FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Projective Spaces - part III

Wojciech Leończuk[1]
Warsaw University
Białystok

Krzysztof Prażmowski[2]
Warsaw University
Białystok

Summary. In the classes of projective spaces, defined in terms of collinearity, introduced in the article [3], we distinguish the subclasses of Desarguesian projective structures. As examples of these types of objects we consider analytical projective spaces defined over suitable real linear spaces; analytical counterpart of the Desargues Axiom is proved without any assumption on the dimension of the underlying linear space.

MML Identifier: ANPROJ_4.

The articles [1], [4], [2], and [3] provide the notation and terminology for this paper. We adopt the following rules: V will denote a real linear space, o, p, p_1, p_2, p_3, q, q_1, q_2, q_3, r, r_1, r_2, r_3 will denote vectors of V, and a, b, c, a_1, b_1, a_2, c_2 will denote real numbers. Let us consider V, p_1, p_2, p_3. We say that p_1, p_2 and p_3 are proper vectors if and only if:

(Def.1) p_1 is a proper vector and p_2 is a proper vector and p_3 is a proper vector.

Next we state the proposition

(1) p_1, p_2 and p_3 are proper vectors if and only if p_1 is a proper vector and p_2 is a proper vector and p_3 is a proper vector.

Let us consider V, p_1, p_2, p_3, r_1, r_2, r_3. We say that p_1, p_2, p_3, r_1, r_2, and r_3 lie on a triangle if and only if:

(Def.2) p_1, p_2 and r_3 are lineary dependent and p_1, p_3 and r_2 are lineary dependent and p_2, p_3 and r_1 are lineary dependent.

Next we state the proposition

(2) p_1, p_2, p_3, r_1, r_2, and r_3 lie on a triangle if and only if p_1, p_2 and r_3 are linreary dependent and p_1, p_3 and r_2 are lineary dependent and p_2, p_3 and r_1 are lineary dependent.

[1]Supported by RPBP.III-24.C6.
[2]Supported by RPBP.III-24.C2.

ISSN 0777–4028

Let us consider V, o, p_1, p_2, p_3, q_1, q_2, q_3. We say that o, p_1, p_2, p_3, q_1, q_2, and q_3 are perspective if and only if:

(Def.3) o, p_1 and q_1 are lineary dependent and o, p_2 and q_2 are lineary dependent and o, p_3 and q_3 are lineary dependent.

The following propositions are true:

(3) o, p_1, p_2, p_3, q_1, q_2, and q_3 are perspective if and only if o, p_1 and q_1 are lineary dependent and o, p_2 and q_2 are lineary dependent and o, p_3 and q_3 are lineary dependent.

(4) Suppose o, p_1 and q_1 are lineary dependent and o and p_1 are not proportional and o and q_1 are not proportional and p_1 and q_1 are not proportional and o, p_1 and q_1 are proper vectors. Then there exist a_1, b_1 such that $b_1 \cdot q_1 = o + a_1 \cdot p_1$ and $a_1 \neq 0$ and $b_1 \neq 0$ and there exist a_2, c_2 such that $q_1 = c_2 \cdot o + a_2 \cdot p_1$ and $c_2 \neq 0$ and $a_2 \neq 0$.

(5) If p, q and r are lineary dependent and p and q are not proportional and p, q and r are proper vectors, then there exist a, b such that $r = a \cdot p + b \cdot q$.

(6) Suppose that

(i) o is a proper vector,
(ii) p_1, p_2 and p_3 are proper vectors,
(iii) q_1, q_2 and q_3 are proper vectors,
(iv) r_1, r_2 and r_3 are proper vectors,
(v) o, p_1, p_2, p_3, q_1, q_2, and q_3 are perspective,
(vi) o and q_1 are not proportional,
(vii) o and q_2 are not proportional,
(viii) o and q_3 are not proportional,
(ix) p_1 and q_1 are not proportional,
(x) p_2 and q_2 are not proportional,
(xi) p_3 and q_3 are not proportional,
(xii) o, p_1 and p_2 are not lineary dependent,
(xiii) o, p_1 and p_3 are not lineary dependent,
(xiv) o, p_2 and p_3 are not lineary dependent,
(xv) p_1, p_2, p_3, r_1, r_2, and r_3 lie on a triangle,
(xvi) q_1, q_2, q_3, r_1, r_2, and r_3 lie on a triangle.

