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Parallelity Spaces [1]

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Summary. In the monography [5] W. Szmielew introduced the parallelity planes $\langle S; \parallel\rangle$, where $\parallel \subseteq S \times S \times S \times S$. In this text we omit upper bound axiom which must be satisfied by the parallelity planes (see also E.Kusak [3]). Further we will list those theorems which remain true when we pass from the parallelity planes to the parallelity spaces. We construct a model of the parallelity space in Abelian group $\langle F \times F \times F; +_F, -_F, \mathbf{0}_F\rangle$, where F is a field.

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The papers [7], [6], [2], [1], and [4] provide the terminology and notation for this paper. We follow the rules: F will denote a field, a, b, c, f, g, h will denote elements of the carrier of F, and x, y will denote elements of [: the carrier of F, the carrier of F, the carrier of F :]. Let us consider F. The functor $+_F$ yields a binary operation on [: the carrier of F, the carrier of F, the carrier of F :] and is defined by:

$(+_F)(x, y) = \langle x_{\mathbf{1}} + y_{\mathbf{1}}, x_{\mathbf{2}} + y_{\mathbf{2}}, x_{\mathbf{3}} + y_{\mathbf{3}}\rangle$.

The following proposition is true

(1) $(+_F)(x, y) = \langle x_{\mathbf{1}} + y_{\mathbf{1}}, x_{\mathbf{2}} + y_{\mathbf{2}}, x_{\mathbf{3}} + y_{\mathbf{3}}\rangle$.

Let us consider F, x, y. The functor $x + y$ yielding an element of [: the carrier of F, the carrier of F, the carrier of F :], is defined by:

$x + y = (+_F)(x, y)$.

One can prove the following three propositions:

(2) $x + y = (+_F)(x, y)$.

(3) $x + y = \langle x_{\mathbf{1}} + y_{\mathbf{1}}, x_{\mathbf{2}} + y_{\mathbf{2}}, x_{\mathbf{3}} + y_{\mathbf{3}}\rangle$.

(4) $\langle a, b, c\rangle + \langle f, g, h\rangle = \langle a + f, b + g, c + h\rangle$.

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Let us consider F. The functor $-_F$ yielding a unary operation on $[\!:$ the carrier of F, the carrier of F, the carrier of $F:\!]$, is defined by:

$(-_F)(x) = \langle -x_{\mathbf{1}}, -x_{\mathbf{2}}, -x_{\mathbf{3}} \rangle$.

The following proposition is true

(5) $(-_F)(x) = \langle -x_{\mathbf{1}}, -x_{\mathbf{2}}, -x_{\mathbf{3}} \rangle$.

Let us consider F, x. The functor $-x$ yields an element of $[\!:$ the carrier of F, the carrier of F, the carrier of $F:\!]$ and is defined by:

$-x = (-_F)(x)$.

We now state two propositions:

(6) $(-_F)(x) = -x$.

(7) $-x = \langle -x_{\mathbf{1}}, -x_{\mathbf{2}}, -x_{\mathbf{3}} \rangle$.

In the sequel S denotes a set. Let us consider S. The mode 4-ary relation over the S, which widens to the type a set, is defined by:

it $\subseteq [\!: S, S, S, S :\!]$.

We now state a proposition

(8) For every set R holds $R \subseteq [\!: S, S, S, S :\!]$ if and only if R is a 4-ary relation over the S.

We consider parallelity structures which are systems

$\langle$ a universum, a parallelity $\rangle$

where the universum is a non-empty set and the parallelity is a 4-ary relation over the the universum. In the sequel F is a field and PS is a parallelity structure. The arguments of the notions defined below are the following: PS which is an object of the type reserved above; a, b, c, d which are elements of the universum of PS. The predicate $a, b \parallel c, d$ is defined by:

$\langle a, b, c, d \rangle \in$the parallelity of PS.

Next we state a proposition

(9) For all elements a, b, c, d of the universum of PS holds $a, b \parallel c, d$ if and only if $\langle a, b, c, d \rangle \in$the parallelity of PS.

