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Subcategories and Products of Categories

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Summary. The *subcategory* of a category and product of categories is defined. The *inclusion functor* is the injection (inclusion) map $\overset{E}{\hookrightarrow}$ which sends each object and each arrow of a Subcategory E of a category C to itself (in C). The inclusion functor is faithful. *Full subcategories* of C, that is, those subcategories E of C such that $\mathrm{Hom}_E(a,b) = \mathrm{Hom}_C(b,b)$ for any objects a, b of E, are defined. A subcategory E of C is full when the inclusion functor $\overset{E}{\hookrightarrow}$ is full. The proposition that a full subcategory is determined by giving the set of objects of a category is proved. The product of two categories B and C is constructed in the usual way. Moreover, some simple facts on *bifunctors* (functors from a product category) are proved. The final notions in this article are that of projection functors and product of two functors (*complex* functors and *product* functors).

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The terminology and notation used in this paper have been introduced in the following articles: [10], [8], [3], [4], [7], [2], [6], [1], [11], [9], and [5]. For simplicity we follow the rules: X denotes a set, C, D, E denote non-empty sets, c denotes an element of C, and d denotes an element of D. Let us consider D, X, E, and let F be a non-empty set of functions from X to E, and let f be a function from D into F, and let d be an element of D. Then $f(d)$ is an element of F.

In the sequel f denotes a function from $[\!:C, D:\!]$ into E. The following propositions are true:

(1) curry f is a function from C into E^D.

(2) curry$'$ f is a function from D into E^C.

Let us consider C, D, E, f. Then curry f is a function from C into E^D. Then curry$'$ f is a function from D into E^C.

The following two propositions are true:

(3) $f(\langle c, d\rangle) = (\mathrm{curry}\, f(c))(d)$.

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(4) $f(\langle c, d\rangle) = (\mathrm{curry}' f(d))(c)$.

In the sequel B, C, D, C', D' denote categories. Let us consider B, C, and let c be an object of C. The functor $B \longmapsto c$ yielding a functor from B to C is defined as follows:

$B \longmapsto c =$ (the morphisms of B) $\longmapsto \mathrm{id}_c$.

One can prove the following propositions:

(5) For every object c of C holds $B \longmapsto c =$ (the morphisms of B) $\longmapsto \mathrm{id}_c$.

(6) For every object c of C and for every morphism f of B holds $(B \longmapsto c)(f) = \mathrm{id}_c$.

(7) For every object c of C and for every object b of B holds $(\mathrm{Obj}(B \longmapsto c))(b) = c$.

Let us consider C, D. The functor $\mathrm{Funct}(C, D)$ yields a non-empty set and is defined by:

for an arbitrary x holds $x \in \mathrm{Funct}(C, D)$ if and only if x is a functor from C to D.

Next we state two propositions:

(8) For every non-empty set F holds $F = \mathrm{Funct}(C, D)$ if and only if for an arbitrary x holds $x \in F$ if and only if x is a functor from C to D.

(9) For every element T of $\mathrm{Funct}(C, D)$ holds T is a functor from C to D.

Let us consider C, D. A non-empty set is called a non-empty set of functors from C into D if:

for every element x of it holds x is a functor from C to D.

The following proposition is true

(10) For every non-empty set F holds F is a non-empty set of functors from C into D if and only if for every element x of F holds x is a functor from C to D.

Let us consider C, D, and let F be a non-empty set of functors from C into D. We see that it makes sense to consider the following mode for restricted scopes of arguments. Then all the objests of the mode element of F are a functor from C to D.

Let A be a non-empty set, and let us consider C, D, and let F be a non-empty set of functors from C into D, and let T be a function from A into F, and let x be an element of A. Then $T(x)$ is an element of F.

Let us consider C, D. Then $\mathrm{Funct}(C, D)$ is a non-empty set of functors from C into D.

Let us consider C. A category is said to be a subcategory of C if:

(i) the objects of it $\subseteq$ the objects of C,

(ii) for all objects a, b of it and for all objects a', b' of C such that $a = a'$ and $b = b'$ holds $\hom(a, b) \subseteq \hom(a', b')$,

(iii) the composition of it $\leq$ the composition of C,

(iv) for every object a of it and for every object a' of C such that $a = a'$ holds $\mathrm{id}_a = \mathrm{id}_{a'}$.

