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Classical Configurations in Affine Planes [1]

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Summary. The classical sequence of implications which hold between Desargues and Pappus Axioms id proved. Formally Minor and Major Desargues Axiom (as suitable properties - predicates - of an affine plane) together with all its indirect forms are introduced; the same procedure is applied to Pappus Axioms. The so called Trapezium Desargues Axiom is also considered.

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The articles [1], and [2] provide the notation and terminology for this paper. We follow the rules: AP will denote an affine plane, a, a', b, b', c, c', o will denote elements of the points of AP, and A, C, K, M, N, P will denote subsets of the points of AP. Let us consider AP. We say that AP satisfies **PPAP** if and only if:

Given M, N, a, b, c, a', b', c'. Then if M is a line and N is a line and $a \in M$ and $b \in M$ and $c \in M$ and $a' \in N$ and $b' \in N$ and $c' \in N$ and $a, b' \parallel b, a'$ and $b, c' \parallel c, b'$, then $a, c' \parallel c, a'$.

We now state the proposition

(1) Given AP. Then AP satisfies **PPAP** if and only if for all M, N, a, b, c, a', b', c' such that M is a line and N is a line and $a \in M$ and $b \in M$ and $c \in M$ and $a' \in N$ and $b' \in N$ and $c' \in N$ and $a, b' \parallel b, a'$ and $b, c' \parallel c, b'$ holds $a, c' \parallel c, a'$.

Let us consider AP. We say that AP satisfies **PAP** if and only if:

Given M, N, o, a, b, c, a', b', c'. Suppose that

(i) M is a line,
(ii) N is a line,
(iii) $M \neq N$,
(iv) $o \in M$,

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(v) $o \in N$,
(vi) $o \neq a$,
(vii) $o \neq a'$,
(viii) $o \neq b$,
(ix) $o \neq b'$,
(x) $o \neq c$,
(xi) $o \neq c'$,
(xii) $a \in M$,
(xiii) $b \in M$,
(xiv) $c \in M$,
(xv) $a' \in N$,
(xvi) $b' \in N$,
(xvii) $c' \in N$,
(xviii) $a, b' \parallel b, a'$,
(xix) $b, c' \parallel c, b'$.

Then $a, c' \parallel c, a'$.

The following proposition is true

(2) Given AP. Then AP satisfies **PAP** if and only if for all M, N, o, a, b, c, a', b', c' such that M is a line and N is a line and $M \neq N$ and $o \in M$ and $o \in N$ and $o \neq a$ and $o \neq a'$ and $o \neq b$ and $o \neq b'$ and $o \neq c$ and $o \neq c'$ and $a \in M$ and $b \in M$ and $c \in M$ and $a' \in N$ and $b' \in N$ and $c' \in N$ and $a, b' \parallel b, a'$ and $b, c' \parallel c, b'$ holds $a, c' \parallel c, a'$.

Let us consider AP. We say that AP satisfies $\mathbf{PAP_1}$ if and only if:

Given M, N, o, a, b, c, a', b', c'. Suppose that

(i) M is a line,
(ii) N is a line,
(iii) $M \neq N$,
(iv) $o \in M$,
(v) $o \in N$,
(vi) $o \neq a$,
(vii) $o \neq a'$,
(viii) $o \neq b$,
(ix) $o \neq b'$,
(x) $o \neq c$,
(xi) $o \neq c'$,
(xii) $a \in M$,
(xiii) $b \in M$,
(xiv) $c \in M$,
(xv) $b' \in N$,
(xvi) $c' \in N$,
(xvii) $a, b' \parallel b, a'$,
(xviii) $b, c' \parallel c, b'$,
(xix) $a, c' \parallel c, a'$,
(xx) $b \neq c$.

Then $a' \in N$.

One can prove the following proposition

(3) Given AP. Then AP satisfies $\mathbf{PAP_1}$ if and only if for all M, N, o, a, b, c, a', b', c' such that M is a line and N is a line and $M \neq N$ and $o \in M$ and $o \in N$ and $o \neq a$ and $o \neq a'$ and $o \neq b$ and $o \neq b'$ and $o \neq c$ and $o \neq c'$ and $a \in M$ and $b \in M$ and $c \in M$ and $b' \in N$ and $c' \in N$ and $a, b' \parallel b, a'$ and $b, c' \parallel c, b'$ and $a, c' \parallel c, a'$ and $b \neq c$ holds $a' \in N$.

Let us consider AP. We say that AP satisfies **DES** if and only if:

Given A, P, C, o, a, b, c, a', b', c'. Suppose that

(i) $o \in A$,
(ii) $o \in P$,
(iii) $o \in C$,
(iv) $o \neq a$,
(v) $o \neq b$,
(vi) $o \neq c$,
(vii) $a \in A$,
(viii) $a' \in A$,
(ix) $b \in P$,
(x) $b' \in P$,
(xi) $c \in C$,
(xii) $c' \in C$,
(xiii) A is a line,
(xiv) P is a line,
(xv) C is a line,
(xvi) $A \neq P$,
(xvii) $A \neq C$,
(xviii) $a, b \parallel a', b'$,
(xix) $a, c \parallel a', c'$.

