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Transformations in Affine Spaces [1]

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Summary. Two classes of bijections of its point universe are correlated with every affine structure. The first class consists of the transformations, called formal isometries, which map every segment onto congruent segment, the second class consists of the automorphisms of such a structure. Each of these two classes of bijections forms a group for a given affine structure, if it satisfies a very weak axiom system (models of these axioms are called congruence spaces); formal isometries form a normal subgroup in the group of automorphism. In particular ordered affine spaces and affine spaces are congruence spaces; therefore formal isometries of these structures can be considered. They are called positive dilatations and dilatations, resp. For convenience the class of negative dilatations, transformations which map every "vector" onto parallel "vector", but with opposite sense, is singled out. The class of translations is distinguished as well. Basic facts concerning all these types of transformations are established, like rigidity, decomposition principle, introductory group-theoretical properties. At the end collineations of affine spaces and their properties are investigated; for affine planes it is proved that the class of collineationms coincides with the class of bijections preserving lines.

MML Identifier: TRANSGEO.

The papers [7], [1], [8], [2], [3], [4], [5], and [6] provide the notation and terminology for this paper. We adopt the following convention: A denotes a non-empty set, a, b, x, y, z, t denote elements of A, and f, f_1, f_2, g, h denote permutations of A. Let us consider A, and let us consider f, and let x be an element of A. Then $f(x)$ is an element of A.

Let us consider A, and let us consider f, and let X be a subset of A. Then $f \circ X$ is a subset of A.

Let us consider A, f, g. Then $g \cdot f$ is a permutation of A.

One can prove the following propositions:

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(1) For all f_1, f_2 such that for every x holds $f_1(x) = f_2(x)$ holds $f_1 = f_2$.
(2) There exists x such that $f(x) = y$.
(3) If $f(x) = f(y)$, then $x = y$.
(4) $f(x) = y$ if and only if $f^{-1}(y) = x$.
(5) $(f \cdot g)(x) = f(g(x))$.

Let us consider A, f, g. The functor $f \setminus g$ yields a permutation of A and is defined by:

$f \setminus g = (g \cdot f) \cdot g^{-1}$.

One can prove the following proposition

(6) $f \setminus g = (g \cdot f) \cdot g^{-1}$.

The scheme *EXPermutation* deals with a non-empty set $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a permutation f of $\mathcal{A}$ such that for all elements x, y of $\mathcal{A}$ holds $f(x) = y$ if and only if $\mathcal{P}[x, y]$

provided the following requirements are met:

- for every element x of $\mathcal{A}$ there exists an element y of $\mathcal{A}$ such that $\mathcal{P}[x, y]$,
- for every element y of $\mathcal{A}$ there exists an element x of $\mathcal{A}$ such that $\mathcal{P}[x, y]$,
- for all elements x, y, x' of $\mathcal{A}$ such that $\mathcal{P}[x, y]$ and $\mathcal{P}[x', y]$ holds $x = x'$,
- for all elements x, y, y' of $\mathcal{A}$ such that $\mathcal{P}[x, y]$ and $\mathcal{P}[x, y']$ holds $y = y'$.

Next we state a number of propositions:

(7) $(\mathrm{id}_A)^{-1} = \mathrm{id}_A$.
(8) $f \cdot f^{-1} = \mathrm{id}_A$ and $f^{-1} \cdot f = \mathrm{id}_A$.
(9) $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.
(10) $\mathrm{id}_A \cdot f = f$ and $f \cdot \mathrm{id}_A = f$.
(11) $f \cdot \mathrm{id}_A = \mathrm{id}_A \cdot f$.
(12) $f \cdot (g \cdot h) = (f \cdot g) \cdot h$.
(13) If $g \cdot f = h \cdot f$ or $f \cdot g = f \cdot h$, then $g = h$.
(14) $(f \cdot g)^{-1} = g^{-1} \cdot f^{-1}$.
(15) $(f^{-1})^{-1} = f$.
(16) $f \cdot g \setminus h = (f \setminus h) \cdot (g \setminus h)$.
(17) $f^{-1} \setminus g = (f \setminus g)^{-1}$.
(18) $f \setminus g \cdot h = (f \setminus h) \setminus g$.
(19) $\mathrm{id}_A \setminus f = \mathrm{id}_A$.
(20) $f \setminus \mathrm{id}_A = f$.
(21) If $f(a) = a$, then $(f \setminus g)(g(a)) = g(a)$.

