FORMALIZED MATHEMATICS
Vol.1, No.**4**, September–October 1990
Université Catholique de Louvain

Real Function Differentiability [1]

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Summary. For a real valued function defined on its domain in real numbers the differentiability in a single point and on a subset of the domain is presented. The main elements of differential calculus are developed. The algebraic properties of differential real functions are shown.

MML Identifier: `FDIFF_1`.

The terminology and notation used here have been introduced in the following articles: [11], [2], [8], [3], [4], [1], [5], [6], [7], [10], and [9]. For simplicity we follow the rules: x, x_0, r, p will be real numbers, n will be a natural number, Y will be a subset of $\mathbb{R}$, Z will be a real open subset, X will be a set, s_1 will be a sequence of real numbers, and f, f_1, f_2 will be partial functions from $\mathbb{R}$ to $\mathbb{R}$. We now state the proposition

(1) For every r holds $r \in Y$ if and only if $r \in \mathbb{R}$ if and only if $Y = \mathbb{R}$.

A sequence of real numbers is called a real sequence convergent to 0 if:
it is non-zero and it is convergent and $\lim \mathrm{it} = 0$.

The following proposition is true

(2) For every s_1 holds s_1 is a real sequence convergent to 0 if and only if s_1 is non-zero and s_1 is convergent and $\lim s_1 = 0$.

A sequence of real numbers is called a constant real sequence if:
it is constant.

We now state the proposition

(3) For every s_1 holds s_1 is a constant real sequence if and only if s_1 is constant.

In the sequel h will be a real sequence convergent to 0 and c will be a constant real sequence. A partial function from $\mathbb{R}$ to $\mathbb{R}$ is called a rest if:

[1] Supported by RPBP.III-24.C8.

ISSN 0777–4028

it is total and for every h holds $h^{-1}\diamond(\mathrm{it}\cdot h)$ is convergent and $\lim(h^{-1}\diamond(\mathrm{it}\cdot h)) = 0$.

One can prove the following proposition

(4) For every f holds f is a rest if and only if f is total and for every h holds $h^{-1} \diamond (f \cdot h)$ is convergent and $\lim(h^{-1} \diamond (f \cdot h)) = 0$.

A partial function from $\mathbb{R}$ to $\mathbb{R}$ is called a linear function if:

it is total and there exists r such that for every p holds $\mathrm{it}(p) = r \cdot p$.

The following proposition is true

(5) For every f holds f is a linear function if and only if f is total and there exists r such that for every p holds $f(p) = r \cdot p$.

We follow the rules: R, R_1, R_2 are rests and L, L_1, L_2 are linear functions. We now state several propositions:

(6) For all L_1, L_2 holds $L_1 + L_2$ is a linear function and $L_1 - L_2$ is a linear function.

(7) For all r, L holds $r \diamond L$ is a linear function.

(8) For all R_1, R_2 holds $R_1 + R_2$ is a rest and $R_1 - R_2$ is a rest and $R_1 \diamond R_2$ is a rest.

(9) For all r, R holds $r \diamond R$ is a rest.

(10) $L_1 \diamond L_2$ is a rest.

(11) $R \diamond L$ is a rest and $L \diamond R$ is a rest.

Let us consider f, x_0. We say that f is differentiable in x_0 if and only if:

there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\, f$ and there exist L, R such that for every x such that $x \in N$ holds $f(x) - f(x_0) = L(x - x_0) + R(x - x_0)$.

The following proposition is true

(12) For all f, x_0 holds f is differentiable in x_0 if and only if there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\, f$ and there exist L, R such that for every x such that $x \in N$ holds $f(x) - f(x_0) = L(x - x_0) + R(x - x_0)$.

Let us consider f, x_0. Let us assume that f is differentiable in x_0. The functor $f'(x_0)$ yields a real number and is defined as follows:

there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\, f$ and there exist L, R such that $f'(x_0) = L(1)$ and for every x such that $x \in N$ holds $f(x) - f(x_0) = L(x - x_0) + R(x - x_0)$.

The following proposition is true

(13) Given r, f, x_0. Suppose f is differentiable in x_0. Then $r = f'(x_0)$ if and only if there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\, f$ and there exist L, R such that $r = L(1)$ and for every x such that $x \in N$ holds $f(x) - f(x_0) = L(x - x_0) + R(x - x_0)$.

Let us consider f, X. We say that f is differentiable on X if and only if:

$X \subseteq \mathrm{dom}\, f$ and for every x such that $x \in X$ holds $f \restriction X$ is differentiable in x.

The following four propositions are true:

(14) For all f, X holds f is differentiable on X if and only if $X \subseteq \operatorname{dom} f$ and for every x such that $x \in X$ holds $f \restriction X$ is differentiable in x.

(15) If f is differentiable on X, then X is a subset of $\mathbb{R}$.

(16) f is differentiable on Z if and only if $Z \subseteq \operatorname{dom} f$ and for every x such that $x \in Z$ holds f is differentiable in x.

