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Partial Functions from a Domain to the Set of Real Numbers

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Summary. Basic operations in the set of partial functions which map a domain to the set of all real numbers are introduced. They include adition, substraction, multiplication, division, multipication by a real number and also module. Main properties of these operations are proved. A definition of the partial function bounded on a set (bounded below and bounded above) is presented. There are theorems showing the laws of conservation of totality and boundeness for operations of partial functions. The characteristic function of a subset of a domain as a partial function is redefined and a few properties are proved.

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The papers [6], [3], [1], [7], [5], [2], and [4] provide the terminology and notation for this paper. For simplicity we follow the rules: X, Y will be sets, C will be a non-empty set, c will be an element of C, f, f_1, f_2, f_3, g, g_1 will be partial functions from C to $\mathbb{R}$, and r, r_1, p, p_1 will be real numbers. We now state two propositions:

(1) $(-1)^{-1} = -1$.

(2) If $0 \leq p$ and $0 \leq r$ and $p \leq p_1$ and $r \leq r_1$, then $p \cdot r \leq p_1 \cdot r_1$.

We now define four new functors. Let us consider C, f_1, f_2. The functor $f_1 + f_2$ yields a partial function from C to $\mathbb{R}$ and is defined as follows:

$\mathrm{dom}(f_1 + f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every c such that $c \in \mathrm{dom}(f_1 + f_2)$ holds $(f_1 + f_2)(c) = f_1(c) + f_2(c)$.

The functor $f_1 - f_2$ yielding a partial function from C to $\mathbb{R}$ is defined as follows:

$\mathrm{dom}(f_1 - f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every c such that $c \in \mathrm{dom}(f_1 - f_2)$ holds $(f_1 - f_2)(c) = f_1(c) - f_2(c)$.

The functor $f_1 \diamond f_2$ yielding a partial function from C to $\mathbb{R}$ is defined by:

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$\operatorname{dom}(f_1 \diamond f_2) = \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and for every c such that $c \in \operatorname{dom}(f_1 \diamond f_2)$ holds $(f_1 \diamond f_2)(c) = f_1(c) \cdot f_2(c)$.
The functor $\frac{f_1}{f_2}$ yielding a partial function from C to $\mathbb{R}$ is defined by:

$\operatorname{dom} \frac{f_1}{f_2} = \operatorname{dom} f_1 \cap (\operatorname{dom} f_2 \setminus f_2{}^{-1}\{0\})$ and for every c such that $c \in \operatorname{dom} \frac{f_1}{f_2}$ holds $\frac{f_1}{f_2}(c) = f_1(c) \cdot (f_2(c))^{-1}$.

Let us consider C, f, r. The functor $r \diamond f$ yields a partial function from C to $\mathbb{R}$ and is defined by:

$\operatorname{dom}(r \diamond f) = \operatorname{dom} f$ and for every c such that $c \in \operatorname{dom}(r \diamond f)$ holds $(r \diamond f)(c) = r \cdot f(c)$.

We now define three new functors. Let us consider C, f. The functor $|f|$ yields a partial function from C to $\mathbb{R}$ and is defined by:

$\operatorname{dom} |f| = \operatorname{dom} f$ and for every c such that $c \in \operatorname{dom} |f|$ holds $|f|(c) = |f(c)|$.
The functor $-f$ yields a partial function from C to $\mathbb{R}$ and is defined by:

$\operatorname{dom}(-f) = \operatorname{dom} f$ and for every c such that $c \in \operatorname{dom}(-f)$ holds $(-f)(c) = -f(c)$.
The functor $\frac{1}{f}$ yielding a partial function from C to $\mathbb{R}$ is defined by:

$\operatorname{dom} \frac{1}{f} = \operatorname{dom} f \setminus f^{-1}\{0\}$ and for every c such that $c \in \operatorname{dom} \frac{1}{f}$ holds $\frac{1}{f}(c) = (f(c))^{-1}$.

One can prove the following propositions:

(3) $f = f_1 + f_2$ if and only if $\operatorname{dom} f = \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and for every c such that $c \in \operatorname{dom} f$ holds $f(c) = f_1(c) + f_2(c)$.

