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Non-contiguous Substrings and One-to-one Finite Sequences

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Summary. This text is a continuation of [3]. We prove a number of theorems concerning both notions introduced there and one-to-one finite sequences. We introduce a function that removes from a string elements of the string that belongs to a given set.

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The notation and terminology used here have been introduced in the following articles: [9], [8], [5], [3], [4], [7], [6], [1], [2], and [10]. For simplicity we follow a convention: p, q, r are finite sequences, u, v, x, y, z are arbitrary, i, j, k, l, m, n are natural numbers, A, X, Y are sets, and D is a non-empty set. The following propositions are true:

(1) $\mathrm{Seg}\,3 = \{1, 2, 3\}$.

(2) $\mathrm{Seg}\,4 = \{1, 2, 3, 4\}$.

(3) $\mathrm{Seg}\,5 = \{1, 2, 3, 4, 5\}$.

(4) $\mathrm{Seg}\,6 = \{1, 2, 3, 4, 5, 6\}$.

(5) $\mathrm{Seg}\,7 = \{1, 2, 3, 4, 5, 6, 7\}$.

(6) $\mathrm{Seg}\,8 = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

(7) $\mathrm{Seg}\,k = \emptyset$ if and only if $k \notin \mathrm{Seg}\,k$.

(8) $0 \notin \mathrm{Seg}\,k$.

(9) $k + 1 \notin \mathrm{Seg}\,k$.

(10) If $k \neq 0$, then $k \in \mathrm{Seg}(k + n)$.

(11) If $k + n \in \mathrm{Seg}\,k$, then $n = 0$.

(12) If $k \in \mathrm{Seg}\,n$ and $k < n$, then $k + 1 \in \mathrm{Seg}\,n$.

(13) If $k \in \mathrm{Seg}\,n$ and $m < k$, then $k - m \in \mathrm{Seg}\,n$.

(14) $k - n \in \mathrm{Seg}\,k$ if and only if $n < k$.

(15) $\mathrm{Seg}\,k$ misses $\{k + 1\}$.

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(16) $\mathrm{Seg}(k+1) \setminus \mathrm{Seg}\,k = \{k+1\}$.
(17) $\mathrm{Seg}\,k \neq \mathrm{Seg}(k+1)$.
(18) If $\mathrm{Seg}\,k = \mathrm{Seg}(k+n)$, then $n = 0$.
(19) $\mathrm{Seg}\,k \subseteq \mathrm{Seg}(k+n)$.
(20) $\mathrm{Seg}\,k \subseteq \mathrm{Seg}\,n$ or $\mathrm{Seg}\,n \subseteq \mathrm{Seg}\,k$.
(21) If $\mathrm{Seg}\,k = \emptyset$, then $k = 0$.
(22) If $\mathrm{Seg}\,k = \{y\}$, then $k = 1$ and $y = 1$.
(23) If $\mathrm{Seg}\,k = \{x, y\}$ and $x \neq y$, then $k = 2$ and $\{x, y\} = \{1, 2\}$.
(24) If $x \in \mathrm{dom}\,p$, then $x \in \mathrm{dom}(p \frown q)$.
(25) If $x \in \mathrm{dom}\,p$, then x is a natural number.
(26) If $x \in \mathrm{dom}\,p$, then $x \neq 0$.
(27) $n \in \mathrm{dom}\,p$ if and only if $1 \leq n$ and $n \leq \mathrm{len}\,p$.
(28) $n \in \mathrm{dom}\,p$ if and only if $n-1$ is a natural number and $\mathrm{len}\,p - n$ is a natural number.
(29) $\mathrm{dom}\langle x, y\rangle = \mathrm{Seg}\,2$.
(30) $\mathrm{dom}\langle x, y, z\rangle = \mathrm{Seg}\,3$.
(31) $\mathrm{len}\,p = \mathrm{len}\,q$ if and only if $\mathrm{dom}\,p = \mathrm{dom}\,q$.
(32) $\mathrm{len}\,p \leq \mathrm{len}\,q$ if and only if $\mathrm{dom}\,p \subseteq \mathrm{dom}\,q$.
(33) If $x \in \mathrm{rng}\,p$, then $1 \in \mathrm{dom}\,p$.
(34) If $\mathrm{rng}\,p \neq \emptyset$, then $1 \in \mathrm{dom}\,p$.
(35) $\mathrm{rng}\langle x, y\rangle = \{x, y\}$.
(36) $\mathrm{rng}\langle x, y, z\rangle = \{x, y, z\}$.
(37) $\varepsilon = \square$.
(38) $\varepsilon \neq \langle x, y\rangle$.
(39) $\varepsilon \neq \langle x, y, z\rangle$.
(40) $\langle x\rangle \neq \langle y, z\rangle$.
(41) $\langle u\rangle \neq \langle x, y, z\rangle$.
(42) $\langle u, v\rangle \neq \langle x, y, z\rangle$.
(43) If $\mathrm{len}\,r = \mathrm{len}\,p + \mathrm{len}\,q$ and for every k such that $k \in \mathrm{dom}\,p$ holds $r(k) = p(k)$ and for every k such that $k \in \mathrm{dom}\,q$ holds $r(\mathrm{len}\,p + k) = q(k)$, then $r = p \frown q$.
(44) If $A \subseteq \mathrm{Seg}\,k$, then $\mathrm{len}(\mathrm{Sgm}\,A) = \mathrm{card}\,A$.
(45) If $A \subseteq \mathrm{Seg}\,k$, then $\mathrm{dom}(\mathrm{Sgm}\,A) = \mathrm{Seg}(\mathrm{card}\,A)$.
(46) If $X \subseteq \mathrm{Seg}\,i$ and $k < l$ and $1 \leq n$ and $m \leq \mathrm{len}(\mathrm{Sgm}\,X)$ and $\mathrm{Sgm}\,X(m) = k$ and $\mathrm{Sgm}\,X(n) = l$, then $m < n$.
(47) If $X \subseteq \mathrm{Seg}\,i$ and $k \leq l$ and $1 \leq n$ and $m \leq \mathrm{len}(\mathrm{Sgm}\,X)$ and $\mathrm{Sgm}\,X(m) = k$ and $\mathrm{Sgm}\,X(n) = l$, then $m \leq n$.
(48) If $X \subseteq \mathrm{Seg}\,i$ and $Y \subseteq \mathrm{Seg}\,j$, then for all m, n such that $m \in X$ and $n \in Y$ holds $m < n$ if and only if $\mathrm{Sgm}(X \cup Y) = \mathrm{Sgm}\,X \frown \mathrm{Sgm}\,Y$.
(49) $\mathrm{Sgm}\,\emptyset = \varepsilon$.

