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Partial Functions

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Summary. In the article we define partial functions. We also define the following notions related to partial functions and functions themselves: the empty function, the restriction of a function to a partial function from a set into a set, the set of all partial functions from a set into a set, the total functions, the relation of tolerance of two functions and the set of all total functions which are tolerated by a partial function. Some simple propositions related to the introduced notions are proved. In the beginning of this article we prove some auxiliary theorems and schemas related to the articles: [1] and [2].

MML Identifier: PARTFUN1.

The terminology and notation used in this paper are introduced in the following articles: [4], [1], [2], and [3]. We adopt the following convention: x, y, y_1, y_2, z, z_1, z_2 will be arbitrary, P, Q, X, X', X_1, X_2, Y, Y', Y_1, Y_2, V, Z will denote sets, and C, D will denote non-empty sets. One can prove the following propositions:

(1) If $P \subseteq [: X_1, Y_1 :]$ and $Q \subseteq [: X_2, Y_2 :]$, then $P \cup Q \subseteq [: X_1 \cup X_2, Y_1 \cup Y_2 :]$.

(2) For all functions f, g such that for every x such that $x \in \operatorname{dom} f \cap \operatorname{dom} g$ holds $f(x) = g(x)$ there exists h being a function such that $\operatorname{graph} f \cup \operatorname{graph} g = \operatorname{graph} h$.

(3) For all functions f, g, h such that $\operatorname{graph} f \cup \operatorname{graph} g = \operatorname{graph} h$ for every x such that $x \in \operatorname{dom} f \cap \operatorname{dom} g$ holds $f(x) = g(x)$.

(4) For arbitrary f such that $f \in Y^X$ holds f is a function from X into Y.

In the article we present several logical schemes. The scheme *LambdaC* deals with a constant $\mathcal{A}$ that is a set, a unary predicate $\mathcal{P}$, a unary functor $\mathcal{F}$ and a unary functor $\mathcal{G}$ and states that:

there exists f being a function such that $\operatorname{dom} f = \mathcal{A}$ and for every x such that $x \in \mathcal{A}$ holds if $\mathcal{P}[x]$, then $f(x) = \mathcal{F}(x)$ but if not $\mathcal{P}[x]$, then $f(x) = \mathcal{G}(x)$
for all values of the parameters.

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The scheme *Lambda1C* deals with a constant $\mathcal{A}$ that is a set, a constant $\mathcal{B}$ that is a set, a unary predicate $\mathcal{P}$, a unary functor $\mathcal{F}$ and a unary functor $\mathcal{G}$ and states that:

there exists f being a function from $\mathcal{A}$ into $\mathcal{B}$ such that for every x such that $x \in \mathcal{A}$ holds if $\mathcal{P}[x]$, then $f(x) = \mathcal{F}(x)$ but if not $\mathcal{P}[x]$, then $f(x) = \mathcal{G}(x)$
provided the parameters satisfy the following condition:

- for every x such that $x \in \mathcal{A}$ holds if $\mathcal{P}[x]$, then $\mathcal{F}(x) \in \mathcal{B}$ but if not $\mathcal{P}[x]$, then $\mathcal{G}(x) \in \mathcal{B}$.

The constant $\Box$ is a function and is defined by:
graph $\Box = \emptyset$.

Next we state a number of propositions:

(5) For every function f such that graph $f = \emptyset$ holds $\Box = f$.

(6) graph $\Box = \emptyset$.

(7) $\Box = \emptyset$.

(8) For every function f such that dom $f = \emptyset$ or rng $f = \emptyset$ holds $\Box = f$.

(9) dom $\Box = \emptyset$.

(10) rng $\Box = \emptyset$.

(11) For every function f holds $f \cdot \Box = \Box$ and $\Box \cdot f = \Box$.

(12) $\mathrm{id}_{\emptyset} = \Box$.

(13) $\Box$ is one-to-one.

(14) $\Box^{-1} = \Box$.

(15) For every function f holds $f \restriction \emptyset = \Box$.

(16) $\Box \restriction X = \Box$.

(17) For every function f holds $\emptyset \restriction f = \Box$.

(18) $Y \restriction \Box = \Box$.

