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Real Sequences and Basic Operations on Them

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Summary. Definition of real sequence and operations on sequences (multiplication of sequences and multiplication by a real number, addition, subtraction, division and absolute value of sequence) are given.

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The notation and terminology used here are introduced in the following articles: [4], [1], [3], and [2]. For simplicity we follow the rules: f will be a function, n will be a natural number, r, p will be real numbers, and x will be arbitrary. We now state a proposition

(1) x is a natural number if and only if $x \in \mathbb{N}$.

The mode sequence of real numbers, which widens to the type a function, is defined by:

$\mathrm{dom}\,\mathrm{it} = \mathbb{N}$ and $\mathrm{rng}\,\mathrm{it} \subseteq \mathbb{R}$.

In the sequel seq, seq_1, seq_2, seq_3, seq', $seq_{1'}$ are sequences of real numbers. Next we state three propositions:

(2) f is a sequence of real numbers if and only if $\mathrm{dom}\,f = \mathbb{N}$ and $\mathrm{rng}\,f \subseteq \mathbb{R}$.

(3) f is a sequence of real numbers if and only if $\mathrm{dom}\,f = \mathbb{N}$ and for every x such that $x \in \mathbb{N}$ holds $f(x)$ is a real number.

(4) f is a sequence of real numbers if and only if $\mathrm{dom}\,f = \mathbb{N}$ and for every n holds $f(n)$ is a real number.

Let us consider seq, n. Then $seq(n)$ is a real number.

Let us consider seq. The predicate seq is non-zero is defined by:

$\mathrm{rng}\,seq \subseteq \mathbb{R} \setminus \{0\}$.

One can prove the following propositions:

(5) seq is non-zero if and only if $\mathrm{rng}\,seq \subseteq \mathbb{R} \setminus \{0\}$.

(6) seq is non-zero if and only if for every x such that $x \in \mathbb{N}$ holds $seq(x) \neq 0$.

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(7) seq is non-zero if and only if for every n holds $seq(n) \neq 0$.

(8) For all seq, seq_1 such that for every x such that $x \in \mathbb{N}$ holds $seq(x) = seq_1(x)$ holds $seq = seq_1$.

(9) For all seq, seq_1 such that for every n holds $seq(n) = seq_1(n)$ holds $seq = seq_1$.

(10) For every r there exists seq such that $\operatorname{rng} seq = \{r\}$.

The scheme *ExRealSeq* concerns a unary functor $\mathcal{F}$ yielding a real number and states that:

there exists seq such that for every n holds $seq(n) = \mathcal{F}(n)$

for all values of the parameter.

We now define two new functors. Let us consider seq_1, seq_2. The functor $seq_1 + seq_2$ yields a sequence of real numbers and is defined by:

for every n holds $(seq_1 + seq_2)(n) = seq_1(n) + seq_2(n)$.

The functor $seq_1 \cdot seq_2$ yielding a sequence of real numbers, is defined by:

for every n holds $(seq_1 \cdot seq_2)(n) = seq_1(n) \cdot seq_2(n)$.

The following two propositions are true:

(11) $seq = seq_1 + seq_2$ if and only if for every n holds $seq(n) = seq_1(n) + seq_2(n)$.

(12) $seq = seq_1 \cdot seq_2$ if and only if for every n holds $seq(n) = seq_1(n) \cdot seq_2(n)$.

Let us consider r, seq. The functor $r \cdot seq$ yielding a sequence of real numbers, is defined by:

for every n holds $(r \cdot seq)(n) = r \cdot seq(n)$.

One can prove the following proposition

(13) $seq = r \cdot seq_1$ if and only if for every n holds $seq(n) = r \cdot seq_1(n)$.

Let us consider seq. The functor $-seq$ yields a sequence of real numbers and is defined by:

for every n holds $(-seq)(n) = -seq(n)$.

We now state a proposition

(14) $seq = -seq_1$ if and only if for every n holds $seq(n) = -seq_1(n)$.

Let us consider seq_1, seq_2. The functor $seq_1 - seq_2$ yields a sequence of real numbers and is defined by:

$seq_1 - seq_2 = seq_1 + (-seq_2)$.

We now state a proposition

(15) $seq = seq_1 - seq_2$ if and only if $seq = seq_1 + (-seq_2)$.

Let us consider seq. Let us assume that seq is non-zero. The functor seq^{-1} yielding a sequence of real numbers, is defined by:

for every n holds $(seq^{-1})(n) = (seq(n))^{-1}$.

One can prove the following proposition

(16) If seq is non-zero, then $seq_1 = seq^{-1}$ if and only if for every n holds $seq_1(n) = (seq(n))^{-1}$.

Let us consider seq_1, seq. Let us assume that seq is non-zero. The functor $\frac{seq_1}{seq}$ yields a sequence of real numbers and is defined by:

$\frac{seq_1}{seq} = seq_1 \cdot seq^{-1}$.

The following proposition is true

(17) If seq_2 is non-zero, then $seq = \frac{seq_1}{seq_2}$ if and only if $seq = seq_1 \cdot {seq_2}^{-1}$.

Let us consider seq. The functor $|seq|$ yielding a sequence of real numbers, is defined by:

for every n holds $|seq|(n) = |seq(n)|$.

The following propositions are true:

(18) $seq = |seq_1|$ if and only if for every n holds $seq(n) = |seq_1(n)|$.

(19) $seq_1 + seq_2 = seq_2 + seq_1$.

(20) $(seq_1 + seq_2) + seq_3 = seq_1 + (seq_2 + seq_3)$.

(21) $seq_1 \cdot seq_2 = seq_2 \cdot seq_1$.

