a quasi order is defined by:

R is pseudo reflexive and R is transitive.

The predicate R is a partial order is defined by:

R is pseudo reflexive and R is transitive and R is antisymmetric.

The predicate R is a linear order is defined by:

R is pseudo reflexive and R is transitive and R is antisymmetric and R is connected.

We now state a number of propositions:

(3) R is a quasi order if and only if R is pseudo reflexive and R is transitive.

(4) R is a partial order if and only if R is pseudo reflexive and R is transitive and R is antisymmetric.

(5) R is a linear order if and only if R is pseudo reflexive and R is transitive and R is antisymmetric and R is connected.

(6) If R is a quasi order, then $R^{\smile}$ is a quasi order.

(7) If R is a partial order, then $R^{\smile}$ is a partial order.

(8) If R is a linear order, then $R^{\smile}$ is a linear order.

(9) If R is well ordering relation, then R is a quasi order and R is a partial order and R is a linear order.

(10) If R is a linear order, then R is a quasi order and R is a partial order.

(11) If R is a partial order, then R is a quasi order.

(12) O is a partial order.

(13) O is a quasi order.

(14) If O is connected, then O is a linear order.

(15) If R is a quasi order, then $R\,|^2\,X$ is a quasi order.

(16) If R is a partial order, then $R\,|^2\,X$ is a partial order.

(17) If R is a linear order, then $R\,|^2\,X$ is a linear order.

(18) field((the order of A) $|^2\,S$) $= S$.

(19) If (the order of A) $|^2\,S$ is a linear order, then S is a chain of A.

(20) (the order of A) $|^2\,C$ is a linear order.

(21) $\varnothing$ is a quasi order and $\varnothing$ is a partial order and $\varnothing$ is a linear order and $\varnothing$ is well ordering relation.

(22) $\triangle_X$ is a quasi order and $\triangle_X$ is a partial order.

We now define three new predicates. Let us consider R, X. The predicate R quasi orders X is defined by:

R is reflexive in X and R is transitive in X.

The predicate R partially orders X is defined by:

R is reflexive in X and R is transitive in X and R is antisymmetric in X.

The predicate R linearly orders X is defined by:

R is reflexive in X and R is transitive in X and R is antisymmetric in X and R is connected in X.

The following propositions are true:

(23) R quasi orders X if and only if R is reflexive in X and R is transitive in X.

(24) R partially orders X if and only if R is reflexive in X and R is transitive in X and R is antisymmetric in X.

(25) R linearly orders X if and only if R is reflexive in X and R is transitive in X and R is antisymmetric in X and R is connected in X.

(26) If R well orders X, then R quasi orders X and R partially orders X and R linearly orders X.

(27) If R linearly orders X, then R quasi orders X and R partially orders X.

(28) If R partially orders X, then R quasi orders X.

(29) If R is a quasi order, then R quasi orders field R.

(30) If R quasi orders Y and $X \subseteq Y$, then R quasi orders X.

(31) If R quasi orders X, then $R \,|^2\, X$ is a quasi order.

(32) If R is a partial order, then R partially orders field R.

(33) If R partially orders Y and $X \subseteq Y$, then R partially orders X.

(34) If R partially orders X, then $R \,|^2\, X$ is a partial order.

(35) If R is a linear order, then R linearly orders field R.

(36) If R linearly orders Y and $X \subseteq Y$, then R linearly orders X.

(37) If R linearly orders X, then $R \,|^2\, X$ is a linear order.

(38) If R quasi orders X, then $R^{\smile}$ quasi orders X.

(39) If R partially orders X, then $R^{\smile}$ partially orders X.

(40) If R linearly orders X, then $R^{\smile}$ linearly orders X.

(41) O quasi orders X.

(42) O partially orders X.

(43) If R partially orders X, then $R \,|^2\, X$ is an order in X.

(44) If R linearly orders X, then $R \,|^2\, X$ is an order in X.

(45) If R well orders X, then $R \,|^2\, X$ is an order in X.

(46) If the order of A linearly orders S, then S is a chain of A.

(47) the order of A linearly orders C.

(48) $\triangle_X$ quasi orders X and $\triangle_X$ partially orders X.

We now define two new predicates. Let us consider R, X. The predicate X has the upper Zorn property w.r.t. R is defined by:

for every Y such that $Y \subseteq X$ and $R \,|^2\, Y$ is a linear order there exists x such that $x \in X$ and for every y such that $y \in Y$ holds $\langle y, x \rangle \in R$.

The predicate X has the lower Zorn property w.r.t. R is defined by:

for every Y such that $Y \subseteq X$ and $R \,|^2\, Y$ is a linear order there exists x such that $x \in X$ and for every y such that $y \in Y$ holds $\langle x, y \rangle \in R$.

We now state several propositions:

(49) X has the upper Zorn property w.r.t. R if and only if for every Y such that $Y \subseteq X$ and $R\,|^2\,Y$ is a linear order there exists x such that $x \in X$ and for every y such that $y \in Y$ holds $\langle y, x \rangle \in R$.

