FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Analytical Metric Affine Spaces and Planes [1]

Henryk Oryszczyszyn
Warsaw University
Białystok

Krzysztof Prażmowski
Warsaw University
Białystok

Summary. We introduce relations of orthogonality of vectors and of orthogonality of segments (considered as pairs of vectors) in real linear space of dimension two. This enables us to show an example of (in fact anisotropic and satisfying theorem on three perpendiculars) metric affine space (and plane as well). These two types of objects are defined formally as "Mizar" modes. They are to be understood as structures consisting of a point universe and two binary relations on segments - a parallelity relation and orthogonality relation, satisfying appropriate axioms. With every such structure we correlate a structure obtained as a reduct of the given one to the parallelity relation only. Some relationships between metric affine spaces and their affine parts are proved; they enable us to use "affine" facts and constructions in investigating metric affine geometry. We define the notions of line, parallelity of lines and two derived relations of orthogonality: between segments and lines, and between lines. Some basic properties of the introduced notions are proved.

MML Identifier: ANALMETR.

The articles [5], [1], [7], [6], [2], [3], and [4] provide the notation and terminology for this paper. For simplicity we follow a convention: V denotes a real linear space, u, u_1, u_2, v, v_1, v_2, w, y denote vectors of V, a, a_1, a_2, b, b_1, b_2 denote real numbers, and x, z are arbitrary. Let us consider V, w, y. We say that w, y span the space if and only if:

(Def.1) for every u there exist a_1, a_2 such that $u = a_1 \cdot w + a_2 \cdot y$ and for all a_1, a_2 such that $a_1 \cdot w + a_2 \cdot y = 0_V$ holds $a_1 = 0$ and $a_2 = 0$.

One can prove the following propositions:

(1) For all w, y holds w, y span the space if and only if for every u there exist a_1, a_2 such that $u = a_1 \cdot w + a_2 \cdot y$ and for all a_1, a_2 such that $a_1 \cdot w + a_2 \cdot y = 0_V$ holds $a_1 = 0$ and $a_2 = 0$.

[1]Supported by RPBP.III-24.C2.

ISSN 0777–4028

(2) If w, y span the space, then there exist a_1, a_2 such that $u = a_1 \cdot w + a_2 \cdot y$.

(3) If w, y span the space and $a_1 \cdot w + a_2 \cdot y = 0_V$, then $a_1 = 0$ and $a_2 = 0$.

Let us consider V, u, v, w, y. We say that u, v are orthogonal w.r.t. w, y if and only if:

(Def.2) there exist a_1, a_2, b_1, b_2 such that $u = a_1 \cdot w + a_2 \cdot y$ and $v = b_1 \cdot w + b_2 \cdot y$ and $a_1 \cdot b_1 + a_2 \cdot b_2 = 0$.

The following propositions are true:

(4) For all u, v, w, y holds u, v are orthogonal w.r.t. w, y if and only if there exist a_1, a_2, b_1, b_2 such that $u = a_1 \cdot w + a_2 \cdot y$ and $v = b_1 \cdot w + b_2 \cdot y$ and $a_1 \cdot b_1 + a_2 \cdot b_2 = 0$.

(5) For all w, y such that w, y span the space holds u, v are orthogonal w.r.t. w, y if and only if for all a_1, a_2, b_1, b_2 such that $u = a_1 \cdot w + a_2 \cdot y$ and $v = b_1 \cdot w + b_2 \cdot y$ holds $a_1 \cdot b_1 + a_2 \cdot b_2 = 0$.

(6) w, y are orthogonal w.r.t. w, y.

(7) There exists V and there exist w, y such that w, y span the space.

(8) If u, v are orthogonal w.r.t. w, y, then v, u are orthogonal w.r.t. w, y.

(9) If w, y span the space, then for all u, v holds u, 0_V are orthogonal w.r.t. w, y and 0_V, v are orthogonal w.r.t. w, y.

(10) If u, v are orthogonal w.r.t. w, y, then $a \cdot u$, $b \cdot v$ are orthogonal w.r.t. w, y.

(11) If u, v are orthogonal w.r.t. w, y, then $a \cdot u$, v are orthogonal w.r.t. w, y and u, $b \cdot v$ are orthogonal w.r.t. w, y.

