FORMALIZED MATHEMATICS
Vol.1, No.**3**, May–August 1990
Université Catholique de Louvain

Some Properties of Real Numbers [1]

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Summary. We define the following operations on real numbers: $max(x,y)$, $min(x,y)$, x^2, $\sqrt{x}$. We prove basic properties of introduced operations. A number of auxiliary theorems absent in [1] and [2] is proved.

MML Identifier: `SQUARE_1`.

The terminology and notation used here are introduced in the papers [1] and [2]. In the sequel a, b, x, y, z will be real numbers. Next we state a number of propositions:

(1) $1 < 2$.

(2) If $1 < x$, then $\frac{1}{x} < 1$.

(3) $\frac{1}{2} < 1$.

(4) $2^{-1} < 1$.

(5) $2 \cdot a = a + a$.

(6) $a = (a - x) + x$.

(7) $a = (a + x) - x$.

(8) If $x - y = 0$, then $x = y$.

(9) $x \leq y$ if and only if $z + x \leq z + y$.

(10) $a \leq a + 1$.

(11) If $x < y$, then $0 < y - x$.

(12) If $x \leq y$, then $0 \leq y - x$.

(13) $1^{-1} = 1$.

(14) $\frac{x}{1} = x$.

(15) $\frac{x+x}{2} = x$.

(16) If $x \neq 0$, then $\frac{1}{\frac{1}{x}} = x$.

[1] Supported by RPBP III.24 C1

ISSN 0777–4028

(17) If $y \neq 0$ and $z \neq 0$, then $\frac{x}{y \cdot z} = \frac{\frac{x}{y}}{z}$.

(18) If $z \neq 0$, then $x \cdot \frac{y}{z} = \frac{x \cdot y}{z}$.

(19) If $0 \leq x$ and $0 \leq y$, then $0 \leq x \cdot y$.

(20) If $x \leq 0$ and $y \leq 0$, then $0 \leq x \cdot y$.

(21) If $0 < x$ and $0 < y$, then $0 < x \cdot y$.

(22) If $x < 0$ and $y < 0$, then $0 < x \cdot y$.

(23) If $0 \leq x$ and $y \leq 0$, then $x \cdot y \leq 0$ and $y \cdot x \leq 0$.

(24) If $0 < x$ and $y < 0$, then $x \cdot y < 0$ and $y \cdot x < 0$.

(25) If $0 \leq x \cdot y$, then $0 \leq x$ and $0 \leq y$ or $x \leq 0$ and $y \leq 0$.

(26) If $0 < x \cdot y$, then $0 < x$ and $0 < y$ or $x < 0$ and $y < 0$.

(27) If $0 \leq a$ and $0 < b$, then $0 \leq \frac{a}{b}$.

(28) If $0 \leq x$, then $y - x \leq y$.

(29) If $0 < x$, then $y - x < y$.

(30) If $x \leq y$, then $z - y \leq z - x$.

The scheme *RealContinuity* deals with two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

there exists z such that for all x, y such that $\mathcal{P}[x]$ and $\mathcal{Q}[y]$ holds $x \leq z$ and $z \leq y$

provided the following requirements are met:

- there exists x such that $\mathcal{P}[x]$,
- there exists x such that $\mathcal{Q}[x]$,
- for all x, y such that $\mathcal{P}[x]$ and $\mathcal{Q}[y]$ holds $x \leq y$.

We now define two new functors. Let us consider x, y. The functor $\min(x, y)$ yields a real number and is defined by:

$\min(x, y) = x$ if $x \leq y$, $\min(x, y) = y$, otherwise.

The functor $\max(x, y)$ yielding a real number, is defined as follows:

$\max(x, y) = x$ if $y \leq x$, $\max(x, y) = y$, otherwise.

We now state a number of propositions:

(31) If $x \leq y$, then $z = x$ if and only if $z = \min(x, y)$ but $x \leq y$ or $z = y$ if and only if $z = \min(x, y)$.

(32) If $y \leq x$, then $\min(x, y) = y$.

(33) If $y \not\leq x$, then $\min(x, y) = x$.

(34) $\min(x, y) = \frac{(x+y)-|x-y|}{2}$.

(35) $\min(x, y) \leq x$ and $\min(y, x) \leq x$.