Next we state the proposition

(11) Given C, D. Then D is a subcategory of C if and only if the following conditions are satisfied:

(i) the objects of $D \subseteq$ the objects of C,

(ii) for all objects a, b of D and for all objects a', b' of C such that $a = a'$ and $b = b'$ holds $\hom(a, b) \subseteq \hom(a', b')$,

(iii) the composition of $D \leq$ the composition of C,

(iv) for every object a of D and for every object a' of C such that $a = a'$ holds $\mathrm{id}_a = \mathrm{id}_{a'}$.

In the sequel E will be a subcategory of C. We now state several propositions:

(12) For every object e of E holds e is an object of C.

(13) The morphisms of $E \subseteq$ the morphisms of C.

(14) For every morphism f of E holds f is a morphism of C.

(15) For every morphism f of E and for every morphism f' of C such that $f = f'$ holds $\operatorname{dom} f = \operatorname{dom} f'$ and $\operatorname{cod} f = \operatorname{cod} f'$.

(16) For all objects a, b of E and for all objects a', b' of C and for every morphism f from a to b such that $a = a'$ and $b = b'$ and $\hom(a, b) \neq \emptyset$ holds f is a morphism from a' to b'.

(17) For all morphisms f, g of E and for all morphisms f', g' of C such that $f = f'$ and $g = g'$ and $\operatorname{dom} g = \operatorname{cod} f$ holds $g \cdot f = g' \cdot f'$.

(18) C is a subcategory of C.

(19) id_E is a functor from E to C.

Let us consider C, E. The functor $\overset{E}{\hookrightarrow}$ yielding a functor from E to C is defined as follows:

$\overset{E}{\hookrightarrow} = \mathrm{id}_E$.

The following propositions are true:

(20) $\overset{E}{\hookrightarrow} = \mathrm{id}_E$.

(21) For every morphism f of E holds $\overset{E}{\hookrightarrow}(f) = f$.

(22) For every object a of E holds $(\mathrm{Obj} \overset{E}{\hookrightarrow})(a) = a$.

(23) For every object a of E holds $\overset{E}{\hookrightarrow}(a) = a$.

(24) $\overset{E}{\hookrightarrow}$ is faithful.

(25) $\overset{E}{\hookrightarrow}$ is full if and only if for all objects a, b of E and for all objects a', b' of C such that $a = a'$ and $b = b'$ holds $\hom(a, b) = \hom(a', b')$.

Let C be a category structure, and let us consider D. We say that C is full subcategory of D if and only if:

C is a subcategory of D and for all objects a, b of C and for all objects a', b' of D such that $a = a'$ and $b = b'$ holds $\hom(a, b) = \hom(a', b')$.

The following propositions are true:

(26) For every C being a category structure and for every D holds C is full subcategory of D if and only if C is a subcategory of D and for all objects

a, b of C and for all objects a', b' of D such that $a = a'$ and $b = b'$ holds $\hom(a, b) = \hom(a', b')$.

(27) E is full subcategory of C if and only if $\overset{E}{\hookrightarrow}$ is full.

(28) For every non-empty subset O of the objects of C holds $\bigcup\{\hom(a, b) : a \in O \wedge b \in O\}$ is a non-empty subset of the morphisms of C.

(29) Let O be a non-empty subset of the objects of C. Let M be a non-empty set. Suppose $M = \bigcup\{\hom(a, b) : a \in O \wedge b \in O\}$. Then (the dom-map of C) $\restriction M$ is a function from M into O and (the cod-map of C) $\restriction M$ is a function from M into O and (the composition of C) $\restriction [\!: M, M :\!]$ is a partial function from $[\!: M, M :\!]$ to M and (the id-map of C) $\restriction O$ is a function from O into M.

(30) Let O be a non-empty subset of the objects of C. Let M be a non-empty set. Let d, c be functions from M into O. Let p be a partial function from $[\!: M, M :\!]$ to M. Let i be a function from O into M. Suppose $M = \bigcup\{\hom(a, b) : a \in O \wedge b \in O\}$ and $d =$ (the dom-map of C) $\restriction M$ and $c =$ (the cod-map of C) $\restriction M$ and $p =$ (the composition of C) $\restriction [\!: M, M :\!]$ and $i =$ (the id-map of C) $\restriction O$. Then $\langle O, M, d, c, p, i\rangle$ is full subcategory of C.