(12) If w, y span the space, then for every u there exists v such that u, v are orthogonal w.r.t. w, y and $v \neq 0_V$.

(13) If w, y span the space and v, u_1 are orthogonal w.r.t. w, y and v, u_2 are orthogonal w.r.t. w, y and $v \neq 0_V$, then there exist a, b such that $a \cdot u_1 = b \cdot u_2$ but $a \neq 0$ or $b \neq 0$.

(14) If w, y span the space and u, v_1 are orthogonal w.r.t. w, y and u, v_2 are orthogonal w.r.t. w, y, then u, $v_1 + v_2$ are orthogonal w.r.t. w, y and u, $v_1 - v_2$ are orthogonal w.r.t. w, y.

(15) If w, y span the space and u, u are orthogonal w.r.t. w, y, then $u = 0_V$.

(16) If w, y span the space and u, $u_1 - u_2$ are orthogonal w.r.t. w, y and u_1, $u_2 - u$ are orthogonal w.r.t. w, y, then u_2, $u - u_1$ are orthogonal w.r.t. w, y.

(17) If w, y span the space and $u \neq 0_V$, then there exists a such that $v - a \cdot u$, u are orthogonal w.r.t. w, y.

(18) $u, v \upuparrows u_1, v_1$ or $u, v \upuparrows v_1, u_1$ if and only if there exist a, b such that $a \cdot (v - u) = b \cdot (v_1 - u_1)$ but $a \neq 0$ or $b \neq 0$.

(19) $\langle\langle u, v\rangle, \langle u_1, v_1\rangle\rangle \in \lambda(\upuparrows_V)$ if and only if there exist a, b such that $a \cdot (v - u) = b \cdot (v_1 - u_1)$ but $a \neq 0$ or $b \neq 0$.

Let us consider V, u, u_1, v, v_1, w, y. We say that u, u_1, v and v_1 are orthogonal w.r.t. w, y if and only if:

(Def.3) $u_1 - u$, $v_1 - v$ are orthogonal w.r.t. w, y.

One can prove the following proposition

(20) For all u, u_1, v, v_1, w, y holds u, u_1, v and v_1 are orthogonal w.r.t. w, y if and only if $u_1 - u$, $v_1 - v$ are orthogonal w.r.t. w, y.

Let us consider V, w, y. The ortogonality determined by w, y in V yielding a binary relation on $[\!:$ the vectors of V, the vectors of $V :\!]$ is defined as follows:

(Def.4) $\langle x, z \rangle \in$ the ortogonality determined by w, y in V if and only if there exist u, u_1, v, v_1 such that $x = \langle u, u_1 \rangle$ and $z = \langle v, v_1 \rangle$ and u, u_1, v and v_1 are orthogonal w.r.t. w, y.

We now state the proposition

(21) For every binary relation R on $[\!:$ the vectors of V, the vectors of $V :\!]$ holds $R =$ the ortogonality determined by w, y in V if and only if for all x, z holds $\langle x, z \rangle \in R$ if and only if there exist u, u_1, v, v_1 such that $x = \langle u, u_1 \rangle$ and $z = \langle v, v_1 \rangle$ and u, u_1, v and v_1 are orthogonal w.r.t. w, y.

In the sequel p, p_1, q, q_1 will denote elements of the points of $\Lambda(\mathrm{OASpace}\,V)$. We now state three propositions:

(22) The points of $\Lambda(\mathrm{OASpace}\,V) =$ the vectors of V.

(23) The congruence of $\Lambda(\mathrm{OASpace}\,V) = \lambda(\Uparrow_V)$.

(24) If $p = u$ and $q = v$ and $p_1 = u_1$ and $q_1 = v_1$, then $p, q \parallel p_1, q_1$ if and only if there exist a, b such that $a \cdot (v - u) = b \cdot (v_1 - u_1)$ but $a \neq 0$ or $b \neq 0$.

