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σ-Fields and Probability

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Summary. This article contains definitions and theorems concerning basic properties of following objects: - a field of subsets of given nonempty set; - a sequence of subsets of given nonempty set; - a σ-field of subsets of given nonempty set and events from this σ-field; - a probability i.e. σ-additive normed measure defined on previously introduced σ-field; - a σ-field generated by family of subsets of given set; - family of Borel Sets.

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The articles [7], [1], [3], [2], [5], [4], [6], and [8] provide the notation and terminology for this paper. For simplicity we adopt the following rules: *Omega* will be a non-empty set, Y, Z, V will be sets, A, B, D will be subsets of *Omega*, f will be a function, m, n will be natural numbers, p, x, y, z will be arbitrary, r, r_1, r_2 will be real numbers, and *seq* will be a sequence of real numbers. We now state three propositions:

(1) For every x holds x is a subset of *Omega* if and only if $x \in 2^{Omega}$.

(2) For all r, r_1, r_2 such that $0 \leq r$ and $r_1 = r_2 - r$ holds $r_1 \leq r_2$.

(3) For all r, *seq* such that there exists n such that for every m such that $n \leq m$ holds $seq(m) = r$ holds *seq* is convergent and $\lim seq = r$.

Let us consider *Omega*. The mode field of subsets of *Omega*, which widens to the type a set, is defined by:

it $\subseteq 2^{Omega}$ and there exists A such that $A \in$ it but if $A \in$ it and $B \in$ it, then $A \cap B \in$ it but if $A \in$ it, then $A^{\rm c} \in$ it.

Next we state a proposition

(4) For all *Omega*, Y holds for all A, B holds $Y \subseteq 2^{Omega}$ and there exists A such that $A \in Y$ but if $A \in Y$ and $B \in Y$, then $A \cap B \in Y$ but if $A \in Y$, then $A^{\rm c} \in Y$ if and only if Y is a field of subsets of *Omega*.

In the sequel *Fld* will be a field of subsets of *Omega*. Next we state a number of propositions:

(5) $Fld \subseteq 2^{Omega}$.

(6) There exists A such that $A \in Fld$.

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(7) If $A \in Fld$ and $B \in Fld$, then $A \cap B \in Fld$.

(8) If $A \in Fld$, then $A^{\rm c} \in Fld$.

(9) If $A \in Fld$ and $B \in Fld$, then $A \cup B \in Fld$.

(10) $\emptyset \in Fld$.

(11) $Omega \in Fld$.

(12) If $A \in Fld$ and $B \in Fld$, then $A \setminus B \in Fld$.

(13) If $A \in Fld$ and $B \in Fld$, then $A \cup B = (A \setminus B) \cup B$ and $(A \setminus B) \cup B \in Fld$ and $A \setminus B$ misses B.

(14) For every $Omega$ holds $\{\emptyset, Omega\}$ is a field of subsets of $Omega$.

(15) For every $Omega$ holds 2^{Omega} is a field of subsets of $Omega$.

(16) $\{\emptyset, Omega\} \subseteq Fld$ and $Fld \subseteq 2^{Omega}$.

(17) For every x such that $x \in Fld$ holds x is a subset of $Omega$.

(18) For every $Omega$ holds for every p such that $p \in [\!: \mathbb{N}, \{Omega\} :\!]$ there exist x, y such that $\langle x, y \rangle = p$ and for all x, y, z such that $\langle x, y \rangle \in [\!: \mathbb{N}, \{Omega\} :\!]$ and $\langle x, z \rangle \in [\!: \mathbb{N}, \{Omega\} :\!]$ holds $y = z$.

(19) For every $Omega$ there exists f such that $\operatorname{dom} f = \mathbb{N}$ and for every n holds $f(n) = Omega$ and $f(n) \in 2^{Omega}$.

Let us consider $Omega$. The mode sequence of subsets of $Omega$, which widens to the type a function, is defined by:

$\operatorname{dom} \mathrm{it} = \mathbb{N}$ and for every n holds $\mathrm{it}(n) \in 2^{Omega}$.

One can prove the following proposition

(20) f is a sequence of subsets of $Omega$ if and only if $\operatorname{dom} f = \mathbb{N}$ and for every n holds $f(n) \in 2^{Omega}$.

In the sequel $ASeq$, $BSeq$ denote sequences of subsets of $Omega$. We now state two propositions:

(21) There exists $ASeq$ such that for every n holds $ASeq(n) = Omega$.