Let us consider F. The functor $F^{\mathbf{3}}$ yields a non-empty set and is defined by:

$F^{\mathbf{3}} = [\!:$ the carrier of F, the carrier of F, the carrier of $F :\!]$.

Next we state a proposition

(10) $F^{\mathbf{3}} = [\!:$ the carrier of F, the carrier of F, the carrier of $F :\!]$.

Let us consider F. The functor $(F^{\mathbf{3}})^{\mathbf{4}}$ yields a non-empty set and is defined by:

$(F^{\mathbf{3}})^{\mathbf{4}} = [\!: F^{\mathbf{3}}, F^{\mathbf{3}}, F^{\mathbf{3}}, F^{\mathbf{3}} :\!]$.

One can prove the following proposition

(11) $(F^{\mathbf{3}})^{\mathbf{4}} = [\!: F^{\mathbf{3}}, F^{\mathbf{3}}, F^{\mathbf{3}}, F^{\mathbf{3}} :\!]$.

We adopt the following convention: x will be arbitrary and a, b, c, d, e, f, g, h will denote elements of $[\!:$ the carrier of F, the carrier of F, the carrier of $F :\!]$. Let us consider F. The functor $\mathbf{Par'}_F$ yielding a set, is defined by:

$x \in \mathbf{Par'}_F$ if and only if the following conditions are satisfied:

(i) $x \in (F^{\mathbf{3}})^{\mathbf{4}}$,

(ii) there exist a, b, c, d such that $x = \langle a, b, c, d \rangle$ and $(a_{\mathbf{1}} - b_{\mathbf{1}}) \cdot (c_{\mathbf{2}} - d_{\mathbf{2}}) - (c_{\mathbf{1}} - d_{\mathbf{1}}) \cdot (a_{\mathbf{2}} - b_{\mathbf{2}}) = 0_F$ and $(a_{\mathbf{1}} - b_{\mathbf{1}}) \cdot (c_{\mathbf{3}} - d_{\mathbf{3}}) - (c_{\mathbf{1}} - d_{\mathbf{1}}) \cdot (a_{\mathbf{3}} - b_{\mathbf{3}}) = 0_F$ and $(a_{\mathbf{2}} - b_{\mathbf{2}}) \cdot (c_{\mathbf{3}} - d_{\mathbf{3}}) - (c_{\mathbf{2}} - d_{\mathbf{2}}) \cdot (a_{\mathbf{3}} - b_{\mathbf{3}}) = 0_F$.

Next we state two propositions:

(12) (i) For every x holds $x \in \mathbf{Par'}_F$ if and only if $x \in (F^{\mathbf{3}})^{\mathbf{4}}$ and there exist a, b, c, d such that $x = \langle a, b, c, d \rangle$ and $(a_{\mathbf{1}} - b_{\mathbf{1}}) \cdot (c_{\mathbf{2}} - d_{\mathbf{2}}) - (c_{\mathbf{1}} - d_{\mathbf{1}}) \cdot (a_{\mathbf{2}} - b_{\mathbf{2}}) = 0_F$ and $(a_{\mathbf{1}} - b_{\mathbf{1}}) \cdot (c_{\mathbf{3}} - d_{\mathbf{3}}) - (c_{\mathbf{1}} - d_{\mathbf{1}}) \cdot (a_{\mathbf{3}} - b_{\mathbf{3}}) = 0_F$ and $(a_{\mathbf{2}} - b_{\mathbf{2}}) \cdot (c_{\mathbf{3}} - d_{\mathbf{3}}) - (c_{\mathbf{2}} - d_{\mathbf{2}}) \cdot (a_{\mathbf{3}} - b_{\mathbf{3}}) = 0_F$,

(ii) $\mathbf{Par'}_F$ is a set.

(13) $\mathbf{Par'}_F \subseteq [\!: F^{\mathbf{3}}, F^{\mathbf{3}}, F^{\mathbf{3}}, F^{\mathbf{3}} :\!]$.