(31) Let O be a non-empty subset of the objects of C. Let M be a non-empty set. Let d, c be functions from M into O. Let p be a partial function from $[\!: M, M :\!]$ to M. Let i be a function from O into M. Suppose $\langle O, M, d, c, p, i\rangle$ is full subcategory of C. Then $M = \bigcup\{\hom(a, b) : a \in O \wedge b \in O\}$ and $d =$ (the dom-map of C) $\restriction M$ and $c =$ (the cod-map of C) $\restriction M$ and $p =$ (the composition of C) $\restriction [\!: M, M :\!]$ and $i =$ (the id-map of C) $\restriction O$.

Let X_1, X_2, Y_1, Y_2 be non-empty sets, and let f_1 be a function from X_1 into Y_1, and let f_2 be a function from X_2 into Y_2. Then $[\!: f_1, f_2 :\!]$ is a function from $[\!: X_1, X_2 :\!]$ into $[\!: Y_1, Y_2 :\!]$.

Let A, B be non-empty sets, and let f be a partial function from $[\!: A, A :\!]$ to A, and let g be a partial function from $[\!: B, B :\!]$ to B. Then $|\!: f, g :\!|$ is a partial function from $[\!: [\!: A, B :\!], [\!: A, B :\!] :\!]$ to $[\!: A, B :\!]$.

Let us consider C, D. The functor $[\!: C, D :\!]$ yielding a category is defined as follows:

$[\!: C, D :\!] = \langle [\!:$ the objects of $C,$ the objects of $D :\!], [\!:$ the morphisms of $C,$ the morphisms of $D :\!], [\!:$ the dom-map of $C,$ the dom-map of $D :\!], [\!:$ the cod-map of $C,$ the cod-map of $D :\!], |\!:$ the composition of $C,$ the composition of $D :\!|, [\!:$ the id-map of $C,$ the id-map of $D :\!]\rangle$.

Next we state three propositions:

(32) $[\!: C, D :\!] = \langle [\!:$ the objects of $C,$ the objects of $D :\!], [\!:$ the morphisms of $C,$ the morphisms of $D :\!], [\!:$ the dom-map of $C,$ the dom-map of $D :\!], [\!:$ the cod-map of $C,$ the cod-map of $D :\!], |\!:$ the composition of $C,$ the composition of $D :\!|, [\!:$ the id-map of $C,$ the id-map of $D :\!]\rangle$.

(33) (i) The objects of $[\!: C, D :\!] = [\!:$ the objects of $C,$ the objects of $D :\!]$,

(ii) the morphisms of $[: C, D :] = [:$ the morphisms of $C,$ the morphisms of $D :]$,

(iii) the dom-map of $[: C, D :] = [:$ the dom-map of $C,$ the dom-map of $D :]$,

(iv) the cod-map of $[: C, D :] = [:$ the cod-map of $C,$ the cod-map of $D :]$,

(v) the composition of $[: C, D :] = |:$ the composition of $C,$ the composition of $D :|$,

(vi) the id-map of $[: C, D :] = [:$ the id-map of $C,$ the id-map of $D :]$.

(34) For every object c of C and for every object d of D holds $\langle c, d\rangle$ is an object of $[: C, D :]$.

Let us consider C, D, and let c be an object of C, and let d be an object of D. Then $\langle c, d\rangle$ is an object of $[: C, D :]$.

One can prove the following propositions:

(35) For every object cd of $[: C, D :]$ there exists an object c of C and there exists an object d of D such that $cd = \langle c, d\rangle$.

(36) For every morphism f of C and for every morphism g of D holds $\langle f, g\rangle$ is a morphism of $[: C, D :]$.

Let us consider C, D, and let f be a morphism of C, and let g be a morphism of D. Then $\langle f, g\rangle$ is a morphism of $[: C, D :]$.

The following propositions are true:

(37) For every morphism fg of $[: C, D :]$ there exists a morphism f of C and there exists a morphism g of D such that $fg = \langle f, g\rangle$.

(38) For every morphism f of C and for every morphism g of D holds $\mathrm{dom}\,\langle f, g\rangle = \langle\,\mathrm{dom}\,f, \mathrm{dom}\,g\rangle$ and $\mathrm{cod}\,\langle f, g\rangle = \langle\,\mathrm{cod}\,f, \mathrm{cod}\,g\rangle$.

