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Many-Argument Relations

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Summary. Definitions of relations based on finite sequences. The arity of relation, the set of logical values *Boolean* consisting of *false* and *true* and the operations of negation and conjunction on them are defined.

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The notation and terminology used in this paper have been introduced in the following papers: [5], [2], [1], [3], and [4]. In the sequel x, y will be arbitrary, k will denote a natural number, and D will denote a non-empty set. Let B, A be non-empty sets, and let b be an element of B. Then $A \longmapsto b$ is an element of B^A.

A set is said to be a relation if:

for an arbitrary x such that $x \in$ it holds x is a finite sequence and for all finite sequences a, b such that $a \in$ it and $b \in$ it holds $\mathrm{len}\, a = \mathrm{len}\, b$.

We follow a convention: X denotes a set, p, r denote relations, and a, b denote finite sequences. We now state several propositions:

(4)[2] For every X such that for every x such that $x \in X$ holds x is a finite sequence and for all a, b such that $a \in X$ and $b \in X$ holds $\mathrm{len}\, a = \mathrm{len}\, b$ holds X is a relation.

(5) If $x \in p$, then x is a finite sequence.

(6) If $a \in p$ and $b \in p$, then $\mathrm{len}\, a = \mathrm{len}\, b$.

(7) If $X \subseteq p$, then X is a relation.

(8) $\{a\}$ is a relation.

(9) $\{\langle x, y \rangle\}$ is a relation.

The scheme *rel_exist* concerns a set $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

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[2]The propositions (1)–(3) became obvious.

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there exists r such that for every a holds $a \in r$ if and only if $a \in \mathcal{A}$ and $\mathcal{P}[a]$ provided the parameters satisfy the following condition:

- for all a, b such that $\mathcal{P}[a]$ and $\mathcal{P}[b]$ holds $\operatorname{len} a = \operatorname{len} b$.

Let us consider p, r. Let us note that one can characterize the predicate $p = r$ by the following (equivalent) condition: for every a holds $a \in p$ if and only if $a \in r$.

We now state the proposition

(10) $p = r$ if and only if for every a holds $a \in p$ if and only if $a \in r$.

The relation $\varnothing$ is defined by:

$a \notin \varnothing$.

One can prove the following propositions:

(11) $a \notin \varnothing$.

(12) $p = \varnothing$ if and only if for no a holds $a \in p$.

(13) $\varnothing = \emptyset$.

Let us consider p. Let us assume that $p \neq \varnothing$. The functor $\operatorname{Arity}(p)$ yielding a natural number is defined by:

for every a such that $a \in p$ holds $\operatorname{Arity}(p) = \operatorname{len} a$.

We now state two propositions:

(14) If $p \neq \varnothing$, then for every k holds $k = \operatorname{Arity}(p)$ if and only if for every a such that $a \in p$ holds $k = \operatorname{len} a$.

(15) If $a \in p$ and $p \neq \varnothing$, then $\operatorname{Arity}(p) = \operatorname{len} a$.

Let us consider k. A relation is called a k-ary relation if:

for every a such that $a \in$ it holds $\operatorname{len} a = k$.

One can prove the following two propositions:

(16) For all k, r such that for every a such that $a \in r$ holds $\operatorname{len} a = k$ holds r is a k-ary relation.

(17) For every k-ary relation r such that $a \in r$ holds $\operatorname{len} a = k$.

Let X be a set. A relation is called a relation on X if:

for every a such that $a \in$ it holds $\operatorname{rng} a \subseteq X$.

In the sequel X denotes a set. Next we state four propositions:

(18) For all X, r such that for every a such that $a \in r$ holds $\operatorname{rng} a \subseteq X$ holds r is a relation on X.

(19) For every relation r on X such that $a \in r$ holds $\operatorname{rng} a \subseteq X$.

(20) $\varnothing$ is a relation on X.

(21) $\varnothing$ is a k-ary relation.

Let us consider X, k. A relation is called a k-ary relation of X if:

it is a relation on X and it is a k-ary relation.

Next we state two propositions:

(22) For every relation r holds r is a k-ary relation of X if and only if r is a relation on X and r is a k-ary relation.

(23) For every k-ary relation R of X holds R is a relation on X and R is a k-ary relation.

Let us consider D. The functor $\mathrm{Rel}(D)$ yielding a non-empty family of sets is defined as follows:

for every X holds $X \in \mathrm{Rel}(D)$ if and only if $X \subseteq D^*$ and for all finite sequences a, b of elements of D such that $a \in X$ and $b \in X$ holds $\mathrm{len}\, a = \mathrm{len}\, b$.

The following propositions are true:

(24) For every non-empty set D and for every non-empty family S of sets holds $S = \mathrm{Rel}(D)$ if and only if for every X holds $X \in S$ if and only if $X \subseteq D^*$ and for all finite sequences a, b of elements of D such that $a \in X$ and $b \in X$ holds $\mathrm{len}\, a = \mathrm{len}\, b$.

(25) $X \in \mathrm{Rel}(D)$ if and only if $X \subseteq D^*$ and for all finite sequences a, b of elements of D such that $a \in X$ and $b \in X$ holds $\mathrm{len}\, a = \mathrm{len}\, b$.