(50) If $0 \neq n$, then $\mathrm{Sgm}\{n\} = \langle n \rangle$.

(51) If $0 < n$ and $n < m$, then $\mathrm{Sgm}\{n, m\} = \langle n, m \rangle$.

(52) $\mathrm{len}(\mathrm{Sgm}(\mathrm{Seg}\, k)) = k$.

(53) $\mathrm{Sgm}(\mathrm{Seg}(k + n)) \restriction \mathrm{Seg}\, k = \mathrm{Sgm}(\mathrm{Seg}\, k)$.

(54) $\mathrm{Sgm}(\mathrm{Seg}\, k) = \mathrm{id}_k$.

(55) $p \restriction \mathrm{Seg}\, n = p$ if and only if $\mathrm{len}\, p \leq n$.

(56) $\mathrm{id}_{n+k} \restriction \mathrm{Seg}\, n = \mathrm{id}_n$.

(57) $\mathrm{id}_n \restriction \mathrm{Seg}\, m = \mathrm{id}_m$ if and only if $m \leq n$.

(58) $\mathrm{id}_n \restriction \mathrm{Seg}\, m = \mathrm{id}_n$ if and only if $n \leq m$.

(59) If $\mathrm{len}\, p = k + l$ and $q = p \restriction \mathrm{Seg}\, k$, then $\mathrm{len}\, q = k$.

(60) If $\mathrm{len}\, p = k + l$ and $q = p \restriction \mathrm{Seg}\, k$, then $\mathrm{dom}\, q = \mathrm{Seg}\, k$.

(61) If $\mathrm{len}\, p = k + 1$ and $q = p \restriction \mathrm{Seg}\, k$, then $p = q \frown \langle p(k + 1) \rangle$.

(62) $p \restriction X$ is a finite sequence if and only if there exists k such that $X \cap \mathrm{dom}\, p = \mathrm{Seg}\, k$.

(63) $\mathrm{card}((p \frown q)^{-1} A) = \mathrm{card}(p^{-1} A) + \mathrm{card}(q^{-1} A)$.

(64) $p^{-1} A \subseteq (p \frown q)^{-1} A$.

Let us consider p, A. The functor $p - A$ yields a finite sequence and is defined by:

$p - A = p \cdot \mathrm{Sgm}(\mathrm{Seg}(\mathrm{len}\, p) \setminus p^{-1} A)$.

The following propositions are true:

(65) $p - A = p \cdot \mathrm{Sgm}(\mathrm{Seg}(\mathrm{len}\, p) \setminus p^{-1} A)$.

(66) $\mathrm{len}(p - A) = \mathrm{len}\, p - \mathrm{card}(p^{-1} A)$.

(67) $\mathrm{len}(p - A) \leq \mathrm{len}\, p$.

(68) If $\mathrm{len}(p - A) = \mathrm{len}\, p$, then A misses $\mathrm{rng}\, p$.

(69) If $n = \mathrm{len}\, p - \mathrm{card}(p^{-1} A)$, then $\mathrm{dom}(p - A) = \mathrm{Seg}\, n$.