In the sequel R will denote a binary relation on $[: A, A :]$. Let us consider A, f, R. We say that f is a formal isometry of R if and only if:

for all x, y holds $\langle\langle x,y\rangle,\langle f(x),f(y)\rangle\rangle \in R$.

The following propositions are true:

(22) f is a formal isometry of R if and only if for all x, y holds $\langle\langle x,y\rangle,\langle f(x),f(y)\rangle\rangle \in R$.

(23) If R is reflexive in $[\!: A, A :\!]$, then id_A is a formal isometry of R.

(24) If R is symmetric in $[\!: A, A :\!]$ and f is a formal isometry of R, then f^{-1} is a formal isometry of R.

(25) If R is transitive in $[\!: A, A :\!]$ and f is a formal isometry of R and g is a formal isometry of R, then $f \cdot g$ is a formal isometry of R.

(26) Suppose that

(i) for all a, b, x, y, z, t such that $\langle\langle x,y\rangle,\langle a,b\rangle\rangle \in R$ and $\langle\langle a,b\rangle,\langle z,t\rangle\rangle \in R$ and $a \neq b$ holds $\langle\langle x,y\rangle,\langle z,t\rangle\rangle \in R$,

(ii) for all x, y, z holds $\langle\langle x,x\rangle,\langle y,z\rangle\rangle \in R$,

(iii) f is a formal isometry of R,

(iv) g is a formal isometry of R.

Then $f \cdot g$ is a formal isometry of R.

Let us consider A, f, R. We say that f is an automorphism of R if and only if:

for all x, y, z, t holds $\langle\langle x,y\rangle,\langle z,t\rangle\rangle \in R$ if and only if $\langle\langle f(x),f(y)\rangle,\langle f(z),f(t)\rangle\rangle \in R$.

The following propositions are true:

(27) For all a, f, R holds f is an automorphism of R if and only if for all x, y, z, t holds $\langle\langle x,y\rangle,\langle z,t\rangle\rangle \in R$ if and only if $\langle\langle f(x),f(y)\rangle,\langle f(z),f(t)\rangle\rangle \in R$.

(28) id_A is an automorphism of R.

(29) If f is an automorphism of R, then f^{-1} is an automorphism of R.

(30) If f is an automorphism of R and g is an automorphism of R, then $g \cdot f$ is an automorphism of R.

(31) If R is symmetric in $[\!: A, A :\!]$ and R is transitive in $[\!: A, A :\!]$ and f is a formal isometry of R, then f is an automorphism of R.

(32) Suppose that

(i) for all a, b, x, y, z, t such that $\langle\langle x,y\rangle,\langle a,b\rangle\rangle \in R$ and $\langle\langle a,b\rangle,\langle z,t\rangle\rangle \in R$ and $a \neq b$ holds $\langle\langle x,y\rangle,\langle z,t\rangle\rangle \in R$,

(ii) for all x, y, z holds $\langle\langle x,x\rangle,\langle y,z\rangle\rangle \in R$,

(iii) R is symmetric in $[\!: A, A :\!]$,

(iv) f is a formal isometry of R.

Then f is an automorphism of R.

(33) If f is a formal isometry of R and g is an automorphism of R, then $f \setminus g$ is a formal isometry of R.

In the sequel AS will be an affine structure. Let us consider AS, and let f be a permutation of the points of AS. We say that f is a dilatation of AS if and only if:

f is a formal isometry of the congruence of AS.

The following proposition is true

(34) For every permutation f of the points of AS holds f is a dilatation of AS if and only if f is a formal isometry of the congruence of AS.

In the sequel a, b denote elements of the points of AS. Next we state the proposition

(35) For every permutation f of the points of AS holds f is a dilatation of AS if and only if for all a, b holds $a, b \upuparrows f(a), f(b)$.

An affine structure is said to be a congruence space if:
(i) for all elements x, y, z, t, a, b of the points of it such that $x, y \upuparrows a, b$ and $a, b \upuparrows z, t$ and $a \neq b$ holds $x, y \upuparrows z, t$,
(ii) for all elements x, y, z of the points of it holds $x, x \upuparrows y, z$,
(iii) for all elements x, y, z, t of the points of it such that $x, y \upuparrows z, t$ holds $z, t \upuparrows x, y$,
(iv) for all elements x, y of the points of it holds $x, y \upuparrows x, y$.