(50) X has the lower Zorn property w.r.t. R if and only if for every Y such that $Y \subseteq X$ and $R\,|^2\,Y$ is a linear order there exists x such that $x \in X$ and for every y such that $y \in Y$ holds $\langle x, y \rangle \in R$.

(51) If X has the upper Zorn property w.r.t. R, then $X \neq \emptyset$.

(52) If X has the lower Zorn property w.r.t. R, then $X \neq \emptyset$.

(53) X has the upper Zorn property w.r.t. R if and only if X has the lower Zorn property w.r.t. $R^{\smile}$.

(54) X has the upper Zorn property w.r.t. $R^{\smile}$ if and only if X has the lower Zorn property w.r.t. R.

We now define four new predicates. Let us consider R, x. The predicate x is maximal in R is defined by:

$x \in \mathrm{field}\,R$ and for no y holds $y \in \mathrm{field}\,R$ and $y \neq x$ and $\langle x, y \rangle \in R$.

The predicate x is minimal in R is defined by:

$x \in \mathrm{field}\,R$ and for no y holds $y \in \mathrm{field}\,R$ and $y \neq x$ and $\langle y, x \rangle \in R$.

The predicate x is superior of R is defined by:

$x \in \mathrm{field}\,R$ and for every y such that $y \in \mathrm{field}\,R$ and $y \neq x$ holds $\langle y, x \rangle \in R$.

The predicate x is inferior of R is defined by:

$x \in \mathrm{field}\,R$ and for every y such that $y \in \mathrm{field}\,R$ and $y \neq x$ holds $\langle x, y \rangle \in R$.

Next we state a number of propositions:

(55) x is maximal in R if and only if $x \in \mathrm{field}\,R$ and for no y holds $y \in \mathrm{field}\,R$ and $y \neq x$ and $\langle x, y \rangle \in R$.

(56) x is minimal in R if and only if $x \in \mathrm{field}\,R$ and for no y holds $y \in \mathrm{field}\,R$ and $y \neq x$ and $\langle y, x \rangle \in R$.

(57) x is superior of R if and only if $x \in \mathrm{field}\,R$ and for every y such that $y \in \mathrm{field}\,R$ and $y \neq x$ holds $\langle y, x \rangle \in R$.

(58) x is inferior of R if and only if $x \in \mathrm{field}\,R$ and for every y such that $y \in \mathrm{field}\,R$ and $y \neq x$ holds $\langle x, y \rangle \in R$.

(59) If x is inferior of R and R is antisymmetric, then x is minimal in R.

(60) If x is superior of R and R is antisymmetric, then x is maximal in R.

(61) If x is minimal in R and R is connected, then x is inferior of R.

(62) If x is maximal in R and R is connected, then x is superior of R.

(63) If $x \in X$ and x is superior of R and $X \subseteq \mathrm{field}\,R$ and R is pseudo reflexive, then X has the upper Zorn property w.r.t. R.

(64) If $x \in X$ and x is inferior of R and $X \subseteq \mathrm{field}\,R$ and R is pseudo reflexive, then X has the lower Zorn property w.r.t. R.

(65) x is minimal in R if and only if x is maximal in $R^{\smile}$.

(66) x is minimal in $R^{\smile}$ if and only if x is maximal in R.

(67) x is inferior of R if and only if x is superior of $R^{\smile}$.

(68) x is inferior of $R^{\smile}$ if and only if x is superior of R.

(69) a is minimal in the order of A if and only if for every b holds $b \not< a$.

(70) a is maximal in the order of A if and only if for every b holds $a \not< b$.

(71) a is superior of the order of A if and only if for every b such that $a \neq b$ holds $b < a$.

(72) a is inferior of the order of A if and only if for every b such that $a \neq b$ holds $a < b$.

(73) If for every C there exists a such that for every b such that $b \in C$ holds $b \leq a$, then there exists a such that for every b holds $a \not< b$.

(74) If for every C there exists a such that for every b such that $b \in C$ holds $a \leq b$, then there exists a such that for every b holds $b \not< a$.

We now state several propositions:

(75) For all R, X such that R partially orders X and field $R = X$ and X has the upper Zorn property w.r.t. R there exists x such that x is maximal in R.

(76) For all R, X such that R partially orders X and field $R = X$ and X has the lower Zorn property w.r.t. R there exists x such that x is minimal in R.

(77) Given X. Suppose $X \neq \emptyset$ and for every Z such that $Z \subseteq X$ and for all X_1, X_2 such that $X_1 \in Z$ and $X_2 \in Z$ holds $X_1 \subseteq X_2$ or $X_2 \subseteq X_1$ there exists Y such that $Y \in X$ and for every X_1 such that $X_1 \in Z$ holds $X_1 \subseteq Y$. Then there exists Y such that $Y \in X$ and for every Z such that $Z \in X$ and $Z \neq Y$ holds $Y \not\subseteq Z$.