We consider metric affine structures which are systems
⟨points, a parallelity, an orthogonality⟩,
where the points constitute a non-empty set, the parallelity is a binary relation on $[\!:$ the points, the points $:\!]$, and the orthogonality is a binary relation on $[\!:$ the points, the points $:\!]$. In the sequel P_1 will denote a metric-affine structure. We now define two new predicates. Let us consider P_1, and let a, b, c, d be elements of the points of P_1. The predicate $a, b \parallel c, d$ is defined as follows:

(Def.5) $\langle \langle a, b \rangle, \langle c, d \rangle \rangle \in$ the parallelity of P_1.

The predicate $a, b \perp c, d$ is defined as follows:

(Def.6) $\langle \langle a, b \rangle, \langle c, d \rangle \rangle \in$ the orthogonality of P_1.

One can prove the following propositions:

(25) For all elements a, b, c, d of the points of P_1 holds $a, b \parallel c, d$ if and only if $\langle \langle a, b \rangle, \langle c, d \rangle \rangle \in$ the parallelity of P_1.

(26) For all elements a, b, c, d of the points of P_1 holds $a, b \perp c, d$ if and only if $\langle \langle a, b \rangle, \langle c, d \rangle \rangle \in$ the orthogonality of P_1.

Let us consider V, w, y. Let us assume that w, y span the space. The functor **AMSp**(V, w, y) yielding a metric-affine structure is defined by:

(Def.7) $\mathbf{AMSp}(V, w, y) = \langle$ the vectors of $V, \lambda(\Uparrow_V)$, the ortogonality determined by w, y in $V\rangle$.

Next we state two propositions:

(27) If w, y span the space, then $P_1 = \mathbf{AMSp}(V, w, y)$ if and only if $P_1 = \langle$ the vectors of $V, \lambda(\Uparrow_V)$, the ortogonality determined by w, y in $V\rangle$.

(28) If w, y span the space, then the points of $\mathbf{AMSp}(V, w, y) =$ the vectors of V and the parallelity of $\mathbf{AMSp}(V, w, y) = \lambda(\Uparrow_V)$ and the orthogonality of $\mathbf{AMSp}(V, w, y) =$ the ortogonality determined by w, y in V.

Let us consider P_1. The affine reduct of P_1 yielding an affine structure is defined by:

(Def.8) the affine reduct of $P_1 = \langle$ the points of P_1, the parallelity of $P_1\rangle$.

We now state two propositions:

(29) For every P_1 and for every A_1 being an affine structure holds $A_1 =$ the affine reduct of P_1 if and only if $A_1 = \langle$ the points of P_1, the parallelity of $P_1\rangle$.

(30) If w, y span the space, then the affine reduct of $\mathbf{AMSp}(V, w, y) = \Lambda(\text{OASpace}\, V)$.

In the sequel p, p_1, p_2, q, q_1, r, r_1, r_2 denote elements of the points of $\mathbf{AMSp}(V, w, y)$. One can prove the following propositions:

(31) If w, y span the space and $p = u$ and $p_1 = u_1$ and $q = v$ and $q_1 = v_1$, then $p, q \perp p_1, q_1$ if and only if u, v, u_1 and v_1 are orthogonal w.r.t. w, y.

(32) If w, y span the space and $p = u$ and $q = v$ and $p_1 = u_1$ and $q_1 = v_1$, then $p, q \parallel p_1, q_1$ if and only if there exist a, b such that $a \cdot (v - u) = b \cdot (v_1 - u_1)$ but $a \neq 0$ or $b \neq 0$.

(33) If w, y span the space and $p, q \perp p_1, q_1$, then $p_1, q_1 \perp p, q$.

(34) If w, y span the space and $p, q \perp p_1, q_1$, then $p, q \perp q_1, p_1$.

(35) If w, y span the space, then for all p, q, r holds $p, q \perp r, r$.

(36) If w, y span the space and $p, p_1 \perp q, q_1$ and $p, p_1 \parallel r, r_1$, then $p = p_1$ or $q, q_1 \perp r, r_1$.

(37) If w, y span the space, then for every p, q, r there exists r_1 such that $p, q \perp r, r_1$ and $r \neq r_1$.

(38) If w, y span the space and $p, p_1 \perp q, q_1$ and $p, p_1 \perp r, r_1$, then $p = p_1$ or $q, q_1 \parallel r, r_1$.

(39) If w, y span the space and $p, q \perp r, r_1$ and $p, q \perp r, r_2$, then $p, q \perp r_1, r_2$.

