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Filters - Part I

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Summary. Filters of a lattice, maximal filters (ultrafilters), operation to create a filter generating by an element or by a nonempty set of elements of the lattice are discussed. Besides, there are introduced implicative lattices such that for every two elements there is an element being pseudo-complement of them. Some facts concerning these concepts are presented too, i.e. for any proper filter there exists an ultrafilter consists it.

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The articles [3], [1], [4], [7], [5], [6], and [2] provide the notation and terminology for this paper. We adopt the following convention: L is a lattice, p, p_1, q, q_1, r, r_1 are elements of the carrier of L, and x is arbitrary. Let E be a non-empty set, and let p be an element of E. Then $\{p\}$ is a non-empty subset of E.

Let E be a non-empty set, and let D_1, D_2 be non-empty subsets of E. Then $D_1 \cup D_2$ is a non-empty subset of E.

The following propositions are true:

(1) If $p \sqsubseteq q$, then $r \sqcup p \sqsubseteq r \sqcup q$ and $p \sqcup r \sqsubseteq q \sqcup r$ and $p \sqcup r \sqsubseteq r \sqcup q$ and $r \sqcup p \sqsubseteq q \sqcup r$.

(2) If $p \sqsubseteq r$, then $p \sqcap q \sqsubseteq r$ and $q \sqcap p \sqsubseteq r$.

(3) If $p \sqsubseteq r$, then $p \sqsubseteq q \sqcup r$ and $p \sqsubseteq r \sqcup q$.

(4) If $p \sqsubseteq p_1$ and $q \sqsubseteq q_1$, then $p \sqcup q \sqsubseteq p_1 \sqcup q_1$ and $p \sqcup q \sqsubseteq q_1 \sqcup p_1$.

(5) If $p \sqsubseteq p_1$ and $q \sqsubseteq q_1$, then $p \sqcap q \sqsubseteq p_1 \sqcap q_1$ and $p \sqcap q \sqsubseteq q_1 \sqcap p_1$.

(6) If $p \sqsubseteq r$ and $q \sqsubseteq r$, then $p \sqcup q \sqsubseteq r$.

(7) If $r \sqsubseteq p$ and $r \sqsubseteq q$, then $r \sqsubseteq p \sqcap q$.

Let us consider L. A non-empty subset of the carrier of L is said to be a filter of L if:

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(Def.1) $p \in$ it and $q \in$ it if and only if $p \sqcap q \in$ it.

One can prove the following two propositions:

(8) For every non-empty subset D of the carrier of L holds D is a filter of L if and only if for all p, q holds $p \in D$ and $q \in D$ if and only if $p \sqcap q \in D$.

(9) For every non-empty subset D of the carrier of L holds D is a filter of L if and only if for all p, q such that $p \in D$ and $q \in D$ holds $p \sqcap q \in D$ and for all p, q such that $p \in D$ and $p \sqsubseteq q$ holds $q \in D$.

In the sequel H, F are filters of L. We now state several propositions:

(10) If $p \in H$, then $p \sqcup q \in H$ and $q \sqcup p \in H$.

(11) There exists p such that $p \in H$.

(12) If L is an upper bound lattice, then $\top_L \in H$.

(13) If L is an upper bound lattice, then $\{\top_L\}$ is a filter of L.

(14) If $\{p\}$ is a filter of L, then L is an upper bound lattice.

(15) The carrier of L is a filter of L.

Let us consider L. The functor $[L)$ yields a filter of L and is defined by:

(Def.2) $[L) =$ the carrier of L.

One can prove the following proposition

(16) $[L) =$ the carrier of L.

Let us consider L, p. The functor $[p)$ yields a filter of L and is defined as follows:

(Def.3) $[p) = \{q : p \sqsubseteq q\}$.

One can prove the following four propositions:

(17) $[p) = \{q : p \sqsubseteq q\}$.

(18) $q \in [p)$ if and only if $p \sqsubseteq q$.

(19) $p \in [p)$ and $p \sqcup q \in [p)$ and $q \sqcup p \in [p)$.

(20) If L is a lower bound lattice, then $[L) = [\bot_L)$.