(36) $\min(x, x) = x$.

(37) $\min(x, y) = \min(y, x)$.

(38) $\min(x, y) = x$ or $\min(x, y) = y$.

(39) $x \leq y$ and $x \leq z$ if and only if $x \leq \min(y, z)$.

(40) $\min(x, \min(y, z)) = \min(\min(x, y), z)$.

(41) If $z < x$ and $z < y$, then $z < \min(x, y)$.

(42) If $y \leq x$, then $z = x$ if and only if $z = \max(x, y)$ but $y \leq x$ or $z = y$ if and only if $z = \max(x, y)$.

(43) If $x \leq y$, then $\max(x, y) = y$.

(44) If $x \not\leq y$, then $\max(x, y) = x$.

(45) $\max(x, y) = \frac{(x+y)+|x-y|}{2}$.

(46) $x \leq \max(x, y)$ and $x \leq \max(y, x)$.

(47) $\max(x, x) = x$.

(48) $\max(x, y) = \max(y, x)$.

(49) $\max(x, y) = x$ or $\max(x, y) = y$.

(50) $y \leq x$ and $z \leq x$ if and only if $\max(y, z) \leq x$.

(51) $\max(x, \max(y, z)) = \max(\max(x, y), z)$.

(52) If $0 < x$ and $0 < y$, then $0 < \max(x, y)$.

(53) $\min(x, y) + \max(x, y) = x + y$.

(54) $\max(x, \min(x, y)) = x$ and $\max(\min(x, y), x) = x$ and $\max(\min(y, x), x) = x$ and $\max(x, \min(y, x)) = x$.

(55) $\min(x, \max(x, y)) = x$ and $\min(\max(x, y), x) = x$ and $\min(\max(y, x), x) = x$ and $\min(x, \max(y, x)) = x$.

(56) $\min(x, \max(y, z)) = \max(\min(x, y), \min(x, z))$ and $\min(\max(y, z), x) = \max(\min(y, x), \min(z, x))$.

(57) $\max(x, \min(y, z)) = \min(\max(x, y), \max(x, z))$ and $\max(\min(y, z), x) = \min(\max(y, x), \max(z, x))$.

Let us consider x. The functor $x^{\mathbf{2}}$ yields an element of $\mathbb{R}$ and is defined by: $x^{\mathbf{2}} = x \cdot x$.

The following proposition is true

(58) $x^{\mathbf{2}} = x \cdot x$.

Let us consider a. Then $a^{\mathbf{2}}$ is a real number.

The following propositions are true:

(59) $1^{\mathbf{2}} = 1$.

(60) $0^{\mathbf{2}} = 0$.

(61) $a^{\mathbf{2}} = (-a)^{\mathbf{2}}$.

(62) $|a|^{\mathbf{2}} = a^{\mathbf{2}}$.

(63) $(a + b)^{\mathbf{2}} = (a^{\mathbf{2}} + (2 \cdot a) \cdot b) + b^{\mathbf{2}}$.

(64) $(a - b)^{\mathbf{2}} = (a^{\mathbf{2}} - (2 \cdot a) \cdot b) + b^{\mathbf{2}}$.

(65) $(a + 1)^{\mathbf{2}} = (a^{\mathbf{2}} + 2 \cdot a) + 1$.

(66) $(a - 1)^{\mathbf{2}} = (a^{\mathbf{2}} - 2 \cdot a) + 1$.

(67) $(a - b) \cdot (a + b) = a^{\mathbf{2}} - b^{\mathbf{2}}$ and $(a + b) \cdot (a - b) = a^{\mathbf{2}} - b^{\mathbf{2}}$.

(68) $(a \cdot b)^{\mathbf{2}} = a^{\mathbf{2}} \cdot b^{\mathbf{2}}$.

(69) If $0 \neq b$, then $\frac{a}{b}^{\mathbf{2}} = \frac{a^{\mathbf{2}}}{b^{\mathbf{2}}}$.

(70) If $a^{\mathbf{2}} - b^{\mathbf{2}} \neq 0$, then $\frac{1}{a+b} = \frac{a-b}{a^{\mathbf{2}}-b^{\mathbf{2}}}$.