(40) If w, y span the space and $p, q \perp p, q$, then $p = q$.

(41) If w, y span the space and $p, q \perp p_1, p_2$ and $p_1, q \perp p_2, p$, then $p_2, q \perp p, p_1$.

(42) If w, y span the space and $p \neq p_1$, then for every q there exists q_1 such that $p, p_1 \parallel p, q_1$ and $p, p_1 \perp q_1, q$.

A metric-affine structure is called a metric affine space if:

(Def.9) (i) $\langle$ the points of it, the parallelity of it$\rangle$ is an affine space,

(ii) for all elements a, b, c, d, p, q, r, s of the points of it holds if $a, b \perp a, b$, then $a = b$ but $a, b \perp c, c$ but if $a, b \perp c, d$, then $a, b \perp d, c$ and $c, d \perp a, b$ but if $a, b \perp p, q$ and $a, b \parallel r, s$, then $p, q \perp r, s$ or $a = b$ but if $a, b \perp p, q$ and $a, b \perp p, s$, then $a, b \perp q, s$,

(iii) for all elements a, b, c of the points of it such that $a \neq b$ there exists an element x of the points of it such that $a, b \parallel a, x$ and $a, b \perp x, c$,

(iv) for every elements a, b, c of the points of it there exists an element x of the points of it such that $a, b \perp c, x$ and $c \neq x$.

We now state two propositions:

(43) Given P_1. Then P_1 is a metric affine space if and only if the following conditions are satisfied:

(i) $\langle$ the points of P_1, the parallelity of $P_1\rangle$ is an affine space,

(ii) for all elements a, b, c, d, p, q, r, s of the points of P_1 holds if $a, b \perp a, b$, then $a = b$ but $a, b \perp c, c$ but if $a, b \perp c, d$, then $a, b \perp d, c$ and $c, d \perp a, b$ but if $a, b \perp p, q$ and $a, b \parallel r, s$, then $p, q \perp r, s$ or $a = b$ but if $a, b \perp p, q$ and $a, b \perp p, s$, then $a, b \perp q, s$,

(iii) for all elements a, b, c of the points of P_1 such that $a \neq b$ there exists an element x of the points of P_1 such that $a, b \parallel a, x$ and $a, b \perp x, c$,

(iv) for every elements a, b, c of the points of P_1 there exists an element x of the points of P_1 such that $a, b \perp c, x$ and $c \neq x$.

(44) If w, y span the space, then $\mathbf{AMSp}(V, w, y)$ is a metric affine space.

A metric-affine structure is said to be a metric affine plane if:

(Def.10) (i) $\langle$ the points of it, the parallelity of it$\rangle$ is an affine plane,

(ii) for all elements a, b, c, d, p, q, r, s of the points of it holds if $a, b \perp a, b$, then $a = b$ but $a, b \perp c, c$ but if $a, b \perp c, d$, then $a, b \perp d, c$ and $c, d \perp a, b$ but if $a, b \perp p, q$ and $a, b \parallel r, s$, then $p, q \perp r, s$ or $a = b$ but if $a, b \perp p, q$ and $a, b \perp r, s$, then $p, q \parallel r, s$ or $a = b$,

(iii) for every elements a, b, c of the points of it there exists an element x of the points of it such that $a, b \perp c, x$ and $c \neq x$.

Next we state four propositions:

(45) Given P_1. Then P_1 is a metric affine plane if and only if the following conditions are satisfied:

(i) $\langle$ the points of P_1, the parallelity of $P_1\rangle$ is an affine plane,

(ii) for all elements a, b, c, d, p, q, r, s of the points of P_1 holds if $a, b \perp a, b$, then $a = b$ but $a, b \perp c, c$ but if $a, b \perp c, d$, then $a, b \perp d, c$ and $c, d \perp a, b$ but if $a, b \perp p, q$ and $a, b \parallel r, s$, then $p, q \perp r, s$ or $a = b$ but if $a, b \perp p, q$ and $a, b \perp r, s$, then $p, q \parallel r, s$ or $a = b$,

(iii) for every elements a, b, c of the points of P_1 there exists an element x of the points of P_1 such that $a, b \perp c, x$ and $c \neq x$.