(39) For all morphisms f, f' of C and for all morphisms g, g' of D such that $\mathrm{dom}\,f' = \mathrm{cod}\,f$ and $\mathrm{dom}\,g' = \mathrm{cod}\,g$ holds $\langle f', g'\rangle \cdot \langle f, g\rangle = \langle f' \cdot f, g' \cdot g\rangle$.

(40) For all morphisms f, f' of C and for all morphisms g, g' of D such that $\mathrm{dom}\,\langle f', g'\rangle = \mathrm{cod}\,\langle f, g\rangle$ holds $\langle f', g'\rangle \cdot \langle f, g\rangle = \langle f' \cdot f, g' \cdot g\rangle$.

(41) For every object c of C and for every object d of D holds $\mathrm{id}_{\langle c,d\rangle} = \langle\,\mathrm{id}_c, \mathrm{id}_d\,\rangle$.

(42) For all objects c, c' of C and for all objects d, d' of D holds $\hom(\langle c, d\rangle, \langle c', d'\rangle) = [: \hom(c, c'), \hom(d, d') :]$.

(43) For all objects c, c' of C and for every morphism f from c to c' and for all objects d, d' of D and for every morphism g from d to d' such that $\hom(c, c') \neq \emptyset$ and $\hom(d, d') \neq \emptyset$ holds $\langle f, g\rangle$ is a morphism from $\langle c, d\rangle$ to $\langle c', d'\rangle$.

(44) For every functor S from $[: C, C' :]$ to D and for every object c of C holds $\mathrm{curry}\,S(\mathrm{id}_c)$ is a functor from C' to D.

(45) For every functor S from $[: C, C' :]$ to D and for every object c' of C' holds $\mathrm{curry}'\,S(\mathrm{id}_{c'})$ is a functor from C to D.

Let us consider C, C', D, and let S be a functor from $[: C, C' :]$ to D, and let c be an object of C. The functor $S(c, -)$ yields a functor from C' to D and is defined as follows:

$S(c,-) = \operatorname{curry} S(\mathrm{id}_c)$.

The following three propositions are true:

(46) For every functor S from $[\colon C, C' \colon]$ to D and for every object c of C holds $S(c,-) = \operatorname{curry} S(\mathrm{id}_c)$.

(47) For every functor S from $[\colon C, C' \colon]$ to D and for every object c of C and for every morphism f of C' holds $S(c,-)(f) = S(\langle \mathrm{id}_c, f \rangle)$.

(48) For every functor S from $[\colon C, C' \colon]$ to D and for every object c of C and for every object c' of C' holds $(\mathrm{Obj}\, S(c,-))(c') = (\mathrm{Obj}\, S)(\langle c, c' \rangle)$.

Let us consider C, C', D, and let S be a functor from $[\colon C, C' \colon]$ to D, and let c' be an object of C'. The functor $S(-,c')$ yielding a functor from C to D is defined by:

$S(-,c') = \operatorname{curry}' S(\mathrm{id}_{c'})$.

We now state several propositions:

(49) For every functor S from $[\colon C, C' \colon]$ to D and for every object c' of C' holds $S(-,c') = \operatorname{curry}' S(\mathrm{id}_{c'})$.

(50) For every functor S from $[\colon C, C' \colon]$ to D and for every object c' of C' and for every morphism f of C holds $S(-,c')(f) = S(\langle f, \mathrm{id}_{c'} \rangle)$.

(51) For every functor S from $[\colon C, C' \colon]$ to D and for every object c of C and for every object c' of C' holds $(\mathrm{Obj}\, S(-,c'))(c) = (\mathrm{Obj}\, S)(\langle c, c' \rangle)$.

(52) Let L be a function from the objects of C into $\mathrm{Funct}(B,D)$. Let M be a function from the objects of B into $\mathrm{Funct}(C,D)$. Suppose that

(i) for every object c of C and for every object b of B holds $(M(b))(\mathrm{id}_c) = (L(c))(\mathrm{id}_b)$,

(ii) for every morphism f of B and for every morphism g of C holds $(M(\operatorname{cod} f))(g) \cdot (L(\operatorname{dom} g))(f) = (L(\operatorname{cod} g))(f) \cdot (M(\operatorname{dom} f))(g)$.