Let us consider F. The functor $\mathbf{Par}_F$ yielding a 4-ary relation over the $F^{\mathbf{3}}$, is defined by:

$\mathbf{Par}_F = \mathbf{Par'}_F$.

We now state a proposition

(14) $\mathbf{Par}_F = \mathbf{Par'}_F$ and $\mathbf{Par}_F$ is a 4-ary relation over the $F^{\mathbf{3}}$.

Let us consider F. The functor Aff_{F^3} yields a parallelity structure and is defined by:

$\mathrm{Aff}_{F^3} = \langle F^{\mathbf{3}}, \mathbf{Par}_F \rangle$.

We now state three propositions:

(15) $\mathrm{Aff}_{F^3} = \langle F^{\mathbf{3}}, \mathbf{Par}_F \rangle$.

(16) the universum of $\mathrm{Aff}_{F^3} = F^{\mathbf{3}}$.

(17) the parallelity of $\mathrm{Aff}_{F^3} = \mathbf{Par}_F$.

In the sequel a, b, c, d, p, q, r, s denote elements of the universum of Aff_{F^3}. One can prove the following propositions:

(18) $a, b \parallel c, d$ if and only if $\langle a, b, c, d \rangle \in \mathbf{Par}_F$.

(19) $\langle a, b, c, d \rangle \in \mathbf{Par}_F$ if and only if the following conditions are satisfied:

(i) $\langle a, b, c, d \rangle \in (F^{\mathbf{3}})^{\mathbf{4}}$,

(ii) there exist e, f, g, h such that $\langle a, b, c, d \rangle = \langle e, f, g, h \rangle$ and $(e_{\mathbf{1}} - f_{\mathbf{1}}) \cdot (g_{\mathbf{2}} - h_{\mathbf{2}}) - (g_{\mathbf{1}} - h_{\mathbf{1}}) \cdot (e_{\mathbf{2}} - f_{\mathbf{2}}) = 0_F$ and $(e_{\mathbf{1}} - f_{\mathbf{1}}) \cdot (g_{\mathbf{3}} - h_{\mathbf{3}}) - (g_{\mathbf{1}} - h_{\mathbf{1}}) \cdot (e_{\mathbf{3}} - f_{\mathbf{3}}) = 0_F$ and $(e_{\mathbf{2}} - f_{\mathbf{2}}) \cdot (g_{\mathbf{3}} - h_{\mathbf{3}}) - (g_{\mathbf{2}} - h_{\mathbf{2}}) \cdot (e_{\mathbf{3}} - f_{\mathbf{3}}) = 0_F$.

(20) $a, b \parallel c, d$ if and only if the following conditions are satisfied:

(i) $\langle a, b, c, d \rangle \in (F^{\mathbf{3}})^{\mathbf{4}}$,

(ii) there exist e, f, g, h such that $\langle a, b, c, d \rangle = \langle e, f, g, h \rangle$ and $(e_{\mathbf{1}} - f_{\mathbf{1}}) \cdot (g_{\mathbf{2}} - h_{\mathbf{2}}) - (g_{\mathbf{1}} - h_{\mathbf{1}}) \cdot (e_{\mathbf{2}} - f_{\mathbf{2}}) = 0_F$ and $(e_{\mathbf{1}} - f_{\mathbf{1}}) \cdot (g_{\mathbf{3}} - h_{\mathbf{3}}) - (g_{\mathbf{1}} - h_{\mathbf{1}}) \cdot (e_{\mathbf{3}} - f_{\mathbf{3}}) = 0_F$ and $(e_{\mathbf{2}} - f_{\mathbf{2}}) \cdot (g_{\mathbf{3}} - h_{\mathbf{3}}) - (g_{\mathbf{2}} - h_{\mathbf{2}}) \cdot (e_{\mathbf{3}} - f_{\mathbf{3}}) = 0_F$.

(21) the universum of $\mathrm{Aff}_{F^3} = [\!:$ the carrier of F, the carrier of F, the carrier of $F :\!]$.