(71) If $a^{\mathbf{2}} - b^{\mathbf{2}} \neq 0$, then $\frac{1}{a-b} = \frac{a+b}{a^{\mathbf{2}}-b^{\mathbf{2}}}$.

(72) $0 \leq a^{\mathbf{2}}$.

(73) If $a^{\mathbf{2}} = 0$, then $a = 0$.

(74) If $0 \neq a$, then $0 < a^{\mathbf{2}}$.

(75) If $0 < a$ and $a < 1$, then $a^{\mathbf{2}} < a$.

(76) If $1 < a$, then $a < a^{\mathbf{2}}$.

(77) If $0 \leq x$ and $x \leq y$, then $x^{\mathbf{2}} \leq y^{\mathbf{2}}$.

(78) If $0 \leq x$ and $x < y$, then $x^{\mathbf{2}} < y^{\mathbf{2}}$.

(79) If $0 \leq x$ and $0 \leq y$ and $x^{\mathbf{2}} \leq y^{\mathbf{2}}$, then $x \leq y$.

(80) If $0 \leq x$ and $0 \leq y$ and $x^{\mathbf{2}} < y^{\mathbf{2}}$, then $x < y$.

Let us consider a. Let us assume that $0 \leq a$. The functor $\sqrt{a}$ yielding a real number, is defined by:

$0 \leq \sqrt{a}$ and $\sqrt{a}^{\mathbf{2}} = a$.

We now state a number of propositions:

(81) If $0 \leq a$, then for every b holds $b = \sqrt{a}$ if and only if $0 \leq b$ and $b^{\mathbf{2}} = a$.

(82) $\sqrt{0} = 0$.

(83) $\sqrt{1} = 1$.

(84) $1 < \sqrt{2}$.

(85) $\sqrt{4} = 2$.

(86) $\sqrt{2} < 2$.

(87) If $0 \leq a$, then $0 \leq \sqrt{a}$.

(88) If $0 \leq a$, then $\sqrt{a}^{\mathbf{2}} = a$.

(89) If $0 \leq a$, then $\sqrt{a^{\mathbf{2}}} = a$.

(90) If $a \leq 0$, then $\sqrt{a^{\mathbf{2}}} = -a$.

(91) $\sqrt{a^{\mathbf{2}}} = |a|$.

(92) If $0 \leq a$ and $\sqrt{a} = 0$, then $a = 0$.

(93) If $0 < a$, then $0 < \sqrt{a}$.

(94) If $0 \leq x$ and $x \leq y$, then $\sqrt{x} \leq \sqrt{y}$.

(95) If $0 \leq x$ and $x < y$, then $\sqrt{x} < \sqrt{y}$.

(96) If $0 \leq x$ and $0 \leq y$ and $\sqrt{x} = \sqrt{y}$, then $x = y$.

(97) If $0 \leq a$ and $0 \leq b$, then $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$.

(98) If $0 \leq a \cdot b$, then $\sqrt{a \cdot b} = \sqrt{|a|} \cdot \sqrt{|b|}$.

(99) If $0 \leq a$ and $0 < b$, then $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.

(100) If $0 < \frac{a}{b}$ and $b \neq 0$, then $\sqrt{\frac{a}{b}} = \frac{\sqrt{|a|}}{\sqrt{|b|}}$.

(101) If $0 < a$, then $\sqrt{\frac{1}{a}} = \frac{1}{\sqrt{a}}$.

(102) If $0 < a$, then $\frac{\sqrt{a}}{a} = \frac{1}{\sqrt{a}}$.

(103) If $0 < a$, then $\frac{a}{\sqrt{a}} = \sqrt{a}$.

(104) If $0 \leq a$ and $0 \leq b$, then $(\sqrt{a} - \sqrt{b}) \cdot (\sqrt{a} + \sqrt{b}) = a - b$.

(105) If $0 \leq a$ and $0 \leq b$ and $a \neq b$, then $\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{a-b}$.

(106) If $0 \leq b$ and $b < a$, then $\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{a-b}$.

(107) If $0 \leq a$ and $0 \leq b$ and $a \neq b$, then $\frac{1}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{a}+\sqrt{b}}{a-b}$.

(108) If $0 \leq b$ and $b < a$, then $\frac{1}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{a}+\sqrt{b}}{a-b}$.

References

[1] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[2] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

Received November 16, 1989