(22) For every A, B there exists $ASeq$ such that $ASeq(0) = A$ and for every n such that $n \neq 0$ holds $ASeq(n) = B$.

Let us consider $Omega$, $ASeq$, n. Then $ASeq(n)$ is a subset of $Omega$.

The following proposition is true

(23) For all $ASeq$, V such that $V = \bigcup(\operatorname{rng} ASeq)$ holds V is a subset of $Omega$.

Let us consider $Omega$, $ASeq$. The functor $\mathrm{Union}\, ASeq$ yields a set and is defined by:

$\mathrm{Union}\, ASeq = \bigcup(\operatorname{rng} ASeq)$.

We now state a proposition

(24) For all $ASeq$, V holds $V = \mathrm{Union}\, ASeq$ if and only if $V = \bigcup(\operatorname{rng} ASeq)$.

Let us consider $Omega$, $ASeq$. Then $\mathrm{Union}\, ASeq$ is a subset of $Omega$.

We now state two propositions:

(25) For all x, $ASeq$ holds $x \in \mathrm{Union}\, ASeq$ if and only if there exists n such that $x \in ASeq(n)$.

(26) For every $ASeq$ there exists $BSeq$ such that for every n holds $BSeq(n) = (ASeq(n))^{\rm c}$.

Let us consider $Omega$, $ASeq$. The functor Complement $ASeq$ yields a sequence of subsets of $Omega$ and is defined by:

for every n holds (Complement $ASeq$)$(n) = (ASeq(n))^{\rm c}$.

One can prove the following proposition

(27) For all $ASeq$, $BSeq$ holds $BSeq =$ Complement $ASeq$ if and only if for every n holds $BSeq(n) = (ASeq(n))^{\rm c}$.

Let us consider $Omega$, $ASeq$. The functor Intersection $ASeq$ yields a subset of $Omega$ and is defined by:

Intersection $ASeq =$ (Union(Complement $ASeq$))$^{\rm c}$.

One can prove the following propositions:

(28) For all $ASeq$, A holds $A =$ Intersection $ASeq$ if and only if $A =$ (Union(Complement $ASeq$))$^{\rm c}$.

(29) For all $ASeq$, x holds $x \in$ Intersection $ASeq$ if and only if for every n holds $x \in ASeq(n)$.

(30) For all A, B, $ASeq$ such that $ASeq(0) = A$ and for every n such that $n \neq 0$ holds $ASeq(n) = B$ holds Intersection $ASeq = A \cap B$.

(31) For every $ASeq$ holds Complement(Complement $ASeq$) $= ASeq$.

We now define two new predicates. Let us consider $Omega$, $ASeq$. The predicate $ASeq$ is nonincreasing is defined by:

for all n, m such that $n \leq m$ holds $ASeq(m) \subseteq ASeq(n)$.

The predicate $ASeq$ is nondecreasing is defined by:

for all n, m such that $n \leq m$ holds $ASeq(n) \subseteq ASeq(m)$.

The following two propositions are true:

(32) For all $Omega$, $ASeq$ holds $ASeq$ is nonincreasing if and only if for all n, m such that $n \leq m$ holds $ASeq(m) \subseteq ASeq(n)$.

(33) For all $Omega$, $ASeq$ holds $ASeq$ is nondecreasing if and only if for all n, m such that $n \leq m$ holds $ASeq(n) \subseteq ASeq(m)$.

Let us consider $Omega$. The mode σ-field of subsets of $Omega$, which widens to the type a set, is defined by:

it $\subseteq 2^{Omega}$ and there exists A such that $A \in$ it and for every $ASeq$ such that for every n holds $ASeq(n) \in$ it holds Intersection $ASeq \in$ it and for every A such that $A \in$ it holds $A^{\rm c} \in$ it.

We now state two propositions:

(34) For all $Omega$, Y holds Y is a σ-field of subsets of $Omega$ if and only if $Y \subseteq 2^{Omega}$ and there exists A such that $A \in Y$ and for every $ASeq$ such that for every n holds $ASeq(n) \in Y$ holds Intersection $ASeq \in Y$ and for every A such that $A \in Y$ holds $A^{\rm c} \in Y$.

(35) For all $Omega$, Y such that Y is a σ-field of subsets of $Omega$ holds Y is a field of subsets of $Omega$.