Let D be a non-empty set. A relation on D is an element of $\mathrm{Rel}(D)$.

In the sequel a will denote a finite sequence of elements of D and p, r will denote elements of $\mathrm{Rel}(D)$. Next we state three propositions:

(26) If $X \subseteq r$, then X is an element of $\mathrm{Rel}(D)$.

(27) $\{a\}$ is an element of $\mathrm{Rel}(D)$.

(28) For all elements x, y of D holds $\{\langle x, y\rangle\}$ is an element of $\mathrm{Rel}(D)$.

Let us consider D, p, r. Let us note that one can characterize the predicate $p = r$ by the following (equivalent) condition: for every a holds $a \in p$ if and only if $a \in r$.

One can prove the following proposition

(29) $p = r$ if and only if for every a holds $a \in p$ if and only if $a \in r$.

The scheme *rel_D_exist* deals with a non-empty set $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

there exists an element r of $\mathrm{Rel}(\mathcal{A})$ such that for every finite sequence a of elements of $\mathcal{A}$ holds $a \in r$ if and only if $\mathcal{P}[a]$

provided the parameters satisfy the following condition:

- for all finite sequences a, b of elements of $\mathcal{A}$ such that $\mathcal{P}[a]$ and $\mathcal{P}[b]$ holds $\mathrm{len}\, a = \mathrm{len}\, b$.

Let us consider D. The functor $\varnothing_D$ yielding an element of $\mathrm{Rel}(D)$ is defined as follows:

$a \notin \varnothing_D$.

The following three propositions are true:

(30) $r = \varnothing_D$ if and only if for no a holds $a \in r$.

(31) $a \notin \varnothing_D$.

(32) $\varnothing_D = \emptyset$.

Let us consider D, p. Let us assume that $p \neq \varnothing_D$. The functor $\mathrm{Arity}(p)$ yielding a natural number is defined by:

if $a \in p$, then $\mathrm{Arity}(p) = \mathrm{len}\, a$.

Next we state two propositions:

(33) If $p \neq \varnothing_D$, then for every k holds $k = \mathrm{Arity}(p)$ if and only if for every a such that $a \in p$ holds $k = \mathrm{len}\, a$.

(34) If $a \in p$ and $p \neq \varnothing_D$, then $\mathrm{Arity}(p) = \mathrm{len}\, a$.

The scheme *rel_D_exist2* concerns a non-empty set $\mathcal{A}$, a natural number $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

there exists an element r of $\mathrm{Rel}(\mathcal{A})$ such that for every finite sequence a of elements of $\mathcal{A}$ such that $\mathrm{len}\, a = \mathcal{B}$ holds $a \in r$ if and only if $\mathcal{P}[a]$
for all values of the parameters.

The non-empty set *Boolean* is defined by:

Boolean $= \{0, 1\}$.

We now define two new functors. The element *false* of *Boolean* is defined by:

false $= 0$.

The element *true* of *Boolean* is defined as follows:

true $= 1$.

The following four propositions are true:

(35) *Boolean* $= \{0, 1\}$.

(36) *false* $= 0$ and *true* $= 1$.

(37) *Boolean* $= \{$*false*, *true*$\}$.

(38) *false* $\neq$ *true*.

In the sequel u, v, w will denote elements of *Boolean*. Next we state the proposition

(39) $v =$ *false* or $v =$ *true*.

We now define two new functors. Let us consider v. The functor $\neg v$ yielding an element of *Boolean* is defined by:

$\neg v =$ *true* if $v =$ *false*, $\neg v =$ *false* if $v =$ *true*.

Let us consider w. The functor $v \wedge w$ yielding an element of *Boolean* is defined by:

$v \wedge w =$ *true* if $v =$ *true* and $w =$ *true*, $v \wedge w =$ *false*, otherwise.

The following propositions are true:

(40) $\neg(\neg v) = v$.

(41) $v =$ *false* if and only if $\neg v =$ *true* but $v =$ *true* if and only if $\neg v =$ *false*.

(42) If $v \neq$ *false*, then $v =$ *true* but if $v \neq$ *true*, then $v =$ *false*.

(43) $v \neq$ *true* if and only if $v =$ *false*.

(44) It is not true that: $v =$ *true* and $w =$ *true* if and only if $v =$ *false* or $w =$ *false*.

(45) $v \wedge w =$ *true* if and only if $v =$ *true* and $w =$ *true* but $v \wedge w =$ *false* if and only if $v =$ *false* or $w =$ *false*.

(46) $v \wedge \neg v =$ *false*.

(47) $\neg(v \wedge \neg v) =$ *true*.

(48) $v \wedge w = w \wedge v$.

(49) *false* $\wedge\, v =$ *false*.

(50) $true \wedge v = v$.

(51) If $v \wedge v = false$, then $v = false$.

(52) $v \wedge (w \wedge u) = (v \wedge w) \wedge u$.

Let us consider X. The functor $Boolean(false \notin X)$ yields an element of *Boolean* and is defined as follows:

$Boolean(false \notin X) = true$ if $false \notin X$, $Boolean(false \notin X) = false$, otherwise.

One can prove the following proposition

(53) $false \notin X$ if and only if $Boolean(false \notin X) = true$ but $false \in X$ if and only if $Boolean(false \notin X) = false$.

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