One can prove the following proposition

(36) Let AS be an affine structure. Then AS is a congruence space if and only if the following conditions are satisfied:
  (i) for all elements x, y, z, t, a, b of the points of AS such that $x, y \upuparrows a, b$ and $a, b \upuparrows z, t$ and $a \neq b$ holds $x, y \upuparrows z, t$,
  (ii) for all elements x, y, z of the points of AS holds $x, x \upuparrows y, z$,
  (iii) for all elements x, y, z, t of the points of AS such that $x, y \upuparrows z, t$ holds $z, t \upuparrows x, y$,
  (iv) for all elements x, y of the points of AS holds $x, y \upuparrows x, y$.

In the sequel CS denotes a congruence space. One can prove the following three propositions:

(37) $\mathrm{id}_{\text{the points of } CS}$ is a dilatation of CS.

(38) For every permutation f of the points of CS such that f is a dilatation of CS holds f^{-1} is a dilatation of CS.

(39) For all permutations f, g of the points of CS such that f is a dilatation of CS and g is a dilatation of CS holds $f \cdot g$ is a dilatation of CS.

We follow the rules: OAS denotes an ordered affine space and a, b, c, d, p, q, x, y, z denote elements of the points of OAS. Next we state the proposition

(40) OAS is a congruence space.

In the sequel f, g are permutations of the points of OAS. Let us consider OAS, and let f be a permutation of the points of OAS. We say that f is a positive dilatation if and only if:
f is a dilatation of OAS.

We now state two propositions:

(41) For every permutation f of the points of OAS holds f is a positive dilatation if and only if f is a dilatation of OAS.

(42) For every permutation f of the points of OAS holds f is a positive dilatation if and only if for all a, b holds $a, b \upuparrows f(a), f(b)$.

Let us consider OAS, and let f be a permutation of the points of OAS. We say that f is a negative dilatation if and only if:

for all a, b holds $a, b \upuparrows f(b), f(a)$.

The following propositions are true:

(43) For every permutation f of the points of OAS holds f is a negative dilatation if and only if for all a, b holds $a, b \upuparrows f(b), f(a)$.

(44) $\mathrm{id}_{\text{the points of } OAS}$ is a positive dilatation.

(45) For every permutation f of the points of OAS such that f is a positive dilatation holds f^{-1} is a positive dilatation.

(46) For all permutations f, g of the points of OAS such that f is a positive dilatation and g is a positive dilatation holds $f \cdot g$ is a positive dilatation.

(47) For no f holds f is a negative dilatation and f is a positive dilatation.

(48) If f is a negative dilatation, then f^{-1} is a negative dilatation.

(49) If f is a positive dilatation and g is a negative dilatation, then $f \cdot g$ is a negative dilatation and $g \cdot f$ is a negative dilatation.

Let us consider OAS, and let f be a permutation of the points of OAS. We say that f is a dilatation if and only if:

f is a formal isometry of $\lambda($ the congruence of $OAS)$.

Next we state a number of propositions:

(50) For every permutation f of the points of OAS holds f is a dilatation if and only if f is a formal isometry of $\lambda($ the congruence of $OAS)$.

(51) For every permutation f of the points of OAS holds f is a dilatation if and only if for all a, b holds $a, b \parallel f(a), f(b)$.

(52) If f is a positive dilatation or f is a negative dilatation, then f is a dilatation.

(53) For every permutation f of the points of OAS such that f is a dilatation there exists a permutation f' of the points of $\Lambda(OAS)$ such that $f = f'$ and f' is a dilatation of $\Lambda(OAS)$.

(54) For every permutation f of the points of $\Lambda(OAS)$ such that f is a dilatation of $\Lambda(OAS)$ there exists a permutation f' of the points of OAS such that $f = f'$ and f' is a dilatation.

(55) $\mathrm{id}_{\text{the points of } OAS}$ is a dilatation.

(56) If f is a dilatation, then f^{-1} is a dilatation.

(57) If f is a dilatation and g is a dilatation, then $f \cdot g$ is a dilatation.

(58) If f is a dilatation, then for all a, b, c, d holds $a, b \parallel c, d$ if and only if $f(a), f(b) \parallel f(c), f(d)$.

(59) If f is a dilatation, then for all a, b, c holds $\mathbf{L}(a, b, c)$ if and only if $\mathbf{L}(f(a), f(b), f(c))$.

(60) If f is a dilatation and $\mathbf{L}(x, f(x), y)$, then $\mathbf{L}(x, f(x), f(y))$.

(61) If $a, b \parallel c, d$, then $a, c \parallel b, d$ or there exists x such that $\mathbf{L}(a, c, x)$ and $\mathbf{L}(b, d, x)$.