(46) If w, y span the space, then $\mathbf{AMSp}(V, w, y)$ is a metric affine plane.

(47) For an arbitrary x holds x is an element of the points of P_1 if and only if x is an element of the points of the affine reduct of P_1.

(48) For all elements a, b, c, d of the points of P_1 and for all elements a', b', c', d' of the points of the affine reduct of P_1 such that $a = a'$ and $b = b'$ and $c = c'$ and $d = d'$ holds $a, b \parallel c, d$ if and only if $a', b' \parallel c', d'$.

Let P_1 be a metric affine space. Then the affine reduct of P_1 is an affine space.

Let P_1 be a metric affine plane. Then the affine reduct of P_1 is an affine plane.

The following proposition is true

(49) For every metric affine plane P_1 holds P_1 is a metric affine space.

We see that the metric affine plane is a metric affine space.

The following two propositions are true:

(50) For every metric affine space P_1 such that the affine reduct of P_1 is an affine plane holds P_1 is a metric affine plane.

(51) Let P_1 be a metric-affine structure. Then P_1 is a metric affine plane if and only if the following conditions are satisfied:

(i) there exist elements a, b of the points of P_1 such that $a \neq b$,

(ii) for all elements a, b, c, d, p, q, r, s of the points of P_1 holds $a, b \parallel b, a$ and $a, b \parallel c, c$ but if $a, b \parallel p, q$ and $a, b \parallel r, s$, then $p, q \parallel r, s$ or $a = b$ but if $a, b \parallel a, c$, then $b, a \parallel b, c$ and there exists an element x of the points of P_1 such that $a, b \parallel c, x$ and $a, c \parallel b, x$ and there exist elements x, y, z of the points of P_1 such that $x, y \nparallel x, z$ and there exists an element x of the points of P_1 such that $a, b \parallel c, x$ and $c \neq x$ but if $a, b \parallel b, d$ and $b \neq a$, then there exists an element x of the points of P_1 such that $c, b \parallel b, x$ and $c, a \parallel d, x$ but if $a, b \perp a, b$, then $a = b$ and $a, b \perp c, c$ but if $a, b \perp c, d$, then $a, b \perp d, c$ and $c, d \perp a, b$ but if $a, b \perp p, q$ and $a, b \parallel r, s$, then $p, q \perp r, s$ or $a = b$ but if $a, b \perp p, q$ and $a, b \perp r, s$, then $p, q \parallel r, s$ or $a = b$ and there exists an element x of the points of P_1 such that $a, b \perp c, x$ and $c \neq x$ but if $a, b \nparallel c, d$, then there exists an element x of the points of P_1 such that $a, b \parallel a, x$ and $c, d \parallel c, x$.

In the sequel x, a, b, c, d, p, q will denote elements of the points of P_1. Let us consider P_1, a, b, c. The predicate $\mathbf{L}(a, b, c)$ is defined as follows:

(Def.11) $a, b \parallel a, c$.

We now state the proposition

(52) For every P_1 and for all a, b, c holds $\mathbf{L}(a, b, c)$ if and only if $a, b \parallel a, c$.

Let us consider P_1, a, b. The functor $\mathrm{Line}(a, b)$ yielding a subset of the points of P_1 is defined by:

(Def.12) for every element x of the points of P_1 holds $x \in \mathrm{Line}(a, b)$ if and only if $\mathbf{L}(a, b, x)$.

In the sequel A, K, M denote subsets of the points of P_1. The following proposition is true

(53) $A = \mathrm{Line}(a, b)$ if and only if for every x holds $x \in A$ if and only if $\mathbf{L}(a, b, x)$.

Let us consider P_1, A. We say that A is a line if and only if:

(Def.13) there exist a, b such that $a \neq b$ and $A = \mathrm{Line}(a, b)$.

Next we state several propositions:

(54) A is a line if and only if there exist a, b such that $a \neq b$ and $A = \mathrm{Line}(a,b)$.

(55) For every metric affine space P_1 and for all elements a, b, c of the points of P_1 and for all elements a', b', c' of the points of the affine reduct of P_1 such that $a = a'$ and $b = b'$ and $c = c'$ holds $\mathbf{L}(a,b,c)$ if and only if $\mathbf{L}(a',b',c')$.