(19) $\Box \circ X = \emptyset$.

(20) $\Box^{-1} Y = \emptyset$.

(21) $\Box$ is a function from $\emptyset$ into Y.

(22) For every function f from $\emptyset$ into Y holds $f = \Box$.

Let us consider X, Y. The mode partial function from X to Y, which widens to the type a function, is defined by:
dom it $\subseteq X$ and rng it $\subseteq Y$.

Next we state a number of propositions:

(23) For every function f holds f is a partial function from X to Y if and only if dom $f \subseteq X$ and rng $f \subseteq Y$.

(24) For every function f holds f is a partial function from dom f to rng f.

(25) For every function f such that rng $f \subseteq Y$ holds f is a partial function from dom f to Y.

(26) For every partial function f from C to D such that $y \in$ rng f there exists x being an element of C such that $x \in$ dom f and $y = f(x)$.

(27) For every partial function f from X to Y such that $x \in \operatorname{dom} f$ holds $f(x) \in Y$.

(28) For every partial function f from X to Y such that $\operatorname{dom} f \subseteq Z$ holds f is a partial function from Z to Y.

(29) For every partial function f from X to Y such that $\operatorname{rng} f \subseteq Z$ holds f is a partial function from X to Z.

(30) For every partial function f from X to Y such that $X \subseteq Z$ holds f is a partial function from Z to Y.

(31) For every partial function f from X to Y such that $Y \subseteq Z$ holds f is a partial function from X to Z.

(32) For every partial function f from X_1 to Y_1 such that $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$ holds f is a partial function from X_2 to Y_2.

(33) For every function f for every partial function g from X to Y such that $\operatorname{graph} f \subseteq \operatorname{graph} g$ holds f is a partial function from X to Y.

(34) For all partial functions f_1, f_2 from C to D such that $X = \operatorname{dom} f_1$ and $X = \operatorname{dom} f_2$ and for every element x of C such that $x \in X$ holds $f_1(x) = f_2(x)$ holds $f_1 = f_2$.

(35) For all partial functions f_1, f_2 from $[\!: X, Y :\!]$ to Z such that $V = \operatorname{dom} f_1$ and $V = \operatorname{dom} f_2$ and for all x, y such that $\langle x, y \rangle \in V$ holds $f_1(\langle x, y \rangle) = f_2(\langle x, y \rangle)$ holds $f_1 = f_2$.

Now we present four schemes. The scheme *PartFuncEx* concerns a constant $\mathcal{A}$ that is a set, a constant $\mathcal{B}$ that is a set and a binary predicate $\mathcal{P}$ and states that:

there exists f being a partial function from $\mathcal{A}$ to $\mathcal{B}$ such that for every x holds $x \in \operatorname{dom} f$ if and only if $x \in \mathcal{A}$ and there exists y such that $\mathcal{P}[x, y]$ and for every x such that $x \in \operatorname{dom} f$ holds $\mathcal{P}[x, f(x)]$

provided the parameters satisfy the following conditions:

- for all x, y such that $x \in \mathcal{A}$ and $\mathcal{P}[x, y]$ holds $y \in \mathcal{B}$,
- for all x, y_1, y_2 such that $x \in \mathcal{A}$ and $\mathcal{P}[x, y_1]$ and $\mathcal{P}[x, y_2]$ holds $y_1 = y_2$.

The scheme *LambdaR* concerns a constant $\mathcal{A}$ that is a set, a constant $\mathcal{B}$ that is a set, a unary functor $\mathcal{F}$ and a unary predicate $\mathcal{P}$ and states that:

there exists f being a partial function from $\mathcal{A}$ to $\mathcal{B}$ such that for every x holds $x \in \operatorname{dom} f$ if and only if $x \in \mathcal{A}$ and $\mathcal{P}[x]$ and for every x such that $x \in \operatorname{dom} f$ holds $f(x) = \mathcal{F}(x)$

provided the parameters satisfy the following condition:

- for every x such that $\mathcal{P}[x]$ holds $\mathcal{F}(x) \in \mathcal{B}$.