(62) If f is a dilatation, then $f = \mathrm{id}_{\text{the points of } OAS}$ or for every x holds $f(x) \neq x$ if and only if for all x, y holds $x, f(x) \parallel y, f(y)$.

(63) If f is a dilatation and $a \neq b$ and $f(a) = a$ and $f(b) = b$ and not $\mathbf{L}(a, b, x)$, then $f(x) = x$.

(64) If f is a dilatation and $f(a) = a$ and $f(b) = b$ and $a \neq b$, then $f = \mathrm{id}_{\text{the points of } OAS}$.

(65) If f is a dilatation and g is a dilatation and $f(a) = g(a)$ and $f(b) = g(b)$, then $a = b$ or $f = g$.

Let us consider OAS, and let f be a permutation of the points of OAS. We say that f is a translation if and only if:

f is a dilatation but $f = \mathrm{id}_{\text{the points of } OAS}$ or for every a holds $a \neq f(a)$.

One can prove the following propositions:

(66) For every permutation f of the points of OAS holds f is a translation if and only if f is a dilatation but $f = \mathrm{id}_{\text{the points of } OAS}$ or for every a holds $a \neq f(a)$.

(67) If f is a dilatation, then f is a translation if and only if for all x, y holds $x, f(x) \parallel y, f(y)$.

(68) If f is a translation and $f(a) = a$, then $f = \mathrm{id}_{\text{the points of } OAS}$.

(69) If f is a translation and g is a translation and $f(a) = g(a)$ and $f(a) \neq a$ and not $\mathbf{L}(a, f(a), x)$, then $f(x) = g(x)$.

(70) If f is a translation and g is a translation and $f(a) = g(a)$, then $f = g$.

(71) If f is a translation, then f^{-1} is a translation.

(72) If f is a translation and g is a translation, then $f \cdot g$ is a translation.

(73) If f is a translation, then f is a positive dilatation.

(74) If f is a dilatation and $f(p) = p$ and $\mathbf{B}(q, p, f(q))$ and not $\mathbf{L}(p, q, x)$, then $\mathbf{B}(x, p, f(x))$.

(75) If f is a dilatation and $f(p) = p$ and $\mathbf{B}(q, p, f(q))$ and $q \neq p$, then $\mathbf{B}(x, p, f(x))$.

(76) If f is a dilatation and $f(p) = p$ and $q \neq p$ and $\mathbf{B}(q, p, f(q))$ and not $\mathbf{L}(p, x, y)$, then $x, y \upharpoonleft\!\upharpoonright f(y), f(x)$.

(77) If f is a dilatation and $f(p) = p$ and $q \neq p$ and $\mathbf{B}(q, p, f(q))$ and $\mathbf{L}(p, x, y)$, then $x, y \upharpoonleft\!\upharpoonright f(y), f(x)$.

(78) If f is a dilatation and $f(p) = p$ and $q \neq p$ and $\mathbf{B}(q, p, f(q))$, then f is a negative dilatation.

(79) If f is a dilatation and $f(p) = p$ and for every x holds $p, x \upharpoonleft\!\upharpoonright p, f(x)$, then for all y, z holds $y, z \upharpoonleft\!\upharpoonright f(y), f(z)$.

(80) If f is a dilatation, then f is a positive dilatation or f is a negative dilatation.

We follow the rules: AFS will be an affine space and a, b, c, d, d_1, d_2, x, y, z, t will be elements of the points of AFS. The following propositions are true:

(81) For all a, b, c, d holds $a, b \upharpoonleft\!\upharpoonright c, d$ if and only if $a, b \parallel c, d$.

(82) AFS is a congruence space.

(83) $\Lambda(OAS)$ is a congruence space.

In the sequel f, g denote permutations of the points of AFS. Let us consider AFS, f. We say that f is a dilatation if and only if:

f is a dilatation of AFS.

Next we state a number of propositions:

(84) For every f holds f is a dilatation if and only if f is a dilatation of AFS.

(85) f is a dilatation if and only if for all a, b holds $a, b \parallel f(a), f(b)$.

(86) $\mathrm{id}_{\text{the points of } AFS}$ is a dilatation.

(87) If f is a dilatation, then f^{-1} is a dilatation.

(88) If f is a dilatation and g is a dilatation, then $f \cdot g$ is a dilatation.

(89) If f is a dilatation, then for all a, b, c, d holds $a, b \parallel c, d$ if and only if $f(a), f(b) \parallel f(c), f(d)$.