Then $b, c \parallel b', c'$.

We now state the proposition

(4) Given AP. Then AP satisfies **DES** if and only if for all A, P, C, o, a, b, c, a', b', c' such that $o \in A$ and $o \in P$ and $o \in C$ and $o \neq a$ and $o \neq b$ and $o \neq c$ and $a \in A$ and $a' \in A$ and $b \in P$ and $b' \in P$ and $c \in C$ and $c' \in C$ and A is a line and P is a line and C is a line and $A \neq P$ and $A \neq C$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$ holds $b, c \parallel b', c'$.

Let us consider AP. We say that AP satisfies $\mathbf{DES_1}$ if and only if:

Given A, P, C, o, a, b, c, a', b', c'. Suppose that

(i) $o \in A$,
(ii) $o \in P$,
(iii) $o \neq a$,
(iv) $o \neq b$,
(v) $o \neq c$,
(vi) $a \in A$,
(vii) $a' \in A$,

(viii) $b \in P$,
(ix) $b' \in P$,
(x) $c \in C$,
(xi) $c' \in C$,
(xii) A is a line,
(xiii) P is a line,
(xiv) C is a line,
(xv) $A \neq P$,
(xvi) $A \neq C$,
(xvii) $a, b \parallel a', b'$,
(xviii) $a, c \parallel a', c'$,
(xix) $b, c \parallel b', c'$,
(xx) not $\mathbf{L}(a, b, c)$,
(xxi) $c \neq c'$.

Then $o \in C$.

One can prove the following proposition

(5) Given AP. Then AP satisfies $\mathbf{DES_1}$ if and only if for all A, P, C, o, a, b, c, a', b', c' such that $o \in A$ and $o \in P$ and $o \neq a$ and $o \neq b$ and $o \neq c$ and $a \in A$ and $a' \in A$ and $b \in P$ and $b' \in P$ and $c \in C$ and $c' \in C$ and A is a line and P is a line and C is a line and $A \neq P$ and $A \neq C$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$ and $b, c \parallel b', c'$ and not $\mathbf{L}(a, b, c)$ and $c \neq c'$ holds $o \in C$.

Let us consider AP. We say that AP satisfies $\mathbf{DES_2}$ if and only if:

Given A, P, C, o, a, b, c, a', b', c'. Suppose that

(i) $o \in A$,
(ii) $o \in P$,
(iii) $o \in C$,
(iv) $o \neq a$,
(v) $o \neq b$,
(vi) $o \neq c$,
(vii) $a \in A$,
(viii) $a' \in A$,
(ix) $b \in P$,
(x) $b' \in P$,
(xi) $c \in C$,
(xii) A is a line,
(xiii) P is a line,
(xiv) C is a line,
(xv) $A \neq P$,
(xvi) $A \neq C$,
(xvii) $a, b \parallel a', b'$,
(xviii) $a, c \parallel a', c'$,
(xix) $b, c \parallel b', c'$.

Then $c' \in C$.

One can prove the following proposition

(6) Given AP. Then AP satisfies $\mathbf{DES_2}$ if and only if for all A, P, C, o, a, b, c, a', b', c' such that $o \in A$ and $o \in P$ and $o \in C$ and $o \neq a$ and $o \neq b$ and $o \neq c$ and $a \in A$ and $a' \in A$ and $b \in P$ and $b' \in P$ and $c \in C$ and A is a line and P is a line and C is a line and $A \neq P$ and $A \neq C$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$ and $b, c \parallel b', c'$ holds $c' \in C$.

Let us consider AP. We say that AP satisfies **TDES** if and only if:
Given K, o, a, b, c, a', b', c'. Suppose that

(i) K is a line,
(ii) $o \in K$,
(iii) $c \in K$,
(iv) $c' \in K$,
(v) $a \notin K$,
(vi) $o \neq c$,
(vii) $a \neq b$,
(viii) $\mathbf{L}(o, a, a')$,
(ix) $\mathbf{L}(o, b, b')$,
(x) $a, b \parallel a', b'$,
(xi) $a, c \parallel a', c'$,
(xii) $a, b \parallel K$.

Then $b, c \parallel b', c'$.

We now state the proposition

(7) Given AP. Then AP satisfies **TDES** if and only if for all K, o, a, b, c, a', b', c' such that K is a line and $o \in K$ and $c \in K$ and $c' \in K$ and $a \notin K$ and $o \neq c$ and $a \neq b$ and $\mathbf{L}(o, a, a')$ and $\mathbf{L}(o, b, b')$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$ and $a, b \parallel K$ holds $b, c \parallel b', c'$.