The scheme *PartFuncEx2* concerns a constant $\mathcal{A}$ that is a set, a constant $\mathcal{B}$ that is a set, a constant $\mathcal{C}$ that is a set and a ternary predicate $\mathcal{P}$ and states that:

there exists f being a partial function from $[\!: \mathcal{A}, \mathcal{B} :\!]$ to $\mathcal{C}$ such that for all x, y holds $\langle x, y \rangle \in \operatorname{dom} f$ if and only if $x \in \mathcal{A}$ and $y \in \mathcal{B}$ and there exists z such that $\mathcal{P}[x, y, z]$ and for all x, y such that $\langle x, y \rangle \in \operatorname{dom} f$ holds $\mathcal{P}[x, y, f(\langle x, y \rangle)]$.

provided the parameters satisfy the following conditions:

- for all x, y, z such that $x \in \mathcal{A}$ and $y \in \mathcal{B}$ and $\mathcal{P}[x, y, z]$ holds $z \in \mathcal{C}$,

- for all x, y, z_1, z_2 such that $x \in \mathcal{A}$ and $y \in \mathcal{B}$ and $\mathcal{P}[x, y, z_1]$ and $\mathcal{P}[x, y, z_2]$ holds $z_1 = z_2$.

The scheme *LambdaR2* concerns a constant $\mathcal{A}$ that is a set, a constant $\mathcal{B}$ that is a set, a constant $\mathcal{C}$ that is a set, a binary functor $\mathcal{F}$ and a binary predicate $\mathcal{P}$ and states that:

there exists f being a partial function from $[: \mathcal{A}, \mathcal{B} :]$ to $\mathcal{C}$ such that for all x, y holds $\langle x, y \rangle \in \operatorname{dom} f$ if and only if $x \in \mathcal{A}$ and $y \in \mathcal{B}$ and $\mathcal{P}[x, y]$ and for all x, y such that $\langle x, y \rangle \in \operatorname{dom} f$ holds $f(\langle x, y \rangle) = \mathcal{F}(x, y)$
provided the parameters satisfy the following condition:

- for all x, y such that $\mathcal{P}[x, y]$ holds $\mathcal{F}(x, y) \in \mathcal{C}$.

The arguments of the notions defined below are the following: X, Y, V, Z which are objects of the type reserved above; f which is a partial function from X to Y; g which is a partial function from V to Z. Then $g \cdot f$ is a partial function from X to Z.

One can prove the following propositions:

(36) For every partial function f from X to Y holds $f \cdot \mathrm{id}_X = f$.

(37) For every partial function f from X to Y holds $\mathrm{id}_Y \cdot f = f$.

(38) For every partial function f from C to D such that for all elements x_1, x_2 of C such that $x_1 \in \operatorname{dom} f$ and $x_2 \in \operatorname{dom} f$ and $f(x_1) = f(x_2)$ holds $x_1 = x_2$ holds f is one-to-one.

(39) For every partial function f from X to Y such that f is one-to-one holds f^{-1} is a partial function from Y to X.

(40) For every function f from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ but f is one-to-one holds f^{-1} is a partial function from Y to X.

(41) For every function f from X into X such that f is one-to-one holds f^{-1} is a partial function from X to X.

(42) For every function f from X into D such that f is one-to-one holds f^{-1} is a partial function from D to X.

(43) For every partial function f from X to Y holds $f \restriction Z$ is a partial function from Z to Y.

(44) For every partial function f from X to Y holds $f \restriction Z$ is a partial function from X to Y.

(45) For every partial function f from X to Y holds $Z \restriction f$ is a partial function from X to Z.

(46) For every partial function f from X to Y holds $Z \restriction f$ is a partial function from X to Y.

(47) For every function f holds $(Y \restriction f) \restriction X$ is a partial function from X to Y.

(48) For every partial function f from X to Y holds $(Y' \restriction f) \restriction X'$ is a partial function from X to Y.

(49) For every partial function f from C to D such that $y \in f^{\circ} X$ there exists x being an element of C such that $x \in \operatorname{dom} f$ and $y = f(x)$.

(50) For every partial function f from X to Y holds $f^{\circ} P \subseteq Y$.

The arguments of the notions defined below are the following: X, Y which are objects of the type reserved above; f which is a partial function from X to Y; P which is an object of the type reserved above. Then $f \circ P$ is a subset of Y.

