FORMALIZED MATHEMATICS
Vol.1, No.3, May–August 1990
Université Catholique de Louvain

Metric Spaces [1]

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Summary. In this paper we define the metric spaces. Two examples of metric spaces are given. We define the discrete metric and the metric on the real axis. Moreover the open ball, the close ball and the sphere in metric spaces are introduced. We also prove some theorems concerning these concepts.

MML Identifier: `METRIC_1`.

The papers [3], [7], [2], [1], [5], [6], and [4] provide the notation and terminology for this paper. We consider metric structures which are systems

$\langle$ a carrier, a distance $\rangle$

where the carrier is a non-empty set and the distance is a function from $[\!:$ the carrier, the carrier $:\!]$ into $\mathbb{R}$. In the sequel M will be a metric structure. Let us consider M. A point of M is an element of the carrier of M.

Next we state a proposition

(1) For every element x of the carrier of M holds x is a point of M.

Let us consider M, and let a, b be elements of the carrier of M. The functor $\rho(a,b)$ yielding a real number, is defined by:

$\rho(a,b) = (\text{the distance of } M)(a,b)$.

We now state a proposition

(2) For all elements x, y of the carrier of M holds $\rho(x,y) = (\text{the distance of } M)(x,y)$.

[1]Supported by RPBP.III.24-B.5

ISSN 0777–4028

In the sequel x will be arbitrary. Let us consider x. Then $\{x\}$ is a non-empty set.

The function $\{[\emptyset, \emptyset]\} \mapsto 0$ from $[: \{\emptyset\}, \{\emptyset\} :]$ into $\mathbb{R}$ is defined by:

$\{[\emptyset, \emptyset]\} \mapsto 0 = [: \{\emptyset\}, \{\emptyset\} :] \longmapsto 0$.

Next we state a proposition

(3) $\{[\emptyset, \emptyset]\} \mapsto 0 = [: \{\emptyset\}, \{\emptyset\} :] \longmapsto 0$.

A metric structure is said to be a metric space if:

for all elements a, b, c of the carrier of it holds $\rho(a, b) = 0$ if and only if $a = b$ but $\rho(a, b) = \rho(b, a)$ and $\rho(a, c) \leq \rho(a, b) + \rho(b, c)$.

We now state three propositions:

(4) For every M being a metric structure holds M is a metric space if and only if for all elements a, b, c of the carrier of M holds $\rho(a, b) = 0$ if and only if $a = b$ but $\rho(a, b) = \rho(b, a)$ and $\rho(a, c) \leq \rho(a, b) + \rho(b, c)$.

(5) For every metric space M and for all elements a, b of the carrier of M holds $\rho(a, b) = \rho(b, a)$.

(6) For every metric space M and for all elements a, b, c of the carrier of M holds $\rho(a, c) \leq \rho(a, b) + \rho(b, c)$.

In the sequel PM denotes a metric space and p_1, p_2 denote elements of the carrier of PM. Next we state a proposition

(7) $0 \leq \rho(p_1, p_2)$.

Let A be a non-empty set. The discrete metric of A yielding a function from $[: A, A :]$ into $\mathbb{R}$, is defined by:

for all elements x, y of A holds (the discrete metric of A)$(x, x) = 0$ but if $x \neq y$, then (the discrete metric of A)$(x, y) = 1$.

In the sequel A denotes a non-empty set and x, y denote elements of A. Next we state two propositions:

(8) (The discrete metric of A)$(x, x) = 0$.

(9) If $x \neq y$, then (the discrete metric of A)$(x, y) = 1$.

Let A be a non-empty set. The discrete space on A yielding a metric space, is defined as follows:

the discrete space on $A = \langle A, \text{the discrete metric of } A \rangle$.

In the sequel x will be an element of $\mathbb{R}$. Let us consider x. The functor $@x$ yielding a real number, is defined by:

$@x = x$.

Next we state a proposition

(10) $x = @x$.