(22) $\langle a, b, c, d \rangle \in (F^{\mathbf{3}})^{\mathbf{4}}$.

(23) $a, b \parallel c, d$ if and only if there exist e, f, g, h such that $\langle a, b, c, d \rangle = \langle e, f, g, h \rangle$ and $(e_{\mathbf{1}} - f_{\mathbf{1}}) \cdot (g_{\mathbf{2}} - h_{\mathbf{2}}) - (g_{\mathbf{1}} - h_{\mathbf{1}}) \cdot (e_{\mathbf{2}} - f_{\mathbf{2}}) = 0_F$ and $(e_{\mathbf{1}} - f_{\mathbf{1}}) \cdot (g_{\mathbf{3}} - h_{\mathbf{3}}) - (g_{\mathbf{1}} - h_{\mathbf{1}}) \cdot (e_{\mathbf{3}} - f_{\mathbf{3}}) = 0_F$ and $(e_{\mathbf{2}} - f_{\mathbf{2}}) \cdot (g_{\mathbf{3}} - h_{\mathbf{3}}) - (g_{\mathbf{2}} - h_{\mathbf{2}}) \cdot (e_{\mathbf{3}} - f_{\mathbf{3}}) = 0_F$.

(24) $a, b \parallel b, a$.

(25) $a, b \parallel c, c$.

(26) If $a, b \parallel p, q$ and $a, b \parallel r, s$, then $p, q \parallel r, s$ or $a = b$.

(27) If $a, b \parallel a, c$, then $b, a \parallel b, c$.

(28) There exists d such that $a, b \parallel c, d$ and $a, c \parallel b, d$.

The mode parallelity space, which widens to the type a parallelity structure, is defined by:

Let a, b, c, d, p, q, r, s be elements of the universum of it . Then

(i) $a, b \parallel b, a$,

(ii) $a, b \parallel c, c$,

(iii) if $a, b \parallel p, q$ and $a, b \parallel r, s$, then $p, q \parallel r, s$ or $a = b$,

(iv) if $a, b \parallel a, c$, then $b, a \parallel b, c$,

(v) there exists x being an element of the universum of it such that $a, b \parallel c, x$ and $a, c \parallel b, x$.

We now state a proposition

(29) Let P be a parallelity structure. Then the following conditions are equivalent:

(i) for all elements a, b, c, d, p, q, r, s of the universum of P holds $a, b \parallel b, a$ and $a, b \parallel c, c$ but if $a, b \parallel p, q$ and $a, b \parallel r, s$, then $p, q \parallel r, s$ or $a = b$ but if $a, b \parallel a, c$, then $b, a \parallel b, c$ and there exists x being an element of the universum of P such that $a, b \parallel c, x$ and $a, c \parallel b, x$,

(ii) P is a parallelity space.

We follow the rules: PS denotes a parallelity space and a, b, c, d, p, q, r, s denote elements of the universum of PS. One can prove the following propositions:

(30) $a, b \parallel b, a$.

(31) $a, b \parallel c, c$.

(32) If $a, b \parallel p, q$ and $a, b \parallel r, s$, then $p, q \parallel r, s$ or $a = b$.

(33) If $a, b \parallel a, c$, then $b, a \parallel b, c$.

(34) There exists d such that $a, b \parallel c, d$ and $a, c \parallel b, d$.

(35) $a, b \parallel a, b$.

(36) If $a, b \parallel c, d$, then $c, d \parallel a, b$.

(37) $a, a \parallel b, c$.

(38) If $a, b \parallel c, d$, then $b, a \parallel c, d$.

(39) If $a, b \parallel c, d$, then $a, b \parallel d, c$.

(40) If $a, b \parallel c, d$, then $b, a \parallel c, d$ and $a, b \parallel d, c$ and $b, a \parallel d, c$ and $c, d \parallel a, b$ and $d, c \parallel a, b$ and $c, d \parallel b, a$ and $d, c \parallel b, a$.