We now state two propositions:

(51) For every partial function f from X to Y holds $f \circ X = \operatorname{rng} f$.

(52) For every partial function f from X to Y holds $f^{-1} Q \subseteq X$.

The arguments of the notions defined below are the following: X, Y which are objects of the type reserved above; f which is a partial function from X to Y; Q which is an object of the type reserved above. Then $f^{-1} Q$ is a subset of X.

Next we state a number of propositions:

(53) For every partial function f from X to Y holds $f^{-1} Y = \operatorname{dom} f$.

(54) For every partial function f from $\emptyset$ to Y holds $\operatorname{dom} f = \emptyset$ and $\operatorname{rng} f = \emptyset$.

(55) For every function f such that $\operatorname{dom} f = \emptyset$ holds f is a partial function from X to Y.

(56) $\square$ is a partial function from X to Y.

(57) For every partial function f from $\emptyset$ to Y holds $f = \square$.

(58) For every partial function f_1 from $\emptyset$ to Y_1 for every partial function f_2 from $\emptyset$ to Y_2 holds $f_1 = f_2$.

(59) For every partial function f from $\emptyset$ to Y holds f is one-to-one.

(60) For every partial function f from $\emptyset$ to Y holds $f \circ P = \emptyset$.

(61) For every partial function f from $\emptyset$ to Y holds $f^{-1} Q = \emptyset$.

(62) For every partial function f from X to $\emptyset$ holds $\operatorname{dom} f = \emptyset$ and $\operatorname{rng} f = \emptyset$.

(63) For every function f such that $\operatorname{rng} f = \emptyset$ holds f is a partial function from X to Y.

(64) For every partial function f from X to $\emptyset$ holds $f = \square$.

(65) For every partial function f_1 from X_1 to $\emptyset$ for every partial function f_2 from X_2 to $\emptyset$ holds $f_1 = f_2$.

(66) For every partial function f from X to $\emptyset$ holds f is one-to-one.

(67) For every partial function f from X to $\emptyset$ holds $f \circ P = \emptyset$.

(68) For every partial function f from X to $\emptyset$ holds $f^{-1} Q = \emptyset$.

(69) For every partial function f from $\{x\}$ to Y holds $\operatorname{rng} f \subseteq \{f(x)\}$.

(70) For every partial function f from $\{x\}$ to Y holds f is one-to-one.

(71) For every partial function f from $\{x\}$ to Y holds $f \circ P \subseteq \{f(x)\}$.

(72) For every function f such that $\operatorname{dom} f = \{x\}$ and $x \in X$ and $f(x) \in Y$ holds f is a partial function from X to Y.

(73) For every partial function f from X to $\{y\}$ such that $x \in \operatorname{dom} f$ holds $f(x) = y$.

(74) For all partial functions f_1, f_2 from X to $\{y\}$ such that $\operatorname{dom} f_1 = \operatorname{dom} f_2$ holds $f_1 = f_2$.

The arguments of the notions defined below are the following: f which is a function; X, Y which are sets. The functor $f_{\restriction X \dot{\to} Y}$ yielding a partial function from X to Y, is defined by:

$f_{\restriction X \dot{\to} Y} = (Y \restriction f) \restriction X$.

We now state a number of propositions:

(75) For every function f for all X, Y holds $f_{\restriction X \dot{\to} Y} = (Y \restriction f) \restriction X$.

(76) For every function f holds $\mathrm{graph}(f_{\restriction X \dot{\to} Y}) \subseteq \mathrm{graph}\, f$.

(77) For every function f holds $\mathrm{dom}(f_{\restriction X \dot{\to} Y}) \subseteq \mathrm{dom}\, f$ and $\mathrm{rng}(f_{\restriction X \dot{\to} Y}) \subseteq \mathrm{rng}\, f$.

(78) For every function f holds $x \in \mathrm{dom}(f_{\restriction X \dot{\to} Y})$ if and only if $x \in \mathrm{dom}\, f$ and $x \in X$ and $f(x) \in Y$.

(79) For every function f such that $x \in \mathrm{dom}\, f$ and $x \in X$ and $f(x) \in Y$ holds $(f_{\restriction X \dot{\to} Y})(x) = f(x)$.