(17) If f is differentiable on Y, then Y is open.

Let us consider f, X. Let us assume that f is differentiable on X. The functor $f'_{\restriction X}$ yielding a partial function from $\mathbb{R}$ to $\mathbb{R}$ is defined by:

$\operatorname{dom}(f'_{\restriction X}) = X$ and for every x such that $x \in X$ holds $(f'_{\restriction X})(x) = f'(x)$.

One can prove the following two propositions:

(18) For all f, X and for every partial function F from $\mathbb{R}$ to $\mathbb{R}$ such that f is differentiable on X holds $F = f'_{\restriction X}$ if and only if $\operatorname{dom} F = X$ and for every x such that $x \in X$ holds $F(x) = f'(x)$.

(19) For all f, Z such that $Z \subseteq \operatorname{dom} f$ and there exists r such that $\operatorname{rng} f = \{r\}$ holds f is differentiable on Z and for every x such that $x \in Z$ holds $(f'_{\restriction Z})(x) = 0$.

Let us consider h, n. Then $h \frown n$ is a real sequence convergent to 0.

Let us consider c, n. Then $c \frown n$ is a constant real sequence.

Next we state a number of propositions:

(20) Given f, x_0. Let N be a neighbourhood of x_0. Suppose f is differentiable in x_0 and $N \subseteq \operatorname{dom} f$. Then for all h, c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h + c) \subseteq N$ holds $h^{-1} \diamond (f \cdot (h + c) - f \cdot c)$ is convergent and $f'(x_0) = \lim(h^{-1} \diamond (f \cdot (h + c) - f \cdot c))$.

(21) For all f_1, f_2, x_0 such that f_1 is differentiable in x_0 and f_2 is differentiable in x_0 holds $f_1 + f_2$ is differentiable in x_0 and $(f_1 + f_2)'(x_0) = f_1'(x_0) + f_2'(x_0)$.

(22) For all f_1, f_2, x_0 such that f_1 is differentiable in x_0 and f_2 is differentiable in x_0 holds $f_1 - f_2$ is differentiable in x_0 and $(f_1 - f_2)'(x_0) = f_1'(x_0) - f_2'(x_0)$.

(23) For all r, f, x_0 such that f is differentiable in x_0 holds $r \diamond f$ is differentiable in x_0 and $(r \diamond f)'(x_0) = r \cdot (f'(x_0))$.

(24) For all f_1, f_2, x_0 such that f_1 is differentiable in x_0 and f_2 is differentiable in x_0 holds $f_1 \diamond f_2$ is differentiable in x_0 and $(f_1 \diamond f_2)'(x_0) = f_2(x_0) \cdot (f_1'(x_0)) + f_1(x_0) \cdot (f_2'(x_0))$.

(25) For all f, Z such that $Z \subseteq \operatorname{dom} f$ and $f \restriction Z = \mathrm{id}_Z$ holds f is differentiable on Z and for every x such that $x \in Z$ holds $(f'_{\restriction Z})(x) = 1$.

(26) For all f_1, f_2, Z such that $Z \subseteq \operatorname{dom}(f_1 + f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z holds $f_1 + f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $((f_1 + f_2)'_{\restriction Z})(x) = f_1'(x) + f_2'(x)$.

(27) For all f_1, f_2, Z such that $Z \subseteq \operatorname{dom}(f_1 - f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z holds $f_1 - f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $((f_1 - f_2)'_{\restriction Z})(x) = f_1'(x) - f_2'(x)$.

(28) For all r, f, Z such that $Z \subseteq \mathrm{dom}(r \diamond f)$ and f is differentiable on Z holds $r \diamond f$ is differentiable on Z and for every x such that $x \in Z$ holds $((r \diamond f)'_{\restriction Z})(x) = r \cdot (f'(x))$.

(29) Given f_1, f_2, Z. Then if $Z \subseteq \mathrm{dom}(f_1 \diamond f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z, then $f_1 \diamond f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $((f_1 \diamond f_2)'_{\restriction Z})(x) = f_2(x) \cdot (f'_1(x)) + f_1(x) \cdot (f'_2(x))$.

(30) If $Z \subseteq \mathrm{dom}\, f$ and f is a constant on Z, then f is differentiable on Z and for every x such that $x \in Z$ holds $(f'_{\restriction Z})(x) = 0$.

(31) If $Z \subseteq \mathrm{dom}\, f$ and for every x such that $x \in Z$ holds $f(x) = r \cdot x + p$, then f is differentiable on Z and for every x such that $x \in Z$ holds $(f'_{\restriction Z})(x) = r$.

(32) If f is differentiable in x_0, then f is continuous in x_0.

(33) If f is differentiable on X, then f is continuous on X.

(34) If f is differentiable on X and $Z \subseteq X$, then f is differentiable on Z.

(35) If f is differentiable in x_0, then there exists R such that $R(0) = 0$ and R is continuous in 0.

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Received June 18, 1990