(22) $(seq_1 \cdot seq_2) \cdot seq_3 = seq_1 \cdot (seq_2 \cdot seq_3)$.

(23) $(seq_1 + seq_2) \cdot seq_3 = seq_1 \cdot seq_3 + seq_2 \cdot seq_3$.

(24) $seq_3 \cdot (seq_1 + seq_2) = seq_3 \cdot seq_1 + seq_3 \cdot seq_2$.

(25) $-seq = (-1) \cdot seq$.

(26) $r \cdot (seq_1 \cdot seq_2) = (r \cdot seq_1) \cdot seq_2$.

(27) $r \cdot (seq_1 \cdot seq_2) = seq_1 \cdot (r \cdot seq_2)$.

(28) $(seq_1 - seq_2) \cdot seq_3 = seq_1 \cdot seq_3 - seq_2 \cdot seq_3$.

(29) $seq_3 \cdot seq_1 - seq_3 \cdot seq_2 = seq_3 \cdot (seq_1 - seq_2)$.

(30) $r \cdot (seq_1 + seq_2) = r \cdot seq_1 + r \cdot seq_2$.

(31) $(r \cdot p) \cdot seq = r \cdot (p \cdot seq)$.

(32) $r \cdot (seq_1 - seq_2) = r \cdot seq_1 - r \cdot seq_2$.

(33) If seq is non-zero, then $r \cdot \frac{seq_1}{seq} = \frac{r \cdot seq_1}{seq}$.

(34) $seq_1 - (seq_2 + seq_3) = (seq_1 - seq_2) - seq_3$.

(35) $1 \cdot seq = seq$.

(36) $-(-seq) = seq$.

(37) $seq_1 - (-seq_2) = seq_1 + seq_2$.

(38) $seq_1 - (seq_2 - seq_3) = (seq_1 - seq_2) + seq_3$.

(39) $seq_1 + (seq_2 - seq_3) = (seq_1 + seq_2) - seq_3$.

(40) $(-seq_1) \cdot seq_2 = -seq_1 \cdot seq_2$ and $seq_1 \cdot (-seq_2) = -seq_1 \cdot seq_2$.

(41) If seq is non-zero, then seq^{-1} is non-zero.

(42) If seq is non-zero, then $(seq^{-1})^{-1} = seq$.

(43) seq is non-zero and seq_1 is non-zero if and only if $seq \cdot seq_1$ is non-zero.

(44) If seq is non-zero and seq_1 is non-zero, then $seq^{-1} \cdot {seq_1}^{-1} = (seq \cdot seq_1)^{-1}$.

(45) If seq is non-zero, then $\frac{seq_1}{seq} \cdot seq = seq_1$.

(46) If seq is non-zero and seq_1 is non-zero, then $\frac{seq'}{seq} \cdot \frac{seq_{1'}}{seq_1} = \frac{seq' \cdot seq_{1'}}{seq \cdot seq_1}$.

(47) If seq is non-zero and seq_1 is non-zero, then $\frac{seq}{seq_1}$ is non-zero.

(48) If seq is non-zero and seq_1 is non-zero, then $\frac{seq}{seq_1}^{-1} = \frac{seq_1}{seq}$.

(49) If seq is non-zero, then $seq_2 \cdot \frac{seq_1}{seq} = \frac{seq_2 \cdot seq_1}{seq}$.

(50) If seq is non-zero and seq_1 is non-zero, then $\frac{seq_2}{\frac{seq}{seq_1}} = \frac{seq_2 \cdot seq_1}{seq}$.

(51) If seq is non-zero and seq_1 is non-zero, then $\frac{seq_2}{seq} = \frac{seq_2 \cdot seq_1}{seq \cdot seq_1}$.

(52) If $r \neq 0$ and seq is non-zero, then $r \cdot seq$ is non-zero.

(53) If seq is non-zero, then $-seq$ is non-zero.

(54) If $r \neq 0$ and seq is non-zero, then $(r \cdot seq)^{-1} = r^{-1} \cdot seq^{-1}$.

(55) If seq is non-zero, then $(-seq)^{-1} = (-1) \cdot seq^{-1}$.

(56) If seq is non-zero, then $-\frac{seq_1}{seq} = \frac{-seq_1}{seq}$ and $\frac{seq_1}{-seq} = -\frac{seq_1}{seq}$.

(57) If seq is non-zero, then $\frac{seq_1}{seq} + \frac{seq_{1'}}{seq} = \frac{seq_1 + seq_{1'}}{seq}$ and $\frac{seq_1}{seq} - \frac{seq_{1'}}{seq} = \frac{seq_1 - seq_{1'}}{seq}$.

(58) If seq is non-zero and seq' is non-zero, then $\frac{seq_1}{seq} + \frac{seq_{1'}}{seq'} = \frac{seq_1 \cdot seq' + seq_{1'} \cdot seq}{seq \cdot seq'}$ and $\frac{seq_1}{seq} - \frac{seq_{1'}}{seq'} = \frac{seq_1 \cdot seq' - seq_{1'} \cdot seq}{seq \cdot seq'}$.

(59) If seq is non-zero and seq' is non-zero and seq_1 is non-zero, then $\frac{\frac{seq_{1'}}{seq}}{\frac{seq'}{seq_1}} = \frac{seq_{1'} \cdot seq_1}{seq \cdot seq'}$.

(60) $|seq \cdot seq'| = |seq| \cdot |seq'|$.

(61) If seq is non-zero, then $|seq|$ is non-zero.

(62) If seq is non-zero, then $|seq|^{-1} = |seq^{-1}|$.

(63) If seq is non-zero, then $|\frac{seq'}{seq}| = \frac{|seq'|}{|seq|}$.

(64) $|r \cdot seq| = |r| \cdot |seq|$.

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