(70) $\mathrm{dom}(p - A) \subseteq \mathrm{dom}\, p$.

(71) If $\mathrm{dom}(p - A) = \mathrm{dom}\, p$, then A misses $\mathrm{rng}\, p$.

(72) $\mathrm{rng}(p - A) = \mathrm{rng}\, p \setminus A$.

(73) $\mathrm{rng}(p - A) \subseteq \mathrm{rng}\, p$.

(74) If $\mathrm{rng}(p - A) = \mathrm{rng}\, p$, then A misses $\mathrm{rng}\, p$.

(75) $p - A = \varepsilon$ if and only if $\mathrm{rng}\, p \subseteq A$.

(76) $p - A = p$ if and only if A misses $\mathrm{rng}\, p$.

(77) $p - \{x\} = p$ if and only if $x \notin \mathrm{rng}\, p$.

(78) $p - \emptyset = p$.

(79) $p - \mathrm{rng}\, p = \varepsilon$.

(80) $p \frown q - A = (p - A) \frown (q - A)$.

(81) $\varepsilon - A = \varepsilon$.

(82) $\langle x \rangle - A = \langle x \rangle$ if and only if $x \notin A$.

(83) $\langle x \rangle - A = \varepsilon$ if and only if $x \in A$.

(84) $\langle x, y\rangle - A = \varepsilon$ if and only if $x \in A$ and $y \in A$.

(85) If $x \in A$ and $y \notin A$, then $\langle x, y\rangle - A = \langle y\rangle$.

(86) If $\langle x, y\rangle - A = \langle y\rangle$ and $x \neq y$, then $x \in A$ and $y \notin A$.

(87) If $x \notin A$ and $y \in A$, then $\langle x, y\rangle - A = \langle x\rangle$.

(88) If $\langle x, y\rangle - A = \langle x\rangle$ and $x \neq y$, then $x \notin A$ and $y \in A$.

(89) $\langle x, y\rangle - A = \langle x, y\rangle$ if and only if $x \notin A$ and $y \notin A$.

(90) If $\operatorname{len} p = k + 1$ and $q = p \restriction \operatorname{Seg} k$, then $p(k + 1) \in A$ if and only if $p - A = q - A$.

(91) If $\operatorname{len} p = k + 1$ and $q = p \restriction \operatorname{Seg} k$, then $p(k + 1) \notin A$ if and only if $p - A = (q - A) \frown \langle p(k + 1)\rangle$.

(92) If $n \in \operatorname{dom} p$, then $p(n) \in A$ or $(p - A)(n - \operatorname{card}\{k : k \in \operatorname{dom} p \wedge k \leq n \wedge p(k) \in A\}) = p(n)$.

(93) If p is a finite sequence of elements of D, then $p - A$ is a finite sequence of elements of D.

(94) If p is one-to-one, then $p - A$ is one-to-one.

(95) If p is one-to-one, then $\operatorname{len}(p - A) = \operatorname{len} p - \operatorname{card}(A \cap \operatorname{rng} p)$.

(96) If p is one-to-one and $A \subseteq \operatorname{rng} p$, then $\operatorname{len}(p - A) = \operatorname{len} p - \operatorname{card} A$.

(97) If p is one-to-one and $x \in \operatorname{rng} p$, then $\operatorname{len}(p - \{x\}) = \operatorname{len} p - 1$.

(98) $\operatorname{rng} p$ misses $\operatorname{rng} q$ and p is one-to-one and q is one-to-one if and only if $p \frown q$ is one-to-one.

(99) If $A \subseteq \operatorname{Seg} k$, then $\operatorname{Sgm} A$ is one-to-one.

(100) id_n is one-to-one.

(101) ε is one-to-one.

(102) $\langle x\rangle$ is one-to-one.

(103) $x \neq y$ if and only if $\langle x, y\rangle$ is one-to-one.

(104) $x \neq y$ and $y \neq z$ and $z \neq x$ if and only if $\langle x, y, z\rangle$ is one-to-one.

(105) If p is one-to-one and $\operatorname{rng} p = \{x\}$, then $\operatorname{len} p = 1$.

(106) If p is one-to-one and $\operatorname{rng} p = \{x\}$, then $p = \langle x\rangle$.

(107) If p is one-to-one and $\operatorname{rng} p = \{x, y\}$ and $x \neq y$, then $\operatorname{len} p = 2$.

(108) If p is one-to-one and $\operatorname{rng} p = \{x, y\}$ and $x \neq y$, then $p = \langle x, y\rangle$ or $p = \langle y, x\rangle$.

(109) If p is one-to-one and $\operatorname{rng} p = \{x, y, z\}$ and $\langle x, y, z\rangle$ is one-to-one, then $\operatorname{len} p = 3$.

(110) If p is one-to-one and $\operatorname{rng} p = \{x, y, z\}$ and $x \neq y$ and $y \neq z$ and $x \neq z$, then $\operatorname{len} p = 3$.

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