(80) For every function f such that $x \in \mathrm{dom}(f_{\restriction X \dot{\to} Y})$ holds $(f_{\restriction X \dot{\to} Y})(x) = f(x)$.

(81) For all functions f, g such that $\mathrm{graph}\, f \subseteq \mathrm{graph}\, g$ holds $\mathrm{graph}(f_{\restriction X \dot{\to} Y}) \subseteq \mathrm{graph}(g_{\restriction X \dot{\to} Y})$.

(82) For every function f such that $Z \subseteq X$ holds $\mathrm{graph}(f_{\restriction Z \dot{\to} Y}) \subseteq \mathrm{graph}(f_{\restriction X \dot{\to} Y})$.

(83) For every function f such that $Z \subseteq Y$ holds $\mathrm{graph}(f_{\restriction X \dot{\to} Z}) \subseteq \mathrm{graph}(f_{\restriction X \dot{\to} Y})$.

(84) For every function f such that $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$ holds $\mathrm{graph}(f_{\restriction X_1 \dot{\to} Y_1}) \subseteq \mathrm{graph}(f_{\restriction X_2 \dot{\to} Y_2})$.

(85) For every function f such that $\mathrm{dom}\, f \subseteq X$ and $\mathrm{rng}\, f \subseteq Y$ holds $f = f_{\restriction X \dot{\to} Y}$.

(86) For every function f holds $f = f_{\restriction \mathrm{dom}\, f \dot{\to} \mathrm{rng}\, f}$.

(87) For every partial function f from X to Y holds $f_{\restriction X \dot{\to} Y} = f$.

(88) For every function f from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ holds $f_{\restriction X \dot{\to} Y} = f$.

(89) For every function f from X into X holds $f_{\restriction X \dot{\to} X} = f$.

(90) For every function f from X into D holds $f_{\restriction X \dot{\to} D} = f$.

(91) $\square_{\restriction X \dot{\to} Y} = \square$.

(92) For all functions f, g holds $\mathrm{graph}((g_{\restriction Y \dot{\to} Z}) \cdot (f_{\restriction X \dot{\to} Y})) \subseteq \mathrm{graph}(g \cdot f_{\restriction X \dot{\to} Z})$.

(93) For all functions f, g such that $\mathrm{rng}\, f \cap \mathrm{dom}\, g \subseteq Y$ holds $(g_{\restriction Y \dot{\to} Z}) \cdot (f_{\restriction X \dot{\to} Y}) = g \cdot f_{\restriction X \dot{\to} Z}$.

(94) For every function f such that f is one-to-one holds $f_{\restriction X \dot{\to} Y}$ is one-to-one.

(95) For every function f such that f is one-to-one holds $(f_{\restriction X \dot{\to} Y})^{-1} = f^{-1}{}_{\restriction Y \dot{\to} X}$.

(96) For every function f holds $(f_{\restriction X \dot{\to} Y}) \restriction Z = f_{\restriction X \cap Z \dot{\to} Y}$.

(97) For every function f holds $Z \restriction (f_{\restriction X \dot{\to} Y}) = f_{\restriction X \dot{\to} Z \cap Y}$.

The arguments of the notions defined below are the following: X, Y which are objects of the type reserved above; f which is a partial function from X to Y. The predicate f is total is defined by:

dom $f = X$.

We now state a number of propositions:

(98) For every partial function f from X to Y holds f is total if and only if dom $f = X$.

(99) For every partial function f from X to Y such that f is total and $Y = \emptyset$ holds $X = \emptyset$.

(100) For every partial function f from X to Y such that dom $f = X$ holds f is a function from X into Y.

(101) For every partial function f from X to Y such that f is total holds f is a function from X into Y.

(102) For every partial function f from X to Y such that if $Y = \emptyset$, then $X = \emptyset$ but f is a function from X into Y holds f is total.

(103) For every function f from X into Y for every partial function f' from X to Y such that if $Y = \emptyset$, then $X = \emptyset$ but $f = f'$ holds f' is total.

(104) For every function f from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ holds $f_{\restriction X \dot\rightarrow Y}$ is total.