(56) For every metric affine space P_1 and for all elements a, b of the points of P_1 and for all elements a', b' of the points of the affine reduct of P_1 such that $a = a'$ and $b = b'$ holds $\mathrm{Line}(a,b) = \mathrm{Line}(a',b')$.

(57) For an arbitrary X holds X is a subset of the points of P_1 if and only if X is a subset of the points of the affine reduct of P_1.

(58) For every metric affine space P_1 and for every subset X of the points of P_1 and for every subset Y of the points of the affine reduct of P_1 such that $X = Y$ holds X is a line if and only if Y is a line.

Let us consider P_1, a, b, K. The predicate $a, b \perp K$ is defined as follows:

(Def.14) there exist p, q such that $p \neq q$ and $K = \mathrm{Line}(p,q)$ and $a, b \perp p, q$.

Let us consider P_1, K, M. The predicate $K \perp M$ is defined by:

(Def.15) there exist p, q such that $p \neq q$ and $K = \mathrm{Line}(p,q)$ and $p, q \perp M$.

Let us consider P_1, K, M. The predicate $K \parallel M$ is defined by:

(Def.16) there exist a, b, c, d such that $a \neq b$ and $c \neq d$ and $K = \mathrm{Line}(a,b)$ and $M = \mathrm{Line}(c,d)$ and $a, b \parallel c, d$.

One can prove the following propositions:

(59) For all a, b, K holds $a, b \perp K$ if and only if there exist p, q such that $p \neq q$ and $K = \mathrm{Line}(p,q)$ and $a, b \perp p, q$.

(60) For all K, M holds $K \perp M$ if and only if there exist p, q such that $p \neq q$ and $K = \mathrm{Line}(p,q)$ and $p, q \perp M$.

(61) For all K, M holds $K \parallel M$ if and only if there exist a, b, c, d such that $a \neq b$ and $c \neq d$ and $K = \mathrm{Line}(a,b)$ and $M = \mathrm{Line}(c,d)$ and $a, b \parallel c, d$.

(62) If $a, b \perp K$, then K is a line but if $K \perp M$, then K is a line and M is a line.

(63) $K \perp M$ if and only if there exist a, b, c, d such that $a \neq b$ and $c \neq d$ and $K = \mathrm{Line}(a,b)$ and $M = \mathrm{Line}(c,d)$ and $a, b \perp c, d$.

(64) For every metric affine space P_1 and for all subsets M, N of the points of P_1 and for all subsets M', N' of the points of the affine reduct of P_1 such that $M = M'$ and $N = N'$ holds $M \parallel N$ if and only if $M' \parallel N'$.

We adopt the following rules: P_1 denotes a metric affine space, A, K, M, N denote subsets of the points of P_1, and a, b, c, d, p, q, r, s denote elements of the points of P_1. The following propositions are true:

(65) If K is a line, then $a, a \perp K$.

(66) If $a, b \perp K$ but $a, b \parallel c, d$ or $c, d \parallel a, b$ and $a \neq b$, then $c, d \perp K$.

(67) If $a, b \perp K$, then $b, a \perp K$.

(68) If $K \parallel M$, then $M \parallel K$.

(69) If $r, s \perp K$ but $K \parallel M$ or $M \parallel K$, then $r, s \perp M$.

(70) If $K \perp M$, then $M \perp K$.

(71) If $a \in K$ and $b \in K$ and $a, b \perp K$, then $a = b$.

(72) If K is a line, then $K \not\perp K$.

(73) If $K \perp M$ or $M \perp K$ but $K \parallel N$ or $N \parallel K$, then $M \perp N$ and $N \perp M$.

(74) If $K \parallel N$, then $K \not\perp N$.

(75) If $a \in K$ and $b \in K$ and $c, d \perp K$, then $c, d \perp a, b$ and $a, b \perp c, d$.

(76) If $a \in K$ and $b \in K$ and $a \neq b$ and K is a line, then $K = \mathrm{Line}(a, b)$.

(77) If $a \in K$ and $b \in K$ and $a \neq b$ and K is a line but $a, b \perp c, d$ or $c, d \perp a, b$, then $c, d \perp K$.

