FORMALIZED MATHEMATICS
Vol.1, No.4, September–October 1990
Université Catholique de Louvain

Increasing and Continuous Ordinal Sequences

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Summary. Concatenation of two ordinal sequences, the mode of all ordinals belonging to a universe and the mode of sequences of them with length equal to the rank of the universe are introduced. Besides, the increasing and continuous transfinite sequences, the limes of ordinal sequences and the power of ordinals, and the fact that every increasing and continuous transfinite sequence has critical numbers (fixed points) are discussed.

MML Identifier: `ORDINAL4`.

The terminology and notation used here have been introduced in the following papers: [6], [4], [2], [3], [1], and [5]. We adopt the following convention: *phi*, fi, *psi* are sequences of ordinal numbers and A, B, C are ordinal numbers. The following proposition is true

(1) If $\mathrm{dom}\, fi = \mathrm{succ}\, A$, then $(\mathrm{last}\, fi)$ (as an ordinal) is the limit of fi and $\lim fi = \mathrm{last}\, fi$.

Let us consider fi, *psi*. The functor $fi \frown psi$ yields a sequence of ordinal numbers and is defined as follows:

$\mathrm{dom}(fi \frown psi) = \mathrm{dom}\, fi + \mathrm{dom}\, psi$ and for every A such that $A \in \mathrm{dom}\, fi$ holds $(fi \frown psi)(A) = fi(A)$ and for every A such that $A \in \mathrm{dom}\, psi$ holds $(fi \frown psi)(\mathrm{dom}\, fi + A) = psi(A)$.

The following propositions are true:

(2) Let *chi* be a sequence of ordinal numbers. Then $chi = fi \frown psi$ if and only if $\mathrm{dom}\, chi = \mathrm{dom}\, fi + \mathrm{dom}\, psi$ and for every A such that $A \in \mathrm{dom}\, fi$ holds $chi(A) = fi(A)$ and for every A such that $A \in \mathrm{dom}\, psi$ holds $chi(\mathrm{dom}\, fi + A) = psi(A)$.

(3) If A is the limit of *psi*, then A is the limit of $fi \frown psi$.

(4) If A is the limit of fi, then $B + A$ is the limit of $B + fi$.

ISSN 0777–4028

(5) If A is the limit of fi, then $A \cdot B$ is the limit of $fi \cdot B$.

(6) If $\mathrm{dom}\, fi = \mathrm{dom}\, psi$ and B is the limit of fi and C is the limit of psi but for every A such that $A \in \mathrm{dom}\, fi$ holds $fi(A) \subseteq psi(A)$ or for every A such that $A \in \mathrm{dom}\, fi$ holds $fi(A) \in psi(A)$, then $B \subseteq C$.

In the sequel f_1, f_2 denote sequences of ordinal numbers. One can prove the following propositions:

(7) If $\mathrm{dom}\, f_1 = \mathrm{dom}\, fi$ and $\mathrm{dom}\, fi = \mathrm{dom}\, f_2$ and A is the limit of f_1 and A is the limit of f_2 and for every A such that $A \in \mathrm{dom}\, fi$ holds $f_1(A) \subseteq fi(A)$ and $fi(A) \subseteq f_2(A)$, then A is the limit of fi.

(8) If $\mathrm{dom}\, fi \neq \mathbf{0}$ and $\mathrm{dom}\, fi$ is a limit ordinal number and fi is increasing, then $\sup fi$ is the limit of fi and $\lim fi = \sup fi$.

(9) If fi is increasing and $A \subseteq B$ and $B \in \mathrm{dom}\, fi$, then $fi(A) \subseteq fi(B)$.

(10) If fi is increasing and $A \in \mathrm{dom}\, fi$, then $A \subseteq fi(A)$.

(11) If phi is increasing, then $phi^{-1} A$ is an ordinal number.

(12) If f_1 is increasing, then $f_2 \cdot f_1$ is a sequence of ordinal numbers.

(13) If f_1 is increasing and f_2 is increasing, then there exists phi such that $phi = f_1 \cdot f_2$ and phi is increasing.

(14) If f_1 is increasing and A is the limit of f_2 and $\sup(\mathrm{rng}\, f_1) = \mathrm{dom}\, f_2$ and $fi = f_2 \cdot f_1$, then A is the limit of fi.

(15) If phi is increasing, then $phi \restriction A$ is increasing.

(16) If phi is increasing and $\mathrm{dom}\, phi$ is a limit ordinal number, then $\sup phi$ is a limit ordinal number.

(17) If fi is increasing and fi is continuous and psi is continuous and $phi = psi \cdot fi$, then phi is continuous.

(18) If for every A such that $A \in \mathrm{dom}\, fi$ holds $fi(A) = C + A$, then fi is increasing.