Let us consider AP. We say that AP satisfies $\mathbf{TDES_1}$ if and only if:
Given K, o, a, b, c, a', b', c'. Suppose that

(i) K is a line,
(ii) $o \in K$,
(iii) $c \in K$,
(iv) $c' \in K$,
(v) $a \notin K$,
(vi) $o \neq c$,
(vii) $a \neq b$,
(viii) $\mathbf{L}(o, a, a')$,
(ix) $a, b \parallel a', b'$,
(x) $b, c \parallel b', c'$,
(xi) $a, c \parallel a', c'$,
(xii) $a, b \parallel K$.

Then $\mathbf{L}(o, b, b')$.

One can prove the following proposition

(8) Given AP. Then AP satisfies $\mathbf{TDES_1}$ if and only if for all K, o, a, b, c, a', b', c' such that K is a line and $o \in K$ and $c \in K$ and $c' \in K$ and $a \notin K$ and $o \neq c$ and $a \neq b$ and $\mathbf{L}(o, a, a')$ and $a, b \parallel a', b'$ and $b, c \parallel b', c'$ and $a, c \parallel a', c'$ and $a, b \parallel K$ holds $\mathbf{L}(o, b, b')$.

Let us consider AP. We say that AP satisfies $\mathbf{TDES_2}$ if and only if:

Given K, o, a, b, c, a', b', c'. Suppose that

(i) K is a line,
(ii) $o \in K$,
(iii) $c \in K$,
(iv) $c' \in K$,
(v) $a \notin K$,
(vi) $o \neq c$,
(vii) $a \neq b$,
(viii) $\mathbf{L}(o, a, a')$,
(ix) $\mathbf{L}(o, b, b')$,
(x) $b, c \parallel b', c'$,
(xi) $a, c \parallel a', c'$,
(xii) $a, b \parallel K$.

Then $a, b \parallel a', b'$.

The following proposition is true

(9) Given AP. Then AP satisfies $\mathbf{TDES_2}$ if and only if for all K, o, a, b, c, a', b', c' such that K is a line and $o \in K$ and $c \in K$ and $c' \in K$ and $a \notin K$ and $o \neq c$ and $a \neq b$ and $\mathbf{L}(o, a, a')$ and $\mathbf{L}(o, b, b')$ and $b, c \parallel b', c'$ and $a, c \parallel a', c'$ and $a, b \parallel K$ holds $a, b \parallel a', b'$.

Let us consider AP. We say that AP satisfies $\mathbf{TDES_3}$ if and only if:

Given K, o, a, b, c, a', b', c'. Suppose that

(i) K is a line,
(ii) $o \in K$,
(iii) $c \in K$,
(iv) $a \notin K$,
(v) $o \neq c$,
(vi) $a \neq b$,
(vii) $\mathbf{L}(o, a, a')$,
(viii) $\mathbf{L}(o, b, b')$,
(ix) $a, b \parallel a', b'$,
(x) $a, c \parallel a', c'$,
(xi) $b, c \parallel b', c'$,
(xii) $a, b \parallel K$.

Then $c' \in K$.

We now state the proposition

(10) Given AP. Then AP satisfies $\mathbf{TDES_3}$ if and only if for all K, o, a, b, c, a', b', c' such that K is a line and $o \in K$ and $c \in K$ and $a \notin K$ and $o \neq c$ and $a \neq b$ and $\mathbf{L}(o, a, a')$ and $\mathbf{L}(o, b, b')$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$ and $b, c \parallel b', c'$ and $a, b \parallel K$ holds $c' \in K$.

Let us consider AP. We say that AP satisfies **des** if and only if:

Given A, P, C, a, b, c, a', b', c'. Suppose that

(i) $A \parallel P$,
(ii) $A \parallel C$,
(iii) $a \in A$,
(iv) $a' \in A$,
(v) $b \in P$,
(vi) $b' \in P$,
(vii) $c \in C$,
(viii) $c' \in C$,
(ix) A is a line,
(x) P is a line,
(xi) C is a line,
(xii) $A \neq P$,
(xiii) $A \neq C$,
(xiv) $a, b \parallel a', b'$,
(xv) $a, c \parallel a', c'$.

Then $b, c \parallel b', c'$.

The following proposition is true

(11) Given AP. Then AP satisfies **des** if and only if for all A, P, C, a, b, c, a', b', c' such that $A \parallel P$ and $A \parallel C$ and $a \in A$ and $a' \in A$ and $b \in P$ and $b' \in P$ and $c \in C$ and $c' \in C$ and A is a line and P is a line and C is a line and $A \neq P$ and $A \neq C$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$ holds $b, c \parallel b', c'$.