(105) For every function f from X into X holds $f_{\restriction X \dot\rightarrow X}$ is total.

(106) For every function f from X into D holds $f_{\restriction X \dot\rightarrow D}$ is total.

(107) For every partial function f from X to Y such that if $Y = \emptyset$, then $X = \emptyset$ there exists g being a function from X into Y such that for every x such that $x \in$ dom f holds $g(x) = f(x)$.

(108) For every partial function f from X to D there exists g being a function from X into D such that for every x such that $x \in$ dom f holds $g(x) = f(x)$.

(109) For every function f from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ holds f is a partial function from X to Y.

(110) For every function f from X into X holds f is a partial function from X to X.

(111) For every function f from X into D holds f is a partial function from X to D.

(112) For every partial function f from $\emptyset$ to Y holds f is total.

(113) For every function f such that $f_{\restriction X \dot\rightarrow Y}$ is total holds $X \subseteq$ dom f.

(114) If $\square_{\restriction X \dot\rightarrow Y}$ is total, then $X = \emptyset$.

(115) For every function f such that $X \subseteq$ dom f and rng $f \subseteq Y$ holds $f_{\restriction X \dot\rightarrow Y}$ is total.

(116) For every function f such that $f_{\restriction X \dot\rightarrow Y}$ is total holds $f \circ X \subseteq Y$.

(117) For every function f such that $X \subseteq$ dom f and $f \circ X \subseteq Y$ holds $f_{\restriction X \dot\rightarrow Y}$ is total.

Let us consider X, Y. The functor $X \dot\rightarrow Y$ yielding a non-empty set, is defined by:

$x \in X \dot{\to} Y$ if and only if there exists f being a function such that $x = f$ and $\operatorname{dom} f \subseteq X$ and $\operatorname{rng} f \subseteq Y$.

We now state a number of propositions:

(118) For every non-empty set F holds $F = X \dot{\to} Y$ if and only if for every x holds $x \in F$ if and only if there exists f being a function such that $x = f$ and $\operatorname{dom} f \subseteq X$ and $\operatorname{rng} f \subseteq Y$.

(119) For every partial function f from X to Y holds $f \in X \dot{\to} Y$.

(120) For arbitrary f such that $f \in X \dot{\to} Y$ holds f is a partial function from X to Y.

(121) For every element f of $X \dot{\to} Y$ holds f is a partial function from X to Y.

(122) $\emptyset \dot{\to} Y = \{\square\}$.

(123) $X \dot{\to} \emptyset = \{\square\}$.

(124) $Y^X \subseteq X \dot{\to} Y$.

(125) If $Z \subseteq X$, then $Z \dot{\to} Y \subseteq X \dot{\to} Y$.

(126) $\emptyset \dot{\to} Y \subseteq X \dot{\to} Y$.

(127) If $Z \subseteq Y$, then $X \dot{\to} Z \subseteq X \dot{\to} Y$.

(128) If $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$, then $X_1 \dot{\to} Y_1 \subseteq X_2 \dot{\to} Y_2$.

Let f, g be functions. The predicate $f \approx g$ is defined by:
for every x such that $x \in \operatorname{dom} f \cap \operatorname{dom} g$ holds $f(x) = g(x)$.

The following propositions are true:

(129) For all functions f, g holds $f \approx g$ if and only if for every x such that $x \in \operatorname{dom} f \cap \operatorname{dom} g$ holds $f(x) = g(x)$.

(130) For all functions f, g holds $f \approx g$ if and only if there exists h being a function such that $\operatorname{graph} f \cup \operatorname{graph} g = \operatorname{graph} h$.

(131) For all functions f, g holds $f \approx g$ if and only if there exists h being a function such that $\operatorname{graph} f \subseteq \operatorname{graph} h$ and $\operatorname{graph} g \subseteq \operatorname{graph} h$.

(132) For all functions f, g such that $\operatorname{dom} f \subseteq \operatorname{dom} g$ holds $f \approx g$ if and only if for every x such that $x \in \operatorname{dom} f$ holds $f(x) = g(x)$.

(133) For all functions f, g holds $f \approx f$.

(134) For all functions f, g such that $f \approx g$ holds $g \approx f$.