(4) $f = f_1 - f_2$ if and only if $\operatorname{dom} f = \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and for every c such that $c \in \operatorname{dom} f$ holds $f(c) = f_1(c) - f_2(c)$.

(5) $f = f_1 \diamond f_2$ if and only if $\operatorname{dom} f = \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and for every c such that $c \in \operatorname{dom} f$ holds $f(c) = f_1(c) \cdot f_2(c)$.

(6) $f = \frac{f_1}{f_2}$ if and only if $\operatorname{dom} f = \operatorname{dom} f_1 \cap (\operatorname{dom} f_2 \setminus f_2{}^{-1}\{0\})$ and for every c such that $c \in \operatorname{dom} f$ holds $f(c) = f_1(c) \cdot (f_2(c))^{-1}$.

(7) $f = r \diamond f_1$ if and only if $\operatorname{dom} f = \operatorname{dom} f_1$ and for every c such that $c \in \operatorname{dom} f$ holds $f(c) = r \cdot f_1(c)$.

(8) $f = |f_1|$ if and only if $\operatorname{dom} f = \operatorname{dom} f_1$ and for every c such that $c \in \operatorname{dom} f$ holds $f(c) = |f_1(c)|$.

(9) $f = -f_1$ if and only if $\operatorname{dom} f = \operatorname{dom} f_1$ and for every c such that $c \in \operatorname{dom} f$ holds $f(c) = -f_1(c)$.

(10) $f_1 = \frac{1}{f}$ if and only if $\operatorname{dom} f_1 = \operatorname{dom} f \setminus f^{-1}\{0\}$ and for every c such that $c \in \operatorname{dom} f_1$ holds $f_1(c) = (f(c))^{-1}$.

(11) $\operatorname{dom} \frac{1}{g} \subseteq \operatorname{dom} g$ and $\operatorname{dom} g \cap (\operatorname{dom} g \setminus g^{-1}\{0\}) = \operatorname{dom} g \setminus g^{-1}\{0\}$.

(12) $\operatorname{dom}(f_1 \diamond f_2) \setminus (f_1 \diamond f_2)^{-1}\{0\} = (\operatorname{dom} f_1 \setminus f_1{}^{-1}\{0\}) \cap (\operatorname{dom} f_2 \setminus f_2{}^{-1}\{0\})$.

(13) If $c \in \operatorname{dom} \frac{1}{f}$, then $f(c) \neq 0$.

(14) $\frac{1}{f}^{-1}\{0\} = \emptyset$.

(15) $|f|^{-1}\{0\} = f^{-1}\{0\}$ and $(-f)^{-1}\{0\} = f^{-1}\{0\}$.

(16) $\mathrm{dom}\,\frac{1}{\frac{1}{f}} = \mathrm{dom}(f \restriction \mathrm{dom}\,\frac{1}{f})$.

(17) If $r \neq 0$, then $(r \diamond f)^{-1}\,\{0\} = f^{-1}\,\{0\}$.

(18) $f_1 + f_2 = f_2 + f_1$.

(19) $(f_1 + f_2) + f_3 = f_1 + (f_2 + f_3)$.

(20) $f_1 \diamond f_2 = f_2 \diamond f_1$.

(21) $(f_1 \diamond f_2) \diamond f_3 = f_1 \diamond (f_2 \diamond f_3)$.

(22) $(f_1 + f_2) \diamond f_3 = f_1 \diamond f_3 + f_2 \diamond f_3$.

(23) $f_3 \diamond (f_1 + f_2) = f_3 \diamond f_1 + f_3 \diamond f_2$.

(24) $r \diamond (f_1 \diamond f_2) = (r \diamond f_1) \diamond f_2$.

(25) $r \diamond (f_1 \diamond f_2) = f_1 \diamond (r \diamond f_2)$.

(26) $(f_1 - f_2) \diamond f_3 = f_1 \diamond f_3 - f_2 \diamond f_3$.

(27) $f_3 \diamond f_1 - f_3 \diamond f_2 = f_3 \diamond (f_1 - f_2)$.

(28) $r \diamond (f_1 + f_2) = r \diamond f_1 + r \diamond f_2$.

