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Convergent Sequences and the Limit of Sequences

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Summary. The article contains definitions and same basic properties of bounded sequences (above and below), convergent sequences and the limit of sequences. In the article there are some properties of real numbers useful in the other theorems of this article.

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The terminology and notation used in this paper have been introduced in the following papers: [1], and [2]. We adopt the following rules: n, m are natural numbers, r, r_1, p, g_1, g, g' are real numbers, and seq, seq', seq_1 are sequences of real numbers. One can prove the following propositions:

(1) $(-1) \cdot (-1) = 1$.

(2) $\frac{g}{2} + \frac{g}{2} = g$ and $\frac{g}{4} + \frac{g}{4} = \frac{g}{2}$.

(3) If $0 < g$, then $0 < \frac{g}{2}$ and $0 < \frac{g}{4}$.

(4) If $0 < g$, then $\frac{g}{2} < g$.

(5) If $g \neq 0$, then $\frac{r}{g \cdot 2} + \frac{r}{g \cdot 2} = \frac{r}{g}$.

(6) If $0 < g$ and $0 < p$, then $0 < \frac{g}{p}$.

(7) If $0 \leq g$ and $0 \leq r$ and $g < g_1$ and $r < r_1$, then $g \cdot r < g_1 \cdot r_1$.

(8) If $g = -g'$, then $-g = g'$.

(9) $-g < r$ and $r < g$ if and only if $|r| < g$.

(10) If $0 < r_1$ and $r_1 < r$ and $0 < g$, then $\frac{g}{r} < \frac{g}{r_1}$.

(11) If $g \neq 0$ and $r \neq 0$, then $|g^{-1} - r^{-1}| = \frac{|g-r|}{|g| \cdot |r|}$.

We now define two new predicates. Let us consider seq. The predicate seq is bounded above is defined by:

there exists r such that for every n holds $seq(n) < r$.

The predicate seq is bounded below is defined by:

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there exists r such that for every n holds $r < seq(n)$.

We now state two propositions:

(12) seq is bounded above if and only if there exists r such that for every n holds $seq(n) < r$.

(13) seq is bounded below if and only if there exists r such that for every n holds $r < seq(n)$.

Let us consider seq. The predicate seq is bounded is defined by:
seq is bounded above and seq is bounded below.

Next we state three propositions:

(14) seq is bounded if and only if seq is bounded above and seq is bounded below.

(15) seq is bounded if and only if there exists r such that $0 < r$ and for every n holds $|seq(n)| < r$.

(16) For every n there exists r such that $0 < r$ and for every m such that $m \leq n$ holds $|seq(m)| < r$.

Let us consider seq. The predicate seq is convergent is defined by:

there exists g such that for every p such that $0 < p$ there exists n such that for every m such that $n \leq m$ holds $|seq(m) - g| < p$.

One can prove the following proposition

(17) seq is convergent if and only if there exists g such that for every p such that $0 < p$ there exists n such that for every m such that $n \leq m$ holds $|seq(m) - g| < p$.

Let us consider seq. Let us assume that seq is convergent. The functor $\lim seq$ yields a real number and is defined by:

for every p such that $0 < p$ there exists n such that for every m such that $n \leq m$ holds $|seq(m) - (\lim seq)| < p$.

The following propositions are true:

(18) If seq is convergent, then $\lim seq = g$ if and only if for every p such that $0 < p$ there exists n such that for every m such that $n \leq m$ holds $|seq(m) - g| < p$.

(19) If seq is convergent and seq' is convergent, then $seq + seq'$ is convergent.

(20) If seq is convergent and seq' is convergent, then $\lim(seq+seq') = \lim seq + \lim seq'$.

(21) If seq is convergent, then $r \cdot seq$ is convergent.

(22) If seq is convergent, then $\lim(r \cdot seq) = r \cdot (\lim seq)$.

(23) If seq is convergent, then $-seq$ is convergent.

(24) If seq is convergent, then $\lim(-seq) = -\lim seq$.

(25) If seq is convergent and seq' is convergent, then $seq - seq'$ is convergent.

(26) If seq is convergent and seq' is convergent, then $\lim(seq-seq') = \lim seq - \lim seq'$.

(27) If seq is convergent, then seq is bounded.

(28) If seq is convergent and seq' is convergent, then $seq \cdot seq'$ is convergent.

(29) If seq is convergent and seq' is convergent, then $\lim(seq \cdot seq') = (\lim seq) \cdot (\lim seq')$.

(30) If seq is convergent, then if $\lim seq \neq 0$, then there exists n such that for every m such that $n \leq m$ holds $\frac{|\lim seq|}{2} < |seq(m)|$.

(31) If seq is convergent and for every n holds $0 \leq seq(n)$, then $0 \leq \lim seq$.

(32) If seq is convergent and seq' is convergent and for every n holds $seq(n) \leq seq'(n)$, then $\lim seq \leq \lim seq'$.

(33) If seq is convergent and seq' is convergent and for every n holds $seq(n) \leq seq_1(n)$ and $seq_1(n) \leq seq'(n)$ and $\lim seq = \lim seq'$, then seq_1 is convergent.

(34) If seq is convergent and seq' is convergent and for every n holds $seq(n) \leq seq_1(n)$ and $seq_1(n) \leq seq'(n)$ and $\lim seq = \lim seq'$, then $\lim seq_1 = \lim seq$.

(35) If seq is convergent and $\lim seq \neq 0$ and seq is non-zero, then seq^{-1} is convergent.

(36) If seq is convergent and $\lim seq \neq 0$ and seq is non-zero, then $\lim seq^{-1} = (\lim seq)^{-1}$.

(37) If seq' is convergent and seq is convergent and $\lim seq \neq 0$ and seq is non-zero, then $\frac{seq'}{seq}$ is convergent.

(38) If seq' is convergent and seq is convergent and $\lim seq \neq 0$ and seq is non-zero, then $\lim \frac{seq'}{seq} = \frac{\lim seq'}{\lim seq}$.

(39) If seq is convergent and seq_1 is bounded and $\lim seq = 0$, then $seq \cdot seq_1$ is convergent.

(40) If seq is convergent and seq_1 is bounded and $\lim seq = 0$, then $\lim(seq \cdot seq_1) = 0$.

References

[1] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[2] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

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