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Probability

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Summary. Some further theorems concerning probability, among them the equivalent definition of probability are discussed, followed by notions of independence of events and conditional probability and basic theorems on them.

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The notation and terminology used in this paper have been introduced in the following papers: [8], [2], [4], [3], [6], [5], [9], [7], and [1]. For simplicity we adopt the following convention: *Omega* denotes a non-empty set, f denotes a function, m, n denote natural numbers, r, r_1, r_2, r_3 denote real numbers, *seq*, seq_1 denote sequences of real numbers, *Sigma* denotes a σ-field of subsets of *Omega*, *ASeq*, *BSeq* denote sequences of subsets of *Sigma*, P, P_1 denote probabilities on *Sigma*, and A, B, C, A_1, A_2, A_3 denote events of *Sigma*. One can prove the following propositions:

(1) $(r - r_1) + r_2 = (r + r_2) - r_1$.

(2) $r \leq r_1$ if and only if $r < r_1$ or $r = r_1$.

(3) For all r, r_1, r_2 such that $0 < r$ and $r_1 \leq r_2$ holds $\frac{r_1}{r} \leq \frac{r_2}{r}$.

(4) For all r, r_1, r_2, r_3 such that $r \neq 0$ and $r_1 \neq 0$ holds $\frac{r_3}{r_1} = \frac{r_2}{r}$ if and only if $r_3 \cdot r = r_2 \cdot r_1$.

(5) If *seq* is convergent and for every n holds $seq_1(n) = r - seq(n)$, then seq_1 is convergent and $\lim seq_1 = r - \lim seq$.

(6) $A \cap Omega = A$ and $Omega \cap A = A$ and $A \cap \Omega_{Sigma} = A$ and $\Omega_{Sigma} \cap A = A$.

(7) If B misses C, then $A \cap B$ misses $A \cap C$ and $B \cap A$ misses $C \cap A$.

The scheme *SeqEx* concerns a unary functor $\mathcal{F}$ and states that:
there exists f such that $\operatorname{dom} f = \mathbb{N}$ and for every n holds $f(n) = \mathcal{F}(n)$

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for all values of the parameter.

Let us consider *Omega*, *Sigma*, *ASeq*, n. Then $ASeq(n)$ is an event of *Sigma*.

Let us consider *Omega*, *Sigma*, *ASeq*. The functor $\bigcap ASeq$ yielding an event of *Sigma* is defined by:

$\bigcap ASeq =$ Intersection $ASeq$.

One can prove the following propositions:

(8) $\bigcap ASeq =$ Intersection $ASeq$.

(9) For every B, $ASeq$ there exists $BSeq$ such that for every n holds $BSeq(n) = ASeq(n) \cap B$.

(10) For all B, $ASeq$, $BSeq$ such that $ASeq$ is nonincreasing and for every n holds $BSeq(n) = ASeq(n) \cap B$ holds $BSeq$ is nonincreasing.

(11) For every function f from *Sigma* into $\mathbb{R}$ and for all $ASeq$, n holds $(f \cdot ASeq)(n) = f(ASeq(n))$.

(12) For all $ASeq$, $BSeq$, B such that for every n holds $BSeq(n) = ASeq(n) \cap B$ holds (Intersection $ASeq$) $\cap B =$ Intersection $BSeq$.

(13) For all P, P_1 such that for every A holds $P(A) = P_1(A)$ holds $P = P_1$.

(14) For every *Omega* and for every sequence $ASeq$ of subsets of *Omega* holds $ASeq$ is nonincreasing if and only if for every n holds $ASeq(n+1) \subseteq ASeq(n)$.

(15) For every sequence $ASeq$ of subsets of *Omega* holds $ASeq$ is nondecreasing if and only if for every n holds $ASeq(n) \subseteq ASeq(n+1)$.

(16) For all sequences $ASeq$, $BSeq$ of subsets of *Omega* such that for every n holds $ASeq(n) = BSeq(n)$ holds $ASeq = BSeq$.

(17) For every sequence $ASeq$ of subsets of *Omega* holds $ASeq$ is nonincreasing if and only if Complement $ASeq$ is nondecreasing.

Let us consider *Omega*, *Sigma*, *ASeq*. The functor $ASeq^{\mathbf{c}}$ yields a sequence of subsets of *Sigma* and is defined by:

$ASeq^{\mathbf{c}} =$ Complement $ASeq$.

The following proposition is true

(18) $ASeq^{\mathbf{c}} =$ Complement $ASeq$.

Let us consider *Omega*, *Sigma*, *ASeq*. We say that $ASeq$ is pairwise disjoint if and only if:

for all m, n such that $m \neq n$ holds $ASeq(m)$ misses $ASeq(n)$.