Let us consider L, F. We say that F is ultrafilter if and only if:

(Def.4) $F \neq$ the carrier of L and for every H such that $F \subseteq H$ and $H \neq$ the carrier of L holds $F = H$.

One can prove the following four propositions:

(21) F is ultrafilter if and only if $F \neq$ the carrier of L and for every H such that $F \subseteq H$ and $H \neq$ the carrier of L holds $F = H$.

(22) If L is a lower bound lattice, then for every F such that $F \neq$ the carrier of L there exists H such that $F \subseteq H$ and H is ultrafilter.

(23) If there exists r such that $p \sqcap r \neq p$, then $[p) \neq$ the carrier of L.

(24) If L is a lower bound lattice and $p \neq \bot_L$, then there exists H such that $p \in H$ and H is ultrafilter.

In the sequel D is a non-empty subset of the carrier of L. Let us consider L, D. The functor $[D)$ yields a filter of L and is defined by:

(Def.5) $D \subseteq [D]$ and for every F such that $D \subseteq F$ holds $[D] \subseteq F$.

One can prove the following two propositions:

(25) $D \subseteq [D]$ and for every F such that $D \subseteq F$ holds $[D] \subseteq F$.

(26) $[F] = F$.

In the sequel D_1, D_2 will be non-empty subsets of the carrier of L. We now state several propositions:

(27) If $D_1 \subseteq D_2$, then $[D_1] \subseteq [D_2]$.

(28) $[[D]] \subseteq [D]$.

(29) If $p \in D$, then $[p] \subseteq [D]$.

(30) If $D = \{p\}$, then $[D] = [p]$.

(31) If L is a lower bound lattice and $\perp_L \in D$, then $[D] = [L]$ and $[D] =$ the carrier of L.

(32) If L is a lower bound lattice and $\perp_L \in F$, then $F = [L]$ and $F =$ the carrier of L.

Let us consider L, F. We say that F is prime if and only if:

(Def.6) $p \sqcup q \in F$ if and only if $p \in F$ or $q \in F$.

One can prove the following two propositions:

(33) F is prime if and only if for all p, q holds $p \sqcup q \in F$ if and only if $p \in F$ or $q \in F$.

(34) If L is a boolean lattice, then for all p, q holds $p \sqcap (p^{\mathrm{c}} \sqcup q) \sqsubseteq q$ and for every r such that $p \sqcap r \sqsubseteq q$ holds $r \sqsubseteq p^{\mathrm{c}} \sqcup q$.

A lattice is called a implicative lattice if:

(Def.7) for every elements p, q of the carrier of it there exists an element r of the carrier of it such that $p \sqcap r \sqsubseteq q$ and for every element r_1 of the carrier of it such that $p \sqcap r_1 \sqsubseteq q$ holds $r_1 \sqsubseteq r$.

One can prove the following proposition

(35) L is a implicative lattice if and only if for every p, q there exists r such that $p \sqcap r \sqsubseteq q$ and for every r_1 such that $p \sqcap r_1 \sqsubseteq q$ holds $r_1 \sqsubseteq r$.

Let us consider L, p, q. Let us assume that L is a implicative lattice. The functor $p \Rightarrow q$ yields an element of the carrier of L and is defined as follows:

(Def.8) $p \sqcap (p \Rightarrow q) \sqsubseteq q$ and for every r such that $p \sqcap r \sqsubseteq q$ holds $r \sqsubseteq p \Rightarrow q$.

The following proposition is true

(36) If L is a implicative lattice, then for all p, q, r holds $r = p \Rightarrow q$ if and only if $p \sqcap r \sqsubseteq q$ and for every r_1 such that $p \sqcap r_1 \sqsubseteq q$ holds $r_1 \sqsubseteq r$.

In the sequel I will denote a implicative lattice and i will denote an element of the carrier of I. The following three propositions are true:

(37) I is an upper bound lattice.

(38) $i \Rightarrow i = \top_I$.

(39) I is a distributive lattice.

In the sequel B is a boolean lattice and F_1, H_1 are filters of B. Next we state the proposition

(40) B is a implicative lattice.

We see that the implicative lattice is a distributive lattice.