(78) Given X. Suppose $X \neq \emptyset$ and for every Z such that $Z \subseteq X$ and for all X_1, X_2 such that $X_1 \in Z$ and $X_2 \in Z$ holds $X_1 \subseteq X_2$ or $X_2 \subseteq X_1$ there exists Y such that $Y \in X$ and for every X_1 such that $X_1 \in Z$ holds $Y \subseteq X_1$. Then there exists Y such that $Y \in X$ and for every Z such that $Z \in X$ and $Z \neq Y$ holds $Z \not\subseteq Y$.

(79) Given X. Suppose $X \neq \emptyset$ and for every Z such that $Z \neq \emptyset$ and $Z \subseteq X$ and for all X_1, X_2 such that $X_1 \in Z$ and $X_2 \in Z$ holds $X_1 \subseteq X_2$ or $X_2 \subseteq X_1$ holds $\bigcup Z \in X$. Then there exists Y such that $Y \in X$ and for every Z such that $Z \in X$ and $Z \neq Y$ holds $Y \not\subseteq Z$.

(80) Given X. Suppose $X \neq \emptyset$ and for every Z such that $Z \neq \emptyset$ and $Z \subseteq X$ and for all X_1, X_2 such that $X_1 \in Z$ and $X_2 \in Z$ holds $X_1 \subseteq X_2$ or $X_2 \subseteq X_1$ holds $\bigcap Z \in X$. Then there exists Y such that $Y \in X$ and for every Z such that $Z \in X$ and $Z \neq Y$ holds $Z \not\subseteq Y$.

Now we present two schemes. The scheme *Zorn_Max* concerns a constant $\mathcal{A}$ that is a non-empty set and a binary predicate $\mathcal{P}$ and states that:

there exists x being an element of $\mathcal{A}$ such that for every element y of $\mathcal{A}$ such that $x \neq y$ holds not $\mathcal{P}[x, y]$

provided the parameters satisfy the following conditions:

- for every element x of $\mathcal{A}$ holds $\mathcal{P}[x, x]$,
- for all elements x, y of $\mathcal{A}$ such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, x]$ holds $x = y$,
- for all elements x, y, z of $\mathcal{A}$ such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, z]$ holds $\mathcal{P}[x, z]$,

- for every X such that $X \subseteq \mathcal{A}$ and for all elements x, y of $\mathcal{A}$ such that $x \in X$ and $y \in X$ holds $\mathcal{P}[x, y]$ or $\mathcal{P}[y, x]$ there exists y being an element of $\mathcal{A}$ such that for every element x of $\mathcal{A}$ such that $x \in X$ holds $\mathcal{P}[x, y]$.

The scheme *Zorn_Min* deals with a constant $\mathcal{A}$ that is a non-empty set and a binary predicate $\mathcal{P}$ and states that:

there exists x being an element of $\mathcal{A}$ such that for every element y of $\mathcal{A}$ such that $x \neq y$ holds not $\mathcal{P}[y, x]$

provided the parameters satisfy the following conditions:

- for every element x of $\mathcal{A}$ holds $\mathcal{P}[x, x]$,
- for all elements x, y of $\mathcal{A}$ such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, x]$ holds $x = y$,
- for all elements x, y, z of $\mathcal{A}$ such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, z]$ holds $\mathcal{P}[x, z]$,
- for every X such that $X \subseteq \mathcal{A}$ and for all elements x, y of $\mathcal{A}$ such that $x \in X$ and $y \in X$ holds $\mathcal{P}[x, y]$ or $\mathcal{P}[y, x]$ there exists y being an element of $\mathcal{A}$ such that for every element x of $\mathcal{A}$ such that $x \in X$ holds $\mathcal{P}[y, x]$.

One can prove the following propositions:

(81) If R partially orders X and field $R = X$, then there exists P such that $R \subseteq P$ and P linearly orders X and field $P = X$.

(82) $R \subseteq [\!: \mathrm{field}\, R, \mathrm{field}\, R :\!]$.

(83) If R is pseudo reflexive and $X \subseteq \mathrm{field}\, R$, then $\mathrm{field}(R \mid^2 X) = X$.

(84) If R is reflexive in X, then $R \mid^2 X$ is pseudo reflexive.

(85) If R is transitive in X, then $R \mid^2 X$ is transitive.

(86) If R is antisymmetric in X, then $R \mid^2 X$ is antisymmetric.

(87) If R is connected in X, then $R \mid^2 X$ is connected.

(88) If R is connected in X and $Y \subseteq X$, then R is connected in Y.

(89) If R well orders X and $Y \subseteq X$, then R well orders Y.

(90) If R is connected, then $R^{\smile}$ is connected.

(91) If R is reflexive in X, then $R^{\smile}$ is reflexive in X.

(92) If R is transitive in X, then $R^{\smile}$ is transitive in X.

(93) If R is antisymmetric in X, then $R^{\smile}$ is antisymmetric in X.

(94) If R is connected in X, then $R^{\smile}$ is connected in X.

(95) $(R \mid^2 X)^{\smile} = R^{\smile} \mid^2 X$.

(96) $R \mid^2 \emptyset = \varnothing$.

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