(29) $(r \cdot p) \diamond f = r \diamond (p \diamond f)$.

(30) $r \diamond (f_1 - f_2) = r \diamond f_1 - r \diamond f_2$.

(31) $f_1 - f_2 = (-1) \diamond (f_2 - f_1)$.

(32) $f_1 - (f_2 + f_3) = (f_1 - f_2) - f_3$.

(33) $1 \diamond f = f$.

(34) $f_1 - (f_2 - f_3) = (f_1 - f_2) + f_3$.

(35) $f_1 + (f_2 - f_3) = (f_1 + f_2) - f_3$.

(36) $|f_1 \diamond f_2| = |f_1| \diamond |f_2|$.

(37) $|r \diamond f| = |r| \diamond |f|$.

(38) $-f = (-1) \diamond f$.

(39) $-(-f) = f$.

(40) $f_1 - f_2 = f_1 + (-f_2)$.

(41) $f_1 - (-f_2) = f_1 + f_2$.

(42) $\frac{1}{\frac{1}{f}} = f \restriction \mathrm{dom}\,\frac{1}{f}$.

(43) $\frac{1}{f_1 \diamond f_2} = \frac{1}{f_1} \diamond \frac{1}{f_2}$.

(44) If $r \neq 0$, then $\frac{1}{r \diamond f} = r^{-1} \diamond \frac{1}{f}$.

(45) $\frac{1}{-f} = (-1) \diamond \frac{1}{f}$.

(46) $\frac{1}{|f|} = |\frac{1}{f}|$.

(47) $\frac{f}{g} = f \diamond \frac{1}{g}$.

(48) $r \diamond \frac{g}{f} = \frac{r \diamond g}{f}$.

(49) $\frac{f}{g} \diamond g = f \restriction \mathrm{dom}\,\frac{1}{g}$.

(50) $\frac{f}{g} \diamond \frac{f_1}{g_1} = \frac{f \diamond f_1}{g \diamond g_1}$.

(51) $\frac{1}{\frac{f_1}{f_2}} = \frac{f_2 \restriction \mathrm{dom} \frac{1}{f_2}}{f_1}$.

(52) $g \diamond \frac{f_1}{f_2} = \frac{g \diamond f_1}{f_2}$.

(53) $\frac{g}{\frac{f_1}{f_2}} = \frac{g \diamond f_2 \restriction \mathrm{dom} \frac{1}{f_2}}{f_1}$.

(54) $-\frac{f}{g} = \frac{-f}{g}$ and $\frac{f}{-g} = -\frac{f}{g}$.

(55) $\frac{f_1}{f} + \frac{f_2}{f} = \frac{f_1 + f_2}{f}$ and $\frac{f_1}{f} - \frac{f_2}{f} = \frac{f_1 - f_2}{f}$.

(56) $\frac{f_1}{f} + \frac{g_1}{g} = \frac{f_1 \diamond g + g_1 \diamond f}{f \diamond g}$.

(57) $\frac{\frac{f}{g}}{\frac{f_1}{g_1}} = \frac{f \diamond g_1 \restriction \mathrm{dom} \frac{1}{g_1}}{g \diamond f_1}$.

(58) $\frac{f_1}{f} - \frac{g_1}{g} = \frac{f_1 \diamond g - g_1 \diamond f}{f \diamond g}$.

(59) $|\frac{f_1}{f_2}| = \frac{|f_1|}{|f_2|}$.

(60) $(f_1 + f_2) \restriction X = f_1 \restriction X + f_2 \restriction X$ and $(f_1 + f_2) \restriction X = f_1 \restriction X + f_2$ and $(f_1 + f_2) \restriction X = f_1 + f_2 \restriction X$.

(61) $(f_1 \diamond f_2) \restriction X = f_1 \restriction X \diamond f_2 \restriction X$ and $(f_1 \diamond f_2) \restriction X = f_1 \restriction X \diamond f_2$ and $(f_1 \diamond f_2) \restriction X = f_1 \diamond f_2 \restriction X$.

(62) $(-f) \restriction X = -f \restriction X$ and $\frac{1}{f} \restriction X = \frac{1}{f \restriction X}$ and $|f| \restriction X = |f \restriction X|$.