(78) If $a \in M$ and $b \in M$ and $c \in N$ and $d \in N$ and $M \perp N$, then $a, b \perp c, d$.

(79) If $p \in M$ and $p \in N$ and $a \in M$ and $b \in N$ and $a \neq b$ and $a \in K$ and $b \in K$ and $A \perp M$ and $A \perp N$ and K is a line, then $A \perp K$.

(80) $b, c \perp a, a$ and $a, a \perp b, c$ and $b, c \parallel a, a$ and $a, a \parallel b, c$.

(81) If $a, b \parallel c, d$, then $a, b \parallel d, c$ and $b, a \parallel c, d$ and $b, a \parallel d, c$ and $c, d \parallel a, b$ and $c, d \parallel b, a$ and $d, c \parallel a, b$ and $d, c \parallel b, a$.

(82) Suppose that

(i) $p \neq q$,

(ii) $p, q \parallel a, b$ and $p, q \parallel c, d$ or $p, q \parallel a, b$ and $c, d \parallel p, q$ or $a, b \parallel p, q$ and $c, d \parallel p, q$ or $a, b \parallel p, q$ and $p, q \parallel c, d$.

Then $a, b \parallel c, d$.

(83) If $a, b \perp c, d$, then $a, b \perp d, c$ and $b, a \perp c, d$ and $b, a \perp d, c$ and $c, d \perp a, b$ and $c, d \perp b, a$ and $d, c \perp a, b$ and $d, c \perp b, a$.

(84) Suppose that

(i) $p \neq q$,

(ii) $p, q \parallel a, b$ and $p, q \perp c, d$ or $p, q \parallel c, d$ and $p, q \perp a, b$ or $p, q \parallel a, b$ and $c, d \perp p, q$ or $p, q \parallel c, d$ and $a, b \perp p, q$ or $a, b \parallel p, q$ and $c, d \perp p, q$ or $c, d \parallel p, q$ and $a, b \perp p, q$ or $a, b \parallel p, q$ and $p, q \perp c, d$ or $c, d \parallel p, q$ and $p, q \perp a, b$.

Then $a, b \perp c, d$.

We follow the rules: P_1 is a metric affine plane, K, M, N are subsets of the points of P_1, and x, a, b, c, d, p, q are elements of the points of P_1. The following propositions are true:

(85) Suppose that

(i) $p \neq q$,

(ii) $p, q \perp a, b$ and $p, q \perp c, d$ or $p, q \perp a, b$ and $c, d \perp p, q$ or $a, b \perp p, q$ and $c, d \perp p, q$ or $a, b \perp p, q$ and $p, q \perp c, d$.

Then $a, b \parallel c, d$.

(86) If $a \in M$ and $b \in M$ and $a \neq b$ and M is a line and $c \in N$ and $d \in N$ and $c \neq d$ and N is a line and $a, b \parallel c, d$, then $M \parallel N$.

(87) If $K \perp M$ or $M \perp K$ but $K \perp N$ or $N \perp K$, then $M \parallel N$ and $N \parallel M$.

(88) If $M \perp N$, then there exists p such that $p \in M$ and $p \in N$.

(89) If $a, b \perp c, d$, then there exists p such that $\mathbf{L}(a, b, p)$ and $\mathbf{L}(c, d, p)$.

(90) If $a, b \perp K$, then there exists p such that $\mathbf{L}(a, b, p)$ and $p \in K$.

(91) There exists x such that $a, x \perp p, q$ and $\mathbf{L}(p, q, x)$.

(92) If K is a line, then there exists x such that $a, x \perp K$ and $x \in K$.

References

[1] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[2] Henryk Oryszczyszyn and Krzysztof Prażmowski. Analytical ordered affine spaces. *Formalized Mathematics*, 1(**3**):601–605, 1990.

[3] Henryk Oryszczyszyn and Krzysztof Prażmowski. Ordered affine spaces defined in terms of directed parallelity - part I. *Formalized Mathematics*, 1(**3**):611–615, 1990.

[4] Henryk Oryszczyszyn and Krzysztof Prażmowski. Parallelity and lines in affine spaces. *Formalized Mathematics*, 1(**3**):617–621, 1990.

[5] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[6] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[7] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received August 10, 1990