For simplicity we follow the rules: I will be a implicative lattice, i, j, k will be elements of the carrier of I, D_3 will be a non-empty subset of the carrier of I, and F_2 will be a filter of I. The following propositions are true:

(41) If $i \in F_2$ and $i \Rightarrow j \in F_2$, then $j \in F_2$.

(42) If $j \in F_2$, then $i \Rightarrow j \in F_2$.

Let us consider L, D_1, D_2. The functor $D_1 \sqcap D_2$ yielding a non-empty subset of the carrier of L is defined as follows:

(Def.9) $D_1 \sqcap D_2 = \{p \sqcap q : p \in D_1 \wedge q \in D_2\}$.

Next we state four propositions:

(43) $D_1 \sqcap D_2 = \{p \sqcap q : p \in D_1 \wedge q \in D_2\}$.

(44) If $p \in D_1$ and $q \in D_2$, then $p \sqcap q \in D_1 \sqcap D_2$ and $q \sqcap p \in D_1 \sqcap D_2$.

(45) If $x \in D_1 \sqcap D_2$, then there exist p, q such that $x = p \sqcap q$ and $p \in D_1$ and $q \in D_2$.

(46) $D_1 \sqcap D_2 = D_2 \sqcap D_1$.

Let L be a distributive lattice, and let F_3, F_4 be filters of L. Then $F_3 \sqcap F_4$ is a filter of L.

Let L be a boolean lattice, and let F_3, F_4 be filters of L. Then $F_3 \sqcap F_4$ is a filter of L.

One can prove the following propositions:

(47) $[D_1 \cup D_2] = [[D_1] \cup D_2]$ and $[D_1 \cup D_2] = [D_1 \cup [D_2]]$.

(48) $[F \cup H] = \{r : \bigvee_{pq}[p \sqcap q \sqsubseteq r \wedge p \in F \wedge q \in H]\}$.

(49) $F \subseteq F \sqcap H$ and $H \subseteq F \sqcap H$.

(50) $[F \cup H] = [F \sqcap H]$.

In the sequel F_3, F_4 are filters of I. The following four propositions are true:

(51) $[F_3 \cup F_4] = F_3 \sqcap F_4$.

(52) $[F_1 \cup H_1] = F_1 \sqcap H_1$.

(53) If $j \in [D_3 \cup \{i\}]$, then $i \Rightarrow j \in [D_3]$.

(54) If $i \Rightarrow j \in F_2$ and $j \Rightarrow k \in F_2$, then $i \Rightarrow k \in F_2$.

In the sequel a, b, c will denote elements of the carrier of B. One can prove the following propositions:

(55) $a \Rightarrow b = a^{\mathrm{c}} \sqcup b$.

(56) $a \sqsubseteq b$ if and only if $a \sqcap b^{\mathrm{c}} = \bot_B$.

(57) F_1 is ultrafilter if and only if $F_1 \neq$ the carrier of B and for every a holds $a \in F_1$ or $a^{\mathrm{c}} \in F_1$.

(58) $F_1 \neq [B]$ and F_1 is prime if and only if F_1 is ultrafilter.

(59) If F_1 is ultrafilter, then for every a holds $a \in F_1$ if and only if $a^{\mathrm{c}} \notin F_1$.

(60) If $a \neq b$, then there exists F_1 such that F_1 is ultrafilter but $a \in F_1$ and $b \notin F_1$ or $a \notin F_1$ and $b \in F_1$.

In the sequel o_1, o_2 are binary operations on F. Let us consider L, F. The functor $\mathbb{L}_F$ yielding a lattice is defined as follows:

(Def.10) there exist o_1, o_2 such that $o_1 =$ (the join operation of L) $\upharpoonright [\!: F, F :\!]$ and $o_2 =$ (the meet operation of L) $\upharpoonright [\!: F, F :\!]$ and $\mathbb{L}_F = \langle F, o_1, o_2 \rangle$.

In the sequel K is a lattice. Next we state a number of propositions:

(61) $K = \mathbb{L}_F$ if and only if there exist o_1, o_2 such that $o_1 =$ (the join operation of L) $\upharpoonright [\!: F, F :\!]$ and $o_2 =$ (the meet operation of L) $\upharpoonright [\!: F, F :\!]$ and $K = \langle F, o_1, o_2 \rangle$.

(62) $\mathbb{L}_{[L]} = L$.

(63) The carrier of $\mathbb{L}_F = F$ and the join operation of $\mathbb{L}_F =$ (the join operation of L) $\upharpoonright [\!: F, F :\!]$ and the meet operation of $\mathbb{L}_F =$ (the meet operation of L) $\upharpoonright [\!: F, F :\!]$.

(64) For all p, q and for all elements p', q' of the carrier of $\mathbb{L}_F$ such that $p = p'$ and $q = q'$ holds $p \sqcup q = p' \sqcup q'$ and $p \sqcap q = p' \sqcap q'$.

(65) For all p, q and for all elements p', q' of the carrier of $\mathbb{L}_F$ such that $p = p'$ and $q = q'$ holds $p \sqsubseteq q$ if and only if $p' \sqsubseteq q'$.

(66) If L is an upper bound lattice, then $\mathbb{L}_F$ is an upper bound lattice.

(67) If L is a modular lattice, then $\mathbb{L}_F$ is a modular lattice.

(68) If L is a distributive lattice, then $\mathbb{L}_F$ is a distributive lattice.

(69) If L is a implicative lattice, then $\mathbb{L}_F$ is a implicative lattice.

(70) $\mathbb{L}_{[p]}$ is a lower bound lattice.

(71) $\perp_{\mathbb{L}_{[p]}} = p$.

(72) If L is an upper bound lattice, then $\top_{\mathbb{L}_{[p]}} = \top_L$.

(73) If L is an upper bound lattice, then $\mathbb{L}_{[p]}$ is a bound lattice.

(74) If L is a complemented lattice and L is a modular lattice, then $\mathbb{L}_{[p]}$ is a complemented lattice.

(75) If L is a boolean lattice, then $\mathbb{L}_{[p]}$ is a boolean lattice.

Let us consider L, p, q. The functor $p \Leftrightarrow q$ yielding an element of the carrier of L is defined by:

(Def.11) $p \Leftrightarrow q = p \Rightarrow q \sqcap q \Rightarrow p$.

Next we state three propositions:

(76) $p \Leftrightarrow q = p \Rightarrow q \sqcap q \Rightarrow p$.

(77) $p \Leftrightarrow q = q \Leftrightarrow p$.

(78) If $i \Leftrightarrow j \in F_2$ and $j \Leftrightarrow k \in F_2$, then $i \Leftrightarrow k \in F_2$.

Let us consider L, F. The functor $\equiv_F$ yielding a binary relation is defined as follows:

(Def.12) field $\equiv_F \subseteq$ the carrier of L and for all p, q holds $\langle p, q \rangle \in \equiv_F$ if and only if $p \Leftrightarrow q \in F$.

In the sequel R will denote a binary relation. We now state several propositions:

(79) $R = \equiv_F$ if and only if field $R \subseteq$ the carrier of L and for all p, q holds $\langle p, q\rangle \in R$ if and only if $p \Leftrightarrow q \in F$.

(80) $\equiv_F$ is a binary relation on the carrier of L.

(81) If L is a implicative lattice, then $\equiv_F$ is reflexive in the carrier of L.

(82) $\equiv_F$ is symmetric in the carrier of L.

(83) If L is a implicative lattice, then $\equiv_F$ is transitive in the carrier of L.

(84) If L is a implicative lattice, then $\equiv_F$ is an equivalence relation of the carrier of L.

(85) If L is a implicative lattice, then field $\equiv_F =$ the carrier of L.

Let us consider I, F_2. Then $\equiv_{F_2}$ is an equivalence relation of the carrier of I.

Let us consider B, F_1. Then $\equiv_{F_1}$ is an equivalence relation of the carrier of B.

Let us consider L, F, p, q. The predicate $p \equiv_F q$ is defined by:

(Def.13) $p \Leftrightarrow q \in F$.

Next we state several propositions:

(86) $p \equiv_F q$ if and only if $p \Leftrightarrow q \in F$.

(87) $p \equiv_F q$ if and only if $\langle p, q\rangle \in \equiv_F$.

(88) $i \