We now state a number of propositions:

(19) $ASeq$ is pairwise disjoint if and only if for all m, n such that $m \neq n$ holds $ASeq(m)$ misses $ASeq(n)$.

(20) Let P be a function from *Sigma* into $\mathbb{R}$. Then P is a probability on *Sigma* if and only if the following conditions are satisfied:

(i) for every A holds $0 \leq P(A)$,

(ii) $P(Omega) = 1$,

(iii) for all A, B such that A misses B holds $P(A \cup B) = P(A) + P(B)$,

(iv) for every $ASeq$ such that $ASeq$ is nondecreasing holds $P \cdot ASeq$ is convergent and $\lim(P \cdot ASeq) = P(\mathrm{Union}\, ASeq)$.

(21) $P((A \cup B) \cup C) = (((P(A) + P(B)) + P(C)) - ((P(A \cap B) + P(B \cap C)) + P(A \cap C))) + P((A \cap B) \cap C)$.

(22) $P(A \setminus A \cap B) = P(A) - P(A \cap B)$.

(23) For all P, A, B holds $P(A \cap B) \leq P(B)$ and $P(A \cap B) \leq P(A)$.

(24) For all P, A, B, C such that $C = B^{c}$ holds $P(A) = P(A \cap B) + P(A \cap C)$.

(25) For all P, A, B holds $(P(A) + P(B)) - 1 \leq P(A \cap B)$.

(26) For all P, A holds $P(A) = 1 - P(\Omega_{Sigma} \setminus A)$.

(27) For all P, A holds $P(A) < 1$ if and only if $0 < P(\Omega_{Sigma} \setminus A)$.

(28) For all P, A holds $P(\Omega_{Sigma} \setminus A) < 1$ if and only if $0 < P(A)$.

We now define two new predicates. Let us consider $Omega$, $Sigma$, P, A, B. We say that A and B are independent w.r.t P if and only if:

$P(A \cap B) = P(A) \cdot P(B)$.

Let us consider C. We say that A, B and C are independent w.r.t P if and only if:

(i) $P((A \cap B) \cap C) = (P(A) \cdot P(B)) \cdot P(C)$,
(ii) $P(A \cap B) = P(A) \cdot P(B)$,
(iii) $P(A \cap C) = P(A) \cdot P(C)$,
(iv) $P(B \cap C) = P(B) \cdot P(C)$.

We now state a number of propositions:

(29) A and B are independent w.r.t P if and only if $P(A \cap B) = P(A) \cdot P(B)$.

(30) A, B and C are independent w.r.t P if and only if the following conditions are satisfied:

(i) $P((A \cap B) \cap C) = (P(A) \cdot P(B)) \cdot P(C)$,
(ii) $P(A \cap B) = P(A) \cdot P(B)$,
(iii) $P(A \cap C) = P(A) \cdot P(C)$,
(iv) $P(B \cap C) = P(B) \cdot P(C)$.

(31) For all A, B, P holds A and B are independent w.r.t P if and only if B and A are independent w.r.t P.

(32) For all A, B, C, P holds A, B and C are independent w.r.t P if and only if $P((A \cap B) \cap C) = (P(A) \cdot P(B)) \cdot P(C)$ and A and B are independent w.r.t P and B and C are independent w.r.t P and A and C are independent w.r.t P.

(33) For all A, B, C, P such that A, B and C are independent w.r.t P holds B, A and C are independent w.r.t P.

(34) For all A, B, C, P such that A, B and C are independent w.r.t P holds A, C and B are independent w.r.t P.

(35) A and $\emptyset_{Sigma}$ are independent w.r.t P.

(36) A and Ω_{Sigma} are independent w.r.t P.

(37) For all A, B, P such that A and B are independent w.r.t P holds A and $\Omega_{Sigma} \setminus B$ are independent w.r.t P.

(38) For all A, B, P such that A and B are independent w.r.t P holds $\Omega_{Sigma} \setminus A$ and $\Omega_{Sigma} \setminus B$ are independent w.r.t P.

(39) For all A, B, C, P such that A and B are independent w.r.t P and A and C are independent w.r.t P and B misses C holds A and $B \cup C$ are independent w.r.t P.

(40) For all P, A, B such that A and B are independent w.r.t P and $P(A) < 1$ and $P(B) < 1$ holds $P(A \cup B) < 1$.

Let us consider *Omega*, *Sigma*, P, B. Let us assume that $0 < P(B)$. The functor $\mathrm{P}(P/B)$ yielding a probability on *Sigma* is defined by:

for every A holds $(\mathrm{P}(P/B))(A) = \frac{P(A \cap B)}{P(B)}$.