In the sequel $Sigma$ is a σ-field of subsets of $Omega$. Next we state several propositions:

(36) $Sigma \subseteq 2^{Omega}$.
(37) For every x such that $x \in Sigma$ holds x is a subset of $Omega$.
(38) There exists A such that $A \in Sigma$.
(39) For all A, B such that $A \in Sigma$ and $B \in Sigma$ holds $A \cap B \in Sigma$.
(40) For every A such that $A \in Sigma$ holds $A^{c} \in Sigma$.
(41) For all A, B such that $A \in Sigma$ and $B \in Sigma$ holds $A \cup B \in Sigma$.
(42) $\emptyset \in Sigma$.
(43) $Omega \in Sigma$.
(44) For all A, B such that $A \in Sigma$ and $B \in Sigma$ holds $A \setminus B \in Sigma$.

Let us consider $Omega$, $Sigma$. The mode sequence of subsets of $Sigma$, which widens to the type a sequence of subsets of $Omega$, is defined by:

for every n holds $\mathrm{it}(n) \in Sigma$.

We now state two propositions:

(45) $ASeq$ is a sequence of subsets of $Sigma$ if and only if for every n holds $ASeq(n) \in Sigma$.
(46) For all $Omega$, $Sigma$ for every sequence $ASeq$ of subsets of $Sigma$ holds Union $ASeq \in Sigma$.

Let us consider $Omega$, $Sigma$. The mode event of $Sigma$, which widens to the type a subset of $Omega$, is defined by:

$\mathrm{it} \in Sigma$.

The following propositions are true:

(47) For all $Sigma$, A holds A is an event of $Sigma$ if and only if $A \in Sigma$.
(48) For all $Sigma$, x such that $x \in Sigma$ holds x is an event of $Sigma$.
(49) For all events A, B of $Sigma$ holds $A \cap B$ is an event of $Sigma$.
(50) For every event A of $Sigma$ holds A^{c} is an event of $Sigma$.
(51) For all events A, B of $Sigma$ holds $A \cup B$ is an event of $Sigma$.
(52) For all $Omega$, $Sigma$ holds $\emptyset$ is an event of $Sigma$.
(53) For all $Omega$, $Sigma$ holds $Omega$ is an event of $Sigma$.
(54) For all events A, B of $Sigma$ holds $A \setminus B$ is an event of $Sigma$.

We now define two new functors. Let us consider $Omega$, $Sigma$. The functor Ω_{Sigma} yields an event of $Sigma$ and is defined by:

$\Omega_{Sigma} = Omega$.

The functor $\emptyset_{Sigma}$ yielding an event of $Sigma$, is defined by:

$\emptyset_{Sigma} = \emptyset$.

Next we state two propositions:

(55) For all $Omega$, $Sigma$ holds $\Omega_{Sigma} = Omega$.
(56) For all $Omega$, $Sigma$ holds $\emptyset_{Sigma} = \emptyset$.

The arguments of the notions defined below are the following: $Omega$, $Sigma$ which are objects of the type reserved above; A, B which are events of $Sigma$. Then $A \cap B$ is an event of $Sigma$. Then $A \cup B$ is an event of $Sigma$. Then $A \setminus B$ is an event of $Sigma$.

We now state two propositions:

(57) For all *Omega*, *Sigma*, *ASeq* holds *ASeq* is a sequence of subsets of *Sigma* if and only if for every n holds $ASeq(n)$ is an event of *Sigma*.

(58) For all *Omega*, *Sigma*, *ASeq* such that *ASeq* is a sequence of subsets of *Sigma* holds Union *ASeq* is an event of *Sigma*.

In the sequel *Sigma* is a σ-field of subsets of *Omega*, A, B are events of *Sigma*, and *ASeq* is a sequence of subsets of *Sigma*. Next we state a proposition

(59) For every *Omega*, *Sigma*, p there exists f such that $\operatorname{dom} f = Sigma$ and for every D such that $D \in Sigma$ holds if $p \in D$, then $f(D) = 1$ but if $p \notin D$, then $f(D) = 0$.

In the sequel P is a function from *Sigma* into $\mathbb{R}$. The following three propositions are true:

(60) For every *Omega*, *Sigma*, p there exists P such that for every D such that $D \in Sigma$ holds if $p \in D$, then $P(D) = 1$ but if $p \notin D$, then $P(D) = 0$.

(61) For every P holds $\operatorname{dom} P = Sigma$ and $\operatorname{rng} P \subseteq \mathbb{R}$.

(62) For all *Sigma*, *ASeq*, P holds $P \cdot ASeq$ is a sequence of real numbers.

Let us consider *Omega*, *Sigma*, *ASeq*, P. Then $P \cdot ASeq$ is a sequence of real numbers.