Then r_1, r_2 and r_3 are lineary dependent.

We adopt the following rules: V will be a non-trivial real linear space and o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 will be elements of the points of the projective space over V. The following proposition is true

(7) Suppose that

(i) $o \neq q_1$,
(ii) $p_1 \neq q_1$,
(iii) $o \neq q_2$,
(iv) $p_2 \neq q_2$,
(v) $o \neq q_3$,
(vi) $p_3 \neq q_3$,

(vii) o, p_1 and p_2 are not collinear,
(viii) o, p_1 and p_3 are not collinear,
(ix) o, p_2 and p_3 are not collinear,
(x) p_1, p_2 and r_3 are collinear,
(xi) q_1, q_2 and r_3 are collinear,
(xii) p_2, p_3 and r_1 are collinear,
(xiii) q_2, q_3 and r_1 are collinear,
(xiv) p_1, p_3 and r_2 are collinear,
(xv) q_1, q_3 and r_2 are collinear,
(xvi) o, p_1 and q_1 are collinear,
(xvii) o, p_2 and q_2 are collinear,
(xviii) o, p_3 and q_3 are collinear.

Then r_1, r_2 and r_3 are collinear.

In the sequel u, v, w, y will denote vectors of V. A projective space defined in terms of collinearity is said to be a Desarguesian projective space defined in terms of collinearity if:

(Def.4) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of the points of it . Suppose that

(i) $o \neq q_1$,
(ii) $p_1 \neq q_1$,
(iii) $o \neq q_2$,
(iv) $p_2 \neq q_2$,
(v) $o \neq q_3$,
(vi) $p_3 \neq q_3$,
(vii) o, p_1 and p_2 are not collinear,
(viii) o, p_1 and p_3 are not collinear,
(ix) o, p_2 and p_3 are not collinear,
(x) p_1, p_2 and r_3 are collinear,
(xi) q_1, q_2 and r_3 are collinear,
(xii) p_2, p_3 and r_1 are collinear,
(xiii) q_2, q_3 and r_1 are collinear,
(xiv) p_1, p_3 and r_2 are collinear,
(xv) q_1, q_3 and r_2 are collinear,
(xvi) o, p_1 and q_1 are collinear,
(xvii) o, p_2 and q_2 are collinear,
(xviii) o, p_3 and q_3 are collinear.

Then r_1, r_2 and r_3 are collinear.

We now state three propositions:

(8) Let C_1 be a projective space defined in terms of collinearity. Then C_1 is a Desarguesian projective space defined in terms of collinearity if and only if for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear

and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear.

(9) If there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$, then the projective space over V is a Desarguesian projective space defined in terms of collinearity.

(10) Let C_1 be a collinearity structure. Then C_1 is a Desarguesian projective space defined in terms of collinearity if and only if the following conditions are satisfied:
- (i) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,
- (ii) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,
- (iii) for all elements p, p_1, p_2, r, r_1 of the points of C_1 such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of C_1 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,
- (iv) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,
- (v) there exist elements p, q, r of the points of C_1 such that p, q and r are not collinear,
- (vi) for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear.