The function $\rho_{\mathbb{R}}$ from $[: \mathbb{R}, \mathbb{R} :]$ into $\mathbb{R}$ is defined as follows:

for all elements x, y of $\mathbb{R}$ holds $\rho_{\mathbb{R}}(x, y) = |@x - @y|$.

Next we state several propositions:

(11) For every function F from $[: \mathbb{R}, \mathbb{R} :]$ into $\mathbb{R}$ holds $F = \rho_{\mathbb{R}}$ if and only if for all elements x, y of $\mathbb{R}$ holds $F(x, y) = |@x - @y|$.

(12) For all real numbers x, y holds $\rho_{\mathbb{R}}(x,y) = |x-y|$.

(13) For all elements x, y of $\mathbb{R}$ holds $\rho_{\mathbb{R}}(x,y) = 0$ if and only if $x = y$.

(14) For all elements x, y of $\mathbb{R}$ holds $\rho_{\mathbb{R}}(x,y) = \rho_{\mathbb{R}}(y,x)$.

(15) For all elements x, y, z of $\mathbb{R}$ holds $\rho_{\mathbb{R}}(x,y) \leq \rho_{\mathbb{R}}(x,z) + \rho_{\mathbb{R}}(z,y)$.

The metric space of real numbers a metric space is defined as follows:
the metric space of real numbers $= \langle \mathbb{R}, \rho_{\mathbb{R}} \rangle$.

Let M be a metric structure, and let p be an element of the carrier of M, and let r be a real number. The functor $\mathrm{Ball}(p,r)$ yielding a subset of the carrier of M, is defined as follows:
$\mathrm{Ball}(p,r) = \{q : \rho(p,q) < r\}$.

We now state a proposition

(16) For every M being a metric structure and for every element p of the carrier of M and for every real number r holds $\mathrm{Ball}(p,r) = \{q : \rho(p,q) < r\}$.

Let M be a metric structure, and let p be an element of the carrier of M, and let r be a real number. The functor $\overline{\mathrm{Ball}}(p,r)$ yields a subset of the carrier of M and is defined as follows:
$\overline{\mathrm{Ball}}(p,r) = \{q : \rho(p,q) \leq r\}$.

We now state a proposition

(17) For every M being a metric structure and for every element p of the carrier of M and for every real number r holds $\overline{\mathrm{Ball}}(p,r) = \{q : \rho(p,q) \leq r\}$.

Let M be a metric structure, and let p be an element of the carrier of M, and let r be a real number. The functor $\mathrm{Sphere}(p,r)$ yielding a subset of the carrier of M, is defined by:
$\mathrm{Sphere}(p,r) = \{q : \rho(p,q) = r\}$.

Next we state several propositions:

(18) For every M being a metric structure and for every element p of the carrier of M and for every real number r holds $\mathrm{Sphere}(p,r) = \{q : \rho(p,q) = r\}$.

(19) For every M being a metric structure and for all elements p, x of the carrier of M and for every real number r holds $x \in \mathrm{Ball}(p,r)$ if and only if $\rho(p,x) < r$.

(20) For every M being a metric structure and for all elements p, x of the carrier of M and for every real number r holds $x \in \overline{\mathrm{Ball}}(p,r)$ if and only if $\rho(p,x) \leq r$.

(21) For every M being a metric structure and for all elements p, x of the carrier of M and for every real number r holds $x \in \mathrm{Sphere}(p,r)$ if and only if $\rho(p,x) = r$.

(22) For every M being a metric structure and for every element p of the carrier of M and for every real number r holds $\mathrm{Ball}(p,r) \subseteq \overline{\mathrm{Ball}}(p,r)$.

(23) For every M being a metric structure and for every element p of the carrier of M and for every real number r holds $\mathrm{Sphere}(p,r) \subseteq \overline{\mathrm{Ball}}(p,r)$.

(24) For every M being a metric structure and for every element p of the carrier of M and for every real number r holds $\mathrm{Sphere}(p,r) \cup \mathrm{Ball}(p,r) = \overline{\mathrm{Ball}}(p,r)$.

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Received May 3, 1990