Let us consider *Omega*, *Sigma*, P, A. Then $P(A)$ is a real number.

Let us consider *Omega*, *Sigma*. The mode probability on *Sigma*, which widens to the type a function from *Sigma* into $\mathbb{R}$, is defined by:

(i) for every A holds $0 \leq \mathrm{it}(A)$,
(ii) $\mathrm{it}(Omega) = 1$,
(iii) for all A, B such that A misses B holds $\mathrm{it}(A \cup B) = \mathrm{it}(A) + \mathrm{it}(B)$,
(iv) for every *ASeq* such that *ASeq* is nonincreasing holds $\mathrm{it} \cdot ASeq$ is convergent and $\lim(\mathrm{it} \cdot ASeq) = \mathrm{it}(\mathrm{Intersection}\, ASeq)$.

Next we state a proposition

(63) Let P be a function from *Sigma* into $\mathbb{R}$. Then P is a probability on *Sigma* if and only if the following conditions are satisfied:
   (i) for every A holds $0 \leq P(A)$,
   (ii) $P(Omega) = 1$,
   (iii) for all A, B such that A misses B holds $P(A \cup B) = P(A) + P(B)$,
   (iv) for every *ASeq* such that *ASeq* is nonincreasing holds $P \cdot ASeq$ is convergent and $\lim(P \cdot ASeq) = P(\mathrm{Intersection}\, ASeq)$.

In the sequel P will be a probability on *Sigma*. One can prove the following propositions:

(64) $P(\emptyset) = 0$.

(65) $P(\emptyset_{Sigma}) = 0$.

(66) $P(\Omega_{Sigma}) = 1$.

(67) For all P, A holds $P(\Omega_{Sigma} \setminus A) + P(A) = 1$.

(68) For all P, A holds $P(\Omega_{Sigma} \setminus A) = 1 - P(A)$.

(69) For all P, A, B such that $A \subseteq B$ holds $P(B \setminus A) = P(B) - P(A)$.

(70) For all P, A, B such that $A \subseteq B$ holds $P(A) \leq P(B)$.

(71) For all P, A holds $P(A) \leq 1$.

(72) For all P, A, B holds $P(A \cup B) = P(A) + P(B \setminus A)$.

(73) For all P, A, B holds $P(A \cup B) = P(A) + P(B \setminus A \cap B)$.

(74) For all P, A, B holds $P(A \cup B) = (P(A) + P(B)) - P(A \cap B)$.

(75) For all P, A, B holds $P(A \cup B) \leq P(A) + P(B)$.

In the sequel D denotes a subset of $\mathbb{R}$ and S denotes a subset of 2^{Omega}. Next we state a proposition

(76) 2^{Omega} is a σ-field of subsets of $Omega$.

The arguments of the notions defined below are the following: $Omega$ which is an object of the type reserved above; X which is a subset of 2^{Omega}. The functor σX yields a σ-field of subsets of $Omega$ and is defined by:

$X \subseteq \sigma X$ and for every Z such that $X \subseteq Z$ and Z is a σ-field of subsets of $Omega$ holds $\sigma X \subseteq Z$.

Next we state a proposition

(77) For all S, $Sigma$ holds $Sigma = \sigma S$ if and only if $S \subseteq Sigma$ and for every Z such that $S \subseteq Z$ and Z is a σ-field of subsets of $Omega$ holds $Sigma \subseteq Z$.

Let us consider r. The functor $\mathrm{HL}(r)$ yielding a subset of $\mathbb{R}$, is defined by:

$\mathrm{HL}(r) = \{r_1 : r_1 < r\}$.

Next we state a proposition

(78) For all r, D holds $D = \mathrm{HL}(r)$ if and only if $D = \{r_1 : r_1 < r\}$.

The constant Halflines is a subset of $2^{\mathbb{R}}$ and is defined by:

$\mathrm{Halflines} = \{D : \bigwedge_r D = \mathrm{HL}(r)\}$.

The following proposition is true

(79) For every subset Z of $2^{\mathbb{R}}$ holds $Z = \mathrm{Halflines}$ if and only if $Z = \{D : \bigwedge_r D = \mathrm{HL}(r)\}$.

The constant the Borel sets is a σ-field of subsets of $\mathbb{R}$ and is defined by:

the Borel sets $= \sigma$ Halflines.

One can prove the following proposition

(80) For every σ-field Z of subsets of $\mathbb{R}$ holds $Z =$ the Borel sets if and only if $Z = \sigma$ Halflines.

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