A Fanoian projective space defined in terms of collinearity is called a Fano-Desarguesian projective space defined in terms of collinearity if:

(Def.5) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of the points of it . Suppose that
- (i) $o \neq q_1$,
- (ii) $p_1 \neq q_1$,
- (iii) $o \neq q_2$,
- (iv) $p_2 \neq q_2$,
- (v) $o \neq q_3$,
- (vi) $p_3 \neq q_3$,
- (vii) o, p_1 and p_2 are not collinear,
- (viii) o, p_1 and p_3 are not collinear,
- (ix) o, p_2 and p_3 are not collinear,

(x) p_1, p_2 and r_3 are collinear,
(xi) q_1, q_2 and r_3 are collinear,
(xii) p_2, p_3 and r_1 are collinear,
(xiii) q_2, q_3 and r_1 are collinear,
(xiv) p_1, p_3 and r_2 are collinear,
(xv) q_1, q_3 and r_2 are collinear,
(xvi) o, p_1 and q_1 are collinear,
(xvii) o, p_2 and q_2 are collinear,
(xviii) o, p_3 and q_3 are collinear.

Then r_1, r_2 and r_3 are collinear.

One can prove the following propositions:

(11) Let C_1 be a Fanoian projective space defined in terms of collinearity. Then C_1 is a Fano-Desarguesian projective space defined in terms of collinearity if and only if for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear.

(12) If there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$, then the projective space over V is a Fano-Desarguesian projective space defined in terms of collinearity.

(13) Let C_1 be a collinearity structure. Then C_1 is a Fano-Desarguesian projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(ii) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(iii) for all elements p, p_1, p_2, r, r_1 of the points of C_1 such that p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear there exists an element r_2 of the points of C_1 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear,

(iv) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(v) there exist elements p, q, r of the points of C_1 such that p, q and r are not collinear,

(vi) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1

are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear,

(vii) for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear.

(14) Let C_1 be a collinearity structure. Then C_1 is a Fano-Desarguesian projective space defined in terms of collinearity if and only if the following conditions are satisfied:

(i) C_1 is a Desarguesian projective space defined in terms of collinearity,

(ii) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

A projective plane defined in terms of collinearity is called a Desarguesian projective plane defined in terms of collinearity if:

(Def.6) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of the points of it . Suppose that

(i) $o \neq q_1$,
(ii) $p_1 \neq q_1$,
(iii) $o \neq q_2$,
(iv) $p_2 \neq q_2$,
(v) $o \neq q_3$,
(vi) $p_3 \neq q_3$,
(vii) o, p_1 and p_2 are not collinear,
(viii) o, p_1 and p_3 are not collinear,
(ix) o, p_2 and p_3 are not collinear,
(x) p_1, p_2 and r_3 are collinear,
(xi) q_1, q_2 and r_3 are collinear,
(xii) p_2, p_3 and r_1 are collinear,
(xiii) q_2, q_3 and r_1 are collinear,
(xiv) p_1, p_3 and r_2 are collinear,
(xv) q_1, q_3 and r_2 are collinear,
(xvi) o, p_1 and q_1 are collinear,
(xvii) o, p_2 and q_2 are collinear,
(xviii) o, p_3 and q_3 are collinear.

Then r_1, r_2 and r_3 are collinear.

We now state four propositions:

(15) Let C_1 be a projective plane defined in terms of collinearity. Then C_1 is a Desarguesian projective plane defined in terms of collinearity if and only if for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear.

(16) Suppose that

(i) there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and for every y there exist a, b, c such that $y = (a \cdot u + b \cdot v) + c \cdot w$.

Then the projective space over V is a Desarguesian projective plane defined in terms of collinearity.

(17) Let C_1 be a collinearity structure. Then C_1 is a Desarguesian projective plane defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(ii) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(iii) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) there exist elements p, q, r of the points of C_1 such that p, q and r are not collinear,

(v) for every elements p, p_1, q, q_1 of the points of C_1 there exists an element r of the points of C_1 such that p, p_1 and r are collinear and q, q_1 and r are collinear,

(vi) for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear.

(18) For every C_1 being a collinearity structure holds C_1 is a Desarguesian projective plane defined in terms of collinearity if and only if C_1 is a Desarguesian projective space defined in terms of collinearity and for every elements p, p_1, q, q_1 of the points of C_1 there exists an element r of the points of C_1 such that p, p_1 and r are collinear and q, q_1 and r are collinear.