(90) If f is a dilatation, then for all a, b, c holds $\mathbf{L}(a, b, c)$ if and only if $\mathbf{L}(f(a), f(b), f(c))$.

(91) If f is a dilatation and $\mathbf{L}(x, f(x), y)$, then $\mathbf{L}(x, f(x), f(y))$.

(92) If $a, b \parallel c, d$, then $a, c \parallel b, d$ or there exists x such that $\mathbf{L}(a, c, x)$ and $\mathbf{L}(b, d, x)$.

(93) If f is a dilatation, then $f = \mathrm{id}_{\text{the points of } AFS}$ or for every x holds $f(x) \neq x$ if and only if for all x, y holds $x, f(x) \parallel y, f(y)$.

(94) If f is a dilatation and $a \neq b$ and $f(a) = a$ and $f(b) = b$ and not $\mathbf{L}(a, b, x)$, then $f(x) = x$.

(95) If f is a dilatation and $f(a) = a$ and $f(b) = b$ and $a \neq b$, then $f = \mathrm{id}_{\text{the points of } AFS}$.

(96) If f is a dilatation and g is a dilatation and $f(a) = g(a)$ and $f(b) = g(b)$, then $a = b$ or $f = g$.

(97) If not $\mathbf{L}(a, b, c)$ and $a, b \parallel c, d_1$ and $a, b \parallel c, d_2$ and $a, c \parallel b, d_1$ and $a, c \parallel b, d_2$, then $d_1 = d_2$.

Let us consider AFS, f. We say that f is a translation if and only if:

f is a dilatation but $f = \mathrm{id}_{\text{the points of } AFS}$ or for every a holds $a \neq f(a)$.

One can prove the following propositions:

(98) For every f holds f is a translation if and only if f is a dilatation but $f = \mathrm{id}_{\text{the points of } AFS}$ or for every a holds $a \neq f(a)$.

(99) $\mathrm{id}_{\text{the points of } AFS}$ is a translation.

(100) If f is a dilatation, then f is a translation if and only if for all x, y holds $x, f(x) \parallel y, f(y)$.

(101) If f is a translation and $f(a) = a$, then $f = \mathrm{id}_{\text{the points of } AFS}$.

(102) If f is a translation and g is a translation and $f(a) = g(a)$ and $f(a) \neq a$ and not $\mathbf{L}(a, f(a), x)$, then $f(x) = g(x)$.

(103) If f is a translation and g is a translation and $f(a) = g(a)$, then $f = g$.

(104) If f is a translation, then f^{-1} is a translation.

(105) If f is a translation and g is a translation, then $f \cdot g$ is a translation.

Let us consider AFS, f. We say that f is a collineation if and only if:
f is an automorphism of the congruence of AFS.

Next we state four propositions:

(106) f is a collineation if and only if f is an automorphism of the congruence of AFS.

(107) f is a collineation if and only if for all x, y, z, t holds $x, y \parallel z, t$ if and only if $f(x), f(y) \parallel f(z), f(t)$.

(108) If f is a collineation, then $\mathbf{L}(x, y, z)$ if and only if $\mathbf{L}(f(x), f(y), f(z))$.

(109) If f is a collineation and g is a collineation, then f^{-1} is a collineation and $f \cdot g$ is a collineation and $\mathrm{id}_{\text{the points of } AFS}$ is a collineation.

In the sequel A, C, K will denote subsets of the points of AFS. Next we state several propositions:

(110) If $a \in A$, then $f(a) \in f \circ A$.

(111) $x \in f \circ A$ if and only if there exists y such that $y \in A$ and $f(y) = x$.

(112) If $f \circ A = f \circ C$, then $A = C$.

(113) If f is a collineation, then $f \circ \mathrm{Line}(a, b) = \mathrm{Line}(f(a), f(b))$.

(114) If f is a collineation and K is a line, then $f \circ K$ is a line.

(115) If f is a collineation and $A \parallel C$, then $f \circ A \parallel f \circ C$.

For simplicity we follow the rules: AFP is an affine plane, A, K are subsets of the points of AFP, p, x are elements of the points of AFP, and f is a permutation of the points of AFP. We now state two propositions:

(116) If for every A such that A is a line holds $f \circ A$ is a line, then f is a collineation.

(117) If f is a collineation and K is a line and for every x such that $x \in K$ holds $f(x) = x$ and $p \notin K$ and $f(p) = p$, then $f = \mathrm{id}_{\text{the points of } AFP}$.

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