Then there exists a functor S from $[\colon B, C \colon]$ to D such that for every morphism f of B and for every morphism g of C holds $S(\langle f, g \rangle) = (L(\operatorname{cod} g))(f) \cdot (M(\operatorname{dom} f))(g)$.

(53) Let L be a function from the objects of C into $\mathrm{Funct}(B,D)$. Let M be a function from the objects of B into $\mathrm{Funct}(C,D)$. Suppose there exists a functor S from $[\colon B, C \colon]$ to D such that for every object c of C and for every object b of B holds $S(-,c) = L(c)$ and $S(b,-) = M(b)$. Then for every morphism f of B and for every morphism g of C holds $(M(\operatorname{cod} f))(g) \cdot (L(\operatorname{dom} g))(f) = (L(\operatorname{cod} g))(f) \cdot (M(\operatorname{dom} f))(g)$.

(54) $\pi_1($ (the morphisms of $C)\times$ (the morphisms of D)) is a functor from $[\colon C, D \colon]$ to C.

(55) $\pi_2($ (the morphisms of $C)\times$ (the morphisms of D)) is a functor from $[\colon C, D \colon]$ to D.

We now define two new functors. Let us consider C, D. The functor $\pi_1(C \times D)$ yields a functor from $[\colon C, D \colon]$ to C and is defined as follows:

$\pi_1(C \times D) = \pi_1($ (the morphisms of $C)\times$ (the morphisms of D)).

The functor $\pi_2(C \times D)$ yielding a functor from $[\colon C, D \colon]$ to D is defined as follows:

$\pi_2(C \times D) = \pi_2($ (the morphisms of $C)\times$ (the morphisms of D)).

One can prove the following propositions:

(56) $\pi_1(C \times D) = \pi_1($ (the morphisms of $C)\times$ (the morphisms of D)).

(57) $\pi_2(C \times D) = \pi_2($ (the morphisms of $C)\times$ (the morphisms of D)).

(58) For every morphism f of C and for every morphism g of D holds $\pi_1(C\times D)(\langle f, g\rangle) = f$.

(59) For every object c of C and for every object d of D holds $(\mathrm{Obj}\,\pi_1(C \times D))(\langle c, d\rangle) = c$.

(60) For every morphism f of C and for every morphism g of D holds $\pi_2(C\times D)(\langle f, g\rangle) = g$.

(61) For every object c of C and for every object d of D holds $(\mathrm{Obj}\,\pi_2(C \times D))(\langle c, d\rangle) = d$.

(62) For every functor T from C to D and for every functor T' from C to D' holds $\langle T, T'\rangle$ is a functor from C to $[\!:D, D'\,:\!]$.

Let us consider C, D, D', and let T be a functor from C to D, and let T' be a functor from C to D'. Then $\langle T, T'\rangle$ is a functor from C to $[\!:D, D'\,:\!]$.

One can prove the following propositions:

(63) For every functor T from C to D and for every functor T' from C to D' and for every object c of C holds $(\mathrm{Obj}\langle T, T'\rangle)(c) = \langle(\mathrm{Obj}\,T)(c), (\mathrm{Obj}\,T')(c)\rangle$.

(64) For every functor T from C to D and for every functor T' from C' to D' holds $[\!:T, T'\,:\!] = \langle T \cdot \pi_1(C \times C'), T' \cdot \pi_2(C \times C')\rangle$.

(65) For every functor T from C to D and for every functor T' from C' to D' holds $[\!:T, T'\,:\!]$ is a functor from $[\!:C, C'\,:\!]$ to $[\!:D, D'\,:\!]$.

Let us consider C, C', D, D', and let T be a functor from C to D, and let T' be a functor from C' to D'. Then $[\!:T, T'\,:\!]$ is a functor from $[\!:C, C'\,:\!]$ to $[\!:D, D'\,:\!]$.

One can prove the following proposition

(66) For every functor T from C to D and for every functor T' from C' to D' and for every object c of C and for every object c' of C' holds $(\mathrm{Obj}[\!:T, T'\,:\!])(\langle c, c'\rangle) = \langle(\mathrm{Obj}\,T)(c), (\mathrm{Obj}\,T')(c')\rangle$.

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