(19) If $C \neq \mathbf{0}$ and for every A such that $A \in \mathrm{dom}\, fi$ holds $fi(A) = A \cdot C$, then fi is increasing.

(20) If $A \neq \mathbf{0}$, then $\mathbf{0}^A = \mathbf{0}$.

(21) If $A \neq \mathbf{0}$ and A is a limit ordinal number, then for every fi such that $\mathrm{dom}\, fi = A$ and for every B such that $B \in A$ holds $fi(B) = C^B$ holds C^A is the limit of fi.

(22) If $C \neq \mathbf{0}$, then $C^A \neq \mathbf{0}$.

(23) If $\mathbf{1} \in C$, then $C^A \in C^{\mathrm{succ}\, A}$.

(24) If $\mathbf{1} \in C$ and $A \in B$, then $C^A \in C^B$.

(25) If $\mathbf{1} \in C$ and for every A such that $A \in \mathrm{dom}\, fi$ holds $fi(A) = C^A$, then fi is increasing.

(26) If $\mathbf{1} \in C$ and $A \neq \mathbf{0}$ and A is a limit ordinal number, then for every fi such that $\mathrm{dom}\, fi = A$ and for every B such that $B \in A$ holds $fi(B) = C^B$ holds $C^A = \sup fi$.

(27) If $C \neq \mathbf{0}$ and $A \subseteq B$, then $C^A \subseteq C^B$.

(28) If $A \subseteq B$, then $A^C \subseteq B^C$.

(29) If $\mathbf{1} \in C$ and $A \neq \mathbf{0}$, then $\mathbf{1} \in C^A$.

(30) $C^{A+B} = (C^B) \cdot (C^A)$.

(31) $(C^A)^B = C^{B \cdot A}$.

(32) If $\mathbf{1} \in C$, then $A \subseteq C^A$.

The scheme *CriticalNumber* concerns a unary functor $\mathcal{F}$ yielding an ordinal number and states that:

there exists A such that $\mathcal{F}(A) = A$

provided the parameter meets the following conditions:

- for all A, B such that $A \in B$ holds $\mathcal{F}(A) \in \mathcal{F}(B)$,
- for every A such that $A \neq \mathbf{0}$ and A is a limit ordinal number for every *phi* such that $\operatorname{dom} fi = A$ and for every B such that $B \in A$ holds $phi(B) = \mathcal{F}(B)$ holds $\mathcal{F}(A)$ is the limit of *phi*.

In the sequel W will be a universal class. We now define two new modes. Let us consider W. An ordinal number is said to be an ordinal of W if:

it $\in W$.

A sequence of ordinal numbers is called a transfinite sequence of ordinals of W if:

dom it $= \operatorname{On} W$ and rng it $\subseteq \operatorname{On} W$.

We now state two propositions:

(33) A is an ordinal of W if and only if $A \in W$.

(34) *phi* is a transfinite sequence of ordinals of W if and only if $\operatorname{dom} phi = \operatorname{On} W$ and $\operatorname{rng} phi \subseteq \operatorname{On} W$.

In the sequel A_1, B_1 will be ordinals of W and *phi* will be a transfinite sequence of ordinals of W. The scheme *UOS_Lambda* concerns a universal class $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding an ordinal of $\mathcal{A}$ and states that:

there exists a transfinite sequence *phi* of ordinals of $\mathcal{A}$ such that for every ordinal a of $\mathcal{A}$ holds $phi(a) = \mathcal{F}(a)$

for all values of the parameters.

We now define two new functors. Let us consider W. The functor $\mathbf{0}_W$ yielding an ordinal of W is defined as follows:

$\mathbf{0}_W = \mathbf{0}$.

The functor $\mathbf{1}_W$ yields an ordinal of W and is defined by:

$\mathbf{1}_W = \mathbf{1}$.

Let us consider *phi*, A_1. Then $phi(A_1)$ is an ordinal of W.

Let us consider W, and let p_2, p_1 be transfinite sequences of ordinals of W. Then $p_1 \cdot p_2$ is a transfinite sequence of ordinals of W.

We now state the proposition

(35) $\mathbf{0}_W = \mathbf{0}$ and $\mathbf{1}_W = \mathbf{1}$.

Let us consider W, A_1. Then $\operatorname{succ} A_1$ is an ordinal of W. Let us consider B_1. Then $A_1 + B_1$ is an ordinal of W.

Let us consider W, A_1, B_1. Then $A_1 \cdot B_1$ is an ordinal of W.

The following propositions are true:

(36) $A_1 \in \mathrm{dom}\, phi$.

(37) If $\mathrm{dom}\, fi \in W$ and $\mathrm{rng}\, fi \subseteq W$, then $\sup fi \in W$.

We now state the proposition

(38) If phi is increasing and phi is continuous and $\omega \in W$, then there exists A such that $A \in \mathrm{dom}\, phi$ and $phi(A) = A$.

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Received May 31, 1990