Next we state a number of propositions:

(41) For all P, B such that $0 < P(B)$ for every A holds $\mathrm{P}(P/B)(A) = \frac{P(A \cap B)}{P(B)}$.

(42) For all P, B, A such that $0 < P(B)$ holds $P(A \cap B) = \mathrm{P}(P/B)(A) \cdot P(B)$.

(43) For all P, A, B, C such that $0 < P(A \cap B)$ holds $P((A \cap B) \cap C) = (P(A) \cdot \mathrm{P}(P/A)(B)) \cdot \mathrm{P}(P/(A \cap B))(C)$.

(44) For all P, A, B, C such that $C = B^{\mathrm{c}}$ and $0 < P(B)$ and $0 < P(C)$ holds $P(A) = \mathrm{P}(P/B)(A) \cdot P(B) + \mathrm{P}(P/C)(A) \cdot P(C)$.

(45) Given P, A, A_1, A_2, A_3. Suppose A_1 misses A_2 and $A_3 = (A_1 \cup A_2)^{\mathrm{c}}$ and $0 < P(A_1)$ and $0 < P(A_2)$ and $0 < P(A_3)$. Then $P(A) = (\mathrm{P}(P/A_1)(A) \cdot P(A_1) + \mathrm{P}(P/A_2)(A) \cdot P(A_2)) + \mathrm{P}(P/A_3)(A) \cdot P(A_3)$.

(46) For all P, A, B such that $0 < P(B)$ holds $\mathrm{P}(P/B)(A) = P(A)$ if and only if A and B are independent w.r.t P.

(47) For all P, A, B such that $0 < P(B)$ and $P(B) < 1$ and $\mathrm{P}(P/B)(A) = \mathrm{P}(P/(\Omega_{Sigma} \setminus B))(A)$ holds A and B are independent w.r.t P.

(48) For all P, A, B such that $0 < P(B)$ holds $\frac{(P(A)+P(B))-1}{P(B)} \leq \mathrm{P}(P/B)(A)$.

(49) For all A, B, P such that $0 < P(A)$ and $0 < P(B)$ holds $\mathrm{P}(P/B)(A) = \frac{\mathrm{P}(P/A)(B) \cdot P(A)}{P(B)}$.

(50) Given B, A_1, A_2, P. Suppose $0 < P(B)$ and $A_2 = {A_1}^{\mathrm{c}}$ and $0 < P(A_1)$ and $0 < P(A_2)$. Then

(i) $\mathrm{P}(P/B)(A_1) = \frac{\mathrm{P}(P/A_1)(B) \cdot P(A_1)}{\mathrm{P}(P/A_1)(B) \cdot P(A_1) + \mathrm{P}(P/A_2)(B) \cdot P(A_2)}$,

(ii) $\mathrm{P}(P/B)(A_2) = \frac{\mathrm{P}(P/A_2)(B) \cdot P(A_2)}{\mathrm{P}(P/A_1)(B) \cdot P(A_1) + \mathrm{P}(P/A_2)(B) \cdot P(A_2)}$.

(51) Given B, A_1, A_2, A_3, P. Suppose $0 < P(B)$ and $0 < P(A_1)$ and $0 < P(A_2)$ and $0 < P(A_3)$ and A_1 misses A_2 and $A_3 = (A_1 \cup A_2)^{\mathrm{c}}$. Then

(i) $\mathrm{P}(P/B)(A_1) = \frac{\mathrm{P}(P/A_1)(B) \cdot P(A_1)}{(\mathrm{P}(P/A_1)(B) \cdot P(A_1) + \mathrm{P}(P/A_2)(B) \cdot P(A_2)) + \mathrm{P}(P/A_3)(B) \cdot P(A_3)}$,

(ii) $\mathrm{P}(P/B)(A_2) = \frac{\mathrm{P}(P/A_2)(B) \cdot P(A_2)}{(\mathrm{P}(P/A_1)(B) \cdot P(A_1) + \mathrm{P}(P/A_2)(B) \cdot P(A_2)) + \mathrm{P}(P/A_3)(B) \cdot P(A_3)}$,

(iii) $\mathrm{P}(P/B)(A_3) = \frac{\mathrm{P}(P/A_3)(B) \cdot P(A_3)}{(\mathrm{P}(P/A_1)(B) \cdot P(A_1) + \mathrm{P}(P/A_2)(B) \cdot P(A_2)) + \mathrm{P}(P/A_3)(B) \cdot P(A_3)}$.

(52) For all A, B, P such that $0 < P(B)$ holds $1 - \frac{P(\Omega_{Sigma} \setminus A)}{P(B)} \leq \mathrm{P}(P/B)(A)$.

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