(41) Suppose $a, b \parallel a, c$. Then $a, c \parallel a, b$ and $b, a \parallel a, c$ and $a, b \parallel c, a$ and $a, c \parallel b, a$ and $b, a \parallel c, a$ and $c, a \parallel a, b$ and $c, a \parallel b, a$ and $b, a \parallel b, c$ and $a, b \parallel b, c$ and $b, a \parallel c, b$ and $b, c \parallel b, a$ and $a, b \parallel c, b$ and $c, b \parallel b, a$ and $b, c \parallel a, b$ and $c, b \parallel a, b$ and $c, a \parallel c, b$ and $a, c \parallel c, b$ and $c, a \parallel b, c$ and $a, c \parallel b, c$ and $c, b \parallel c, a$ and $b, c \parallel c, a$ and $c, b \parallel a, c$ and $b, c \parallel a, c$.

(42) If $a = b$ or $c = d$ or $a = c$ and $b = d$ or $a = d$ and $b = c$, then $a, b \parallel c, d$.

(43) If $a \neq b$ and $p, q \parallel a, b$ and $a, b \parallel r, s$, then $p, q \parallel r, s$.

(44) If $a, b \nparallel a, c$, then $a \neq b$ and $b \neq c$ and $c \neq a$.

(45) If $a, b \nparallel c, d$, then $a \neq b$ and $c \neq d$.

(46) Suppose $a, b \nparallel c, d$. Then $b, a \nparallel c, d$ and $a, b \nparallel d, c$ and $b, a \nparallel d, c$ and $c, d \nparallel a, b$ and $d, c \nparallel a, b$ and $c, d \nparallel b, a$ and $d, c \nparallel b, a$.

(47) Suppose $a, b \nparallel a, c$. Then $a, c \nparallel a, b$ and $b, a \nparallel a, c$ and $a, b \nparallel c, a$ and $a, c \nparallel b, a$ and $b, a \nparallel c, a$ and $c, a \nparallel a, b$ and $c, a \nparallel b, a$ and $b, a \nparallel b, c$ and $a, b \nparallel b, c$ and $b, a \nparallel c, b$ and $b, c \nparallel b, a$ and $b, a \nparallel c, b$ and $c, b \nparallel b, a$ and $b, c \nparallel a, b$ and $c, b \nparallel a, b$ and $c, a \nparallel c, b$ and $a, c \nparallel c, b$ and $c, a \nparallel b, c$ and $a, c \nparallel b, c$ and $c, b \nparallel c, a$ and $b, c \nparallel c, a$ and $c, b \nparallel a, c$ and $b, c \nparallel a, c$.

(48) If $a, b \nparallel c, d$ and $a, b \parallel p, q$ and $c, d \parallel r, s$ and $p \neq q$ and $r \neq s$, then $p, q \nparallel r, s$.

(49) If $a, b \nparallel a, c$ and $a, b \parallel p, q$ and $a, c \parallel p, r$ and $b, c \parallel q, r$ and $p \neq q$, then $p, q \nparallel p, r$.

(50) If $a, b \nparallel a, c$ and $a, c \parallel p, r$ and $b, c \parallel p, r$, then $p = r$.

(51) If $p, q \nparallel p, r$ and $p, r \parallel p, s$ and $q, r \parallel q, s$, then $r = s$.

(52) If $a, b \nparallel a, c$ and $a, b \parallel p, q$ and $a, c \parallel p, r$ and $a, c \parallel p, s$ and $b, c \parallel q, r$ and $b, c \parallel q, s$, then $r = s$.

(53) If $a, b \parallel a, c$ and $a, b \parallel a, d$, then $a, b \parallel c, d$.

(54) If for all a, b holds $a = b$, then for all p, q, r, s holds $p, q \parallel r, s$.

(55) If there exist a, b such that $a \neq b$ and for every c holds $a, b \parallel a, c$, then for all p, q, r, s holds $p, q \parallel r, s$.

(56) If $a, b \nparallel a, c$ and $p \neq q$, then $p, q \nparallel p, a$ or $p, q \nparallel p, b$ or $p, q \nparallel p, c$.

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