Let us consider AP. We say that AP satisfies $\mathbf{des_1}$ if and only if:

Given A, P, C, a, b, c, a', b', c'. Suppose that

(i) $A \parallel P$,
(ii) $a \in A$,
(iii) $a' \in A$,
(iv) $b \in P$,
(v) $b' \in P$,
(vi) $c \in C$,
(vii) $c' \in C$,
(viii) A is a line,
(ix) P is a line,
(x) C is a line,
(xi) $A \neq P$,
(xii) $A \neq C$,
(xiii) $a, b \parallel a', b'$,
(xiv) $a, c \parallel a', c'$,
(xv) $b, c \parallel b', c'$,
(xvi) not $\mathbf{L}(a, b, c)$,
(xvii) $c \neq c'$.

Then $A \parallel C$.

The following proposition is true

(12) Given AP. Then AP satisfies $\mathbf{des_1}$ if and only if for all A, P, C, a, b, c, a', b', c' such that $A \parallel P$ and $a \in A$ and $a' \in A$ and $b \in P$ and $b' \in P$ and $c \in C$ and $c' \in C$ and A is a line and P is a line and C is a line and $A \neq P$ and $A \neq C$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$ and $b, c \parallel b', c'$ and not $\mathbf{L}(a, b, c)$ and $c \neq c'$ holds $A \parallel C$.

Let us consider AP. We say that AP satisfies **pap** if and only if:

Given M, N, a, b, c, a', b', c'. Suppose M is a line and N is a line and $a \in M$ and $b \in M$ and $c \in M$ and $M \parallel N$ and $M \neq N$ and $a' \in N$ and $b' \in N$ and $c' \in N$ and $a, b' \parallel b, a'$ and $b, c' \parallel c, b'$. Then $a, c' \parallel c, a'$.

The following proposition is true

(13) Given AP. Then AP satisfies **pap** if and only if for all M, N, a, b, c, a', b', c' such that M is a line and N is a line and $a \in M$ and $b \in M$ and $c \in M$ and $M \parallel N$ and $M \neq N$ and $a' \in N$ and $b' \in N$ and $c' \in N$ and $a, b' \parallel b, a'$ and $b, c' \parallel c, b'$ holds $a, c' \parallel c, a'$.

Let us consider AP. We say that AP satisfies $\mathbf{pap_1}$ if and only if:

Given M, N, a, b, c, a', b', c'. Suppose that

(i) M is a line,
(ii) N is a line,
(iii) $a \in M$,
(iv) $b \in M$,
(v) $c \in M$,
(vi) $M \parallel N$,
(vii) $M \neq N$,
(viii) $a' \in N$,
(ix) $b' \in N$,
(x) $a, b' \parallel b, a'$,
(xi) $b, c' \parallel c, b'$,
(xii) $a, c' \parallel c, a'$,
(xiii) $a' \neq b'$.

Then $c' \in N$.

We now state a number of propositions:

(14) Given AP. Then AP satisfies $\mathbf{pap_1}$ if and only if for all M, N, a, b, c, a', b', c' such that M is a line and N is a line and $a \in M$ and $b \in M$ and $c \in M$ and $M \parallel N$ and $M \neq N$ and $a' \in N$ and $b' \in N$ and $a, b' \parallel b, a'$ and $b, c' \parallel c, b'$ and $a, c' \parallel c, a'$ and $a' \neq b'$ holds $c' \in N$.

(15) AP satisfies **PAP** if and only if AP satisfies $\mathbf{PAP_1}$.

(16) AP satisfies **DES** if and only if AP satisfies $\mathbf{DES_1}$.

(17) If AP satisfies **TDES**, then AP satisfies $\mathbf{TDES_1}$.

(18) If AP satisfies $\mathbf{TDES_1}$, then AP satisfies $\mathbf{TDES_2}$.

(19) If AP satisfies $\mathbf{TDES_2}$, then AP satisfies $\mathbf{TDES_3}$.

(20) If AP satisfies $\mathbf{TDES_3}$, then AP satisfies **TDES**.

(21) AP satisfies **des** if and only if AP satisfies $\mathbf{des_1}$.

(22) AP satisfies **pap** if and only if AP satisfies $\mathbf{pap_1}$.

(23) If AP satisfies **PAP**, then AP satisfies **pap**.

(24) AP satisfies **PPAP** if and only if AP satisfies **PAP** and AP satisfies **pap**.

(25) If AP satisfies **PAP**, then AP satisfies **DES**.

(26) If AP satisfies **DES**, then AP satisfies **TDES**.

(27) If AP satisfies $\mathbf{TDES_1}$, then AP satisfies $\mathbf{des_1}$.

(28) If AP satisfies **TDES**, then AP satisfies **des**.

(29) If AP satisfies **des**, then AP satisfies **pap**.

References

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