(135) For all functions f, g such that $\operatorname{graph} f \subseteq \operatorname{graph} g$ holds $f \approx g$.

(136) For all functions f, g such that $\operatorname{dom} f = \operatorname{dom} g$ and $f \approx g$ holds $f = g$.

(137) For all functions f, g such that $f = g$ holds $f \approx g$.

(138) For all functions f, g such that $\operatorname{dom} f \cap \operatorname{dom} g = \emptyset$ holds $f \approx g$.

(139) For all functions f, g, h such that $\operatorname{graph} f \subseteq \operatorname{graph} h$ and $\operatorname{graph} g \subseteq \operatorname{graph} h$ holds $f \approx g$.

(140) For all partial functions f, g from X to Y for every function h such that $f \approx h$ and $\operatorname{graph} g \subseteq \operatorname{graph} f$ holds $g \approx h$.

(141) For every function f holds $\square \approx f$ and $f \approx \square$.

(142) For every function f holds $\square_{\restriction X \dot{\to} Y} \approx f$ and $f \approx \square_{\restriction X \dot{\to} Y}$.

(143) For all partial functions f, g from X to $\{y\}$ holds $f \approx g$.

(144) For every function f holds $f \restriction X \approx f$ and $f \restriction X \approx f$.

(145) For every function f holds $Y \restriction f \approx f$ and $f \approx Y \restriction f$.

(146) For every function f holds $(Y \restriction f) \restriction X \approx f$ and $f \approx (Y \restriction f) \restriction X$.

(147) For every function f holds $f_{\restriction X \dot{\to} Y} \approx f$ and $f \approx f_{\restriction X \dot{\to} Y}$.

(148) For all partial functions f, g from X to Y such that f is total and g is total and $f \approx g$ holds $f = g$.

(149) For all functions f, g from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ but $f \approx g$ holds $f = g$.

(150) For all functions f, g from X into X such that $f \approx g$ holds $f = g$.

(151) For all functions f, g from X into D such that $f \approx g$ holds $f = g$.

(152) For every partial function f from X to Y for every function g from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ holds $f \approx g$ if and only if for every x such that $x \in \operatorname{dom} f$ holds $f(x) = g(x)$.

(153) For every partial function f from X to X for every function g from X into X holds $f \approx g$ if and only if for every x such that $x \in \operatorname{dom} f$ holds $f(x) = g(x)$.

(154) For every partial function f from X to D for every function g from X into D holds $f \approx g$ if and only if for every x such that $x \in \operatorname{dom} f$ holds $f(x) = g(x)$.

(155) For every partial function f from X to Y such that if $Y = \emptyset$, then $X = \emptyset$ there exists g being a function from X into Y such that $f \approx g$.

(156) For every partial function f from X to X there exists g being a function from X into X such that $f \approx g$.

(157) For every partial function f from X to D there exists g being a function from X into D such that $f \approx g$.

(158) For all partial functions f, g, h from X to Y such that $f \approx h$ and $g \approx h$ and h is total holds $f \approx g$.

(159) For all partial functions f, g from X to Y for every function h from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ but $f \approx h$ and $g \approx h$ holds $f \approx g$.

(160) For all partial functions f, g from X to X for every function h from X into X such that $f \approx h$ and $g \approx h$ holds $f \approx g$.

(161) For all partial functions f, g from X to D for every function h from X into D such that $f \approx h$ and $g \approx h$ holds $f \approx g$.

(162) For all partial functions f, g from X to Y such that if $Y = \emptyset$, then $X = \emptyset$ but $f \approx g$ there exists h being a partial function from X to Y such that h is total and $f \approx h$ and $g \approx h$.

(163) For all partial functions f, g from X to Y such that if $Y = \emptyset$, then $X = \emptyset$ but $f \approx g$ there exists h being a function from X into Y such that $f \approx h$ and $g \approx h$.

The arguments of the notions defined below are the following: X, Y which are objects of the type reserved above; f which is a partial function from X to Y. The functor $\operatorname{TotFuncs} f$ yields a set and is defined by:

$x \in \mathrm{TotFuncs}\, f$ if and only if there exists g being a partial function from X to Y such that $g = x$ and g is total and $f \approx g$.