(63) $(f_1 - f_2) \restriction X = f_1 \restriction X - f_2 \restriction X$ and $(f_1 - f_2) \restriction X = f_1 \restriction X - f_2$ and $(f_1 - f_2) \restriction X = f_1 - f_2 \restriction X$.

(64) $\frac{f_1}{f_2} \restriction X = \frac{f_1 \restriction X}{f_2 \restriction X}$ and $\frac{f_1}{f_2} \restriction X = \frac{f_1 \restriction X}{f_2}$ and $\frac{f_1}{f_2} \restriction X = \frac{f_1}{f_2 \restriction X}$.

(65) $(r \diamond f) \restriction X = r \diamond f \restriction X$.

(66) f_1 is total and f_2 is total if and only if $f_1 + f_2$ is total but f_1 is total and f_2 is total if and only if $f_1 - f_2$ is total but f_1 is total and f_2 is total if and only if $f_1 \diamond f_2$ is total.

(67) f is total if and only if $r \diamond f$ is total.

(68) f is total if and only if $-f$ is total.

(69) f is total if and only if $|f|$ is total.

(70) $\frac{1}{f}$ is total if and only if $f^{-1} \{0\} = \emptyset$ and f is total.

(71) f_1 is total and $f_2{}^{-1} \{0\} = \emptyset$ and f_2 is total if and only if $\frac{f_1}{f_2}$ is total.

(72) If f_1 is total and f_2 is total, then $(f_1 + f_2)(c) = f_1(c) + f_2(c)$ and $(f_1 - f_2)(c) = f_1(c) - f_2(c)$ and $(f_1 \diamond f_2)(c) = f_1(c) \cdot f_2(c)$.

(73) If f is total, then $(r \diamond f)(c) = r \cdot f(c)$.

(74) If f is total, then $(-f)(c) = -f(c)$ and $|f|(c) = |f(c)|$.

(75) If $\frac{1}{f}$ is total, then $\frac{1}{f}(c) = (f(c))^{-1}$.

(76) If f_1 is total and $\frac{1}{f_2}$ is total, then $\frac{f_1}{f_2}(c) = f_1(c) \cdot (f_2(c))^{-1}$.

Let us consider X, C. Then $\chi_{X,C}$ is a partial function from C to $\mathbb{R}$.

Next we state a number of propositions:

(77) $f = \chi_{X,C}$ if and only if $\mathrm{dom}\, f = C$ and for every c holds if $c \in X$, then $f(c) = 1$ but if $c \notin X$, then $f(c) = 0$.

(78) $\chi_{X,C}$ is total.

(79) $c \in X$ if and only if $\chi_{X,C}(c) = 1$.

(80) $c \notin X$ if and only if $\chi_{X,C}(c) = 0$.

(81) $c \in C \setminus X$ if and only if $\chi_{X,C}(c) = 0$.

(82) $\chi_{\emptyset,C}(c) = 0$.

(83) $\chi_{C,C}(c) = 1$.

(84) $\chi_{X,C}(c) \neq 1$ if and only if $\chi_{X,C}(c) = 0$.

(85) If $X \cap Y = \emptyset$, then $\chi_{X,C} + \chi_{Y,C} = \chi_{X \cup Y,C}$.

(86) $\chi_{X,C} \diamond \chi_{Y,C} = \chi_{X \cap Y,C}$.

We now define two new predicates. Let us consider C, f, Y. We say that f is upper bounded on Y if and only if:

there exists r such that for every c such that $c \in Y \cap \mathrm{dom}\, f$ holds $f(c) \leq r$.

We say that f is lower bounded on Y if and only if:

there exists r such that for every c such that $c \in Y \cap \mathrm{dom}\, f$ holds $r \leq f(c)$.

Let us consider C, f, Y. We say that f is bounded on Y if and only if:

f is upper bounded on Y and f is lower bounded on Y.

The following propositions are true:

(87) f is upper bounded on Y if and only if there exists r such that for every c such that $c \in Y \cap \mathrm{dom}\, f$ holds $f(c) \leq r$.

(88) f is lower bounded on Y if and only if there exists r such that for every c such that $c \in Y \cap \mathrm{dom}\, f$ holds $r \leq f(c)$.