A Fanoian projective plane defined in terms of collinearity is called a Fano-Desarguesian projective plane defined in terms of collinearity if:

(Def.7) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of the points of it . Suppose that

(i) $o \neq q_1$,
(ii) $p_1 \neq q_1$,
(iii) $o \neq q_2$,
(iv) $p_2 \neq q_2$,
(v) $o \neq q_3$,
(vi) $p_3 \neq q_3$,
(vii) o, p_1 and p_2 are not collinear,
(viii) o, p_1 and p_3 are not collinear,
(ix) o, p_2 and p_3 are not collinear,
(x) p_1, p_2 and r_3 are collinear,
(xi) q_1, q_2 and r_3 are collinear,
(xii) p_2, p_3 and r_1 are collinear,
(xiii) q_2, q_3 and r_1 are collinear,
(xiv) p_1, p_3 and r_2 are collinear,
(xv) q_1, q_3 and r_2 are collinear,
(xvi) o, p_1 and q_1 are collinear,
(xvii) o, p_2 and q_2 are collinear,
(xviii) o, p_3 and q_3 are collinear.

Then r_1, r_2 and r_3 are collinear.

One can prove the following propositions:

(19) Let C_1 be a Fanoian projective plane defined in terms of collinearity. Then C_1 is a Fano-Desarguesian projective plane defined in terms of collinearity if and only if for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear.

(20) Suppose that

(i) there exist u, v, w such that for all a, b, c such that $(a \cdot u + b \cdot v) + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and for every y there exist a, b, c such that $y = (a \cdot u + b \cdot v) + c \cdot w$.

Then the projective space over V is a Fano-Desarguesian projective plane defined in terms of collinearity.

(21) Let C_1 be a collinearity structure. Then C_1 is a Fano-Desarguesian projective plane defined in terms of collinearity if and only if the following conditions are satisfied:

(i) for all elements p, q, r, r_1, r_2 of the points of C_1 such that $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear holds r, r_1 and r_2 are collinear,

(ii) for all elements p, q, r of the points of C_1 holds p, q and p are collinear and p, p and q are collinear and p, q and q are collinear,

(iii) for every elements p, q of the points of C_1 there exists an element r of the points of C_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear,

(iv) there exist elements p, q, r of the points of C_1 such that p, q and r are not collinear,

(v) for every elements p, p_1, q, q_1 of the points of C_1 there exists an element r of the points of C_1 such that p, p_1 and r are collinear and q, q_1 and r are collinear,

(vi) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear,

(vii) for all elements o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 of the points of C_1 such that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear holds r_1, r_2 and r_3 are collinear.

(22) Let C_1 be a collinearity structure. Then C_1 is a Fano-Desarguesian projective plane defined in terms of collinearity if and only if the following conditions are satisfied:

(i) C_1 is a Desarguesian projective plane defined in terms of collinearity,

(ii) for all elements p_1, r_2, q, r_1, q_1, p, r of the points of C_1 such that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear holds p_1, r_2 and q_1 are collinear or p_1, r_2 and r_1 are collinear or p_1, r_1 and q_1 are collinear or r_2, r_1 and q_1 are collinear.

(23) For every C_1 being a collinearity structure holds

C_1

is a Fano-Desarguesian projective plane defined in terms of collinearity if and only if C_1 is a Fano-Desarguesian projective space defined in terms of collinearity and for every elements p, p_1, q, q_1 of the points of C_1 there exists an element r of the points of C_1 such that p, p_1 and r are collinear and q, q_1 and r are collinear.

References

[1] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[2] Wojciech Leończuk and Krzysztof Prażmowski. A construction of analytical projective space. *Formalized Mathematics*, 1(**4**):761–766, 1990.

[3] Wojciech Leończuk and Krzysztof Prażmowski. Projective spaces - part I. *Formalized Mathematics*, 1(**4**):767–776, 1990.

[4] Wojciech Skaba. The collinearity structure. *Formalized Mathematics*, 1(**4**):657–659, 1990.

Received August 10, 1990