The following propositions are true:

(164) For all X, Y for every partial function f from X to Y for every Z holds $Z = \mathrm{TotFuncs}\, f$ if and only if for every x holds $x \in Z$ if and only if there exists g being a partial function from X to Y such that $g = x$ and g is total and $f \approx g$.

(165) For every partial function f from X to Y for every function g from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ but $f \approx g$ holds $g \in \mathrm{TotFuncs}\, f$.

(166) For every partial function f from X to X for every function g from X into X such that $f \approx g$ holds $g \in \mathrm{TotFuncs}\, f$.

(167) For every partial function f from X to D for every function g from X into D such that $f \approx g$ holds $g \in \mathrm{TotFuncs}\, f$.

(168) For every partial function f from X to Y for arbitrary g such that $g \in \mathrm{TotFuncs}\, f$ holds g is a partial function from X to Y.

(169) For all partial functions f, g from X to Y such that $g \in \mathrm{TotFuncs}\, f$ holds g is total.

(170) For every partial function f from X to Y for arbitrary g such that $g \in \mathrm{TotFuncs}\, f$ holds g is a function from X into Y.

(171) For every partial function f from X to Y for every function g such that $g \in \mathrm{TotFuncs}\, f$ holds $f \approx g$ and $g \approx f$.

(172) For every partial function f from X to $\emptyset$ such that $X \neq \emptyset$ holds $\mathrm{TotFuncs}\, f = \emptyset$.

(173) For every partial function f from X to Y holds $\mathrm{TotFuncs}\, f \subseteq Y^X$.

(174) For every partial function f from X to Y holds f is total if and only if $\mathrm{TotFuncs}\, f = \{f\}$.

(175) For every partial function f from $\emptyset$ to Y holds $\mathrm{TotFuncs}\, f = \{f\}$.

(176) For every partial function f from $\emptyset$ to Y holds $\mathrm{TotFuncs}\, f = \{\square\}$.

(177) $\mathrm{TotFuncs}(\square_{\restriction X \dot\to Y}) = Y^X$.

(178) For every function f from X into Y such that if $Y = \emptyset$, then $X = \emptyset$ holds $\mathrm{TotFuncs}(f_{\restriction X \dot\to Y}) = \{f\}$.

(179) For every function f from X into X holds $\mathrm{TotFuncs}(f_{\restriction X \dot\to X}) = \{f\}$.

(180) For every function f from X into D holds $\mathrm{TotFuncs}(f_{\restriction X \dot\to D}) = \{f\}$.

(181) For every partial function f from X to $\{y\}$ for every function g from X into $\{y\}$ holds $\mathrm{TotFuncs}\, f = \{g\}$.

(182) For all partial functions f, g from X to Y such that $\mathrm{graph}\, g \subseteq \mathrm{graph}\, f$ holds $\mathrm{TotFuncs}\, f \subseteq \mathrm{TotFuncs}\, g$.

(183) For all partial functions f, g from X to Y such that $\mathrm{dom}\, g \subseteq \mathrm{dom}\, f$ and $\mathrm{TotFuncs}\, f \subseteq \mathrm{TotFuncs}\, g$ holds $\mathrm{graph}\, g \subseteq \mathrm{graph}\, f$.

(184) For all partial functions f, g from X to Y such that $\mathrm{TotFuncs}\, f \subseteq \mathrm{TotFuncs}\, g$ and for every y holds $Y \neq \{y\}$ holds $\mathrm{graph}\, g \subseteq \mathrm{graph}\, f$.

(185) For all partial functions f, g from X to Y such that $\mathrm{TotFuncs}\, f \cap \mathrm{TotFuncs}\, g \neq \emptyset$ holds $f \approx g$.

(186) For all partial functions f, g from X to Y such that if $Y = \emptyset$, then $X = \emptyset$ but $f \approx g$ holds $\mathrm{TotFuncs}\, f \cap \mathrm{TotFuncs}\, g \neq \emptyset$.

(187) For all partial functions f, g from X to Y such that for every y holds $Y \neq \{y\}$ and $\mathrm{TotFuncs}\, f = \mathrm{TotFuncs}\, g$ holds $f = g$.

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