(89) f is bounded on Y if and only if f is upper bounded on Y and f is lower bounded on Y.

(90) f is bounded on Y if and only if there exists r such that for every c such that $c \in Y \cap \mathrm{dom}\, f$ holds $|f(c)| \leq r$.

(91) If $Y \subseteq X$ and f is upper bounded on X, then f is upper bounded on Y but if $Y \subseteq X$ and f is lower bounded on X, then f is lower bounded on Y but if $Y \subseteq X$ and f is bounded on X, then f is bounded on Y.

(92) If f is upper bounded on X and f is lower bounded on Y, then f is bounded on $X \cap Y$.

(93) If $X \cap \mathrm{dom}\, f = \emptyset$, then f is bounded on X.

(94) If $0 = r$, then $r \diamond f$ is bounded on Y.

(95) If f is upper bounded on Y and $0 \leq r$, then $r \diamond f$ is upper bounded on Y but if f is upper bounded on Y and $r \leq 0$, then $r \diamond f$ is lower bounded on Y.

(96) If f is lower bounded on Y and $0 \leq r$, then $r \diamond f$ is lower bounded on Y but if f is lower bounded on Y and $r \leq 0$, then $r \diamond f$ is upper bounded on Y.

(97) If f is bounded on Y, then $r \diamond f$ is bounded on Y.

(98) $|f|$ is lower bounded on X.

(99) If f is bounded on Y, then $|f|$ is bounded on Y and $-f$ is bounded on Y.

(100) If f_1 is upper bounded on X and f_2 is upper bounded on Y, then f_1+f_2 is upper bounded on $X \cap Y$ but if f_1 is lower bounded on X and f_2 is lower bounded on Y, then $f_1 + f_2$ is lower bounded on $X \cap Y$ but if f_1 is bounded on X and f_2 is bounded on Y, then $f_1 + f_2$ is bounded on $X \cap Y$.

(101) If f_1 is bounded on X and f_2 is bounded on Y, then $f_1 \diamond f_2$ is bounded on $X \cap Y$ and $f_1 - f_2$ is bounded on $X \cap Y$.

(102) If f is upper bounded on X and f is upper bounded on Y, then f is upper bounded on $X \cup Y$.

(103) If f is lower bounded on X and f is lower bounded on Y, then f is lower bounded on $X \cup Y$.

(104) If f is bounded on X and f is bounded on Y, then f is bounded on $X \cup Y$.

(105) If f_1 is a constant on X and f_2 is a constant on Y, then $f_1 + f_2$ is a constant on $X \cap Y$ and $f_1 - f_2$ is a constant on $X \cap Y$ and $f_1 \diamond f_2$ is a constant on $X \cap Y$.

(106) If f is a constant on Y, then $p \diamond f$ is a constant on Y.

(107) If f is a constant on Y, then $|f|$ is a constant on Y and $-f$ is a constant on Y.

(108) If f is a constant on Y, then f is bounded on Y.

(109) If f is a constant on Y, then for every r holds $r \diamond f$ is bounded on Y and $-f$ is bounded on Y and $|f|$ is bounded on Y.

(110) If f_1 is upper bounded on X and f_2 is a constant on Y, then $f_1 + f_2$ is upper bounded on $X \cap Y$ but if f_1 is lower bounded on X and f_2 is a constant on Y, then $f_1 + f_2$ is lower bounded on $X \cap Y$ but if f_1 is bounded on X and f_2 is a constant on Y, then $f_1 + f_2$ is bounded on $X \cap Y$.

(111) (i) If f_1 is upper bounded on X and f_2 is a constant on Y, then $f_1 - f_2$ is upper bounded on $X \cap Y$,

(ii) if f_1 is lower bounded on X and f_2 is a constant on Y, then $f_1 - f_2$ is lower bounded on $X \cap Y$,

(iii) if f_1 is bounded on X and f_2 is a constant on Y, then $f_1 - f_2$ is bounded on $X \cap Y$ and $f_2 - f_1$ is bounded on $X \cap Y$ and $f_1 \diamond f_2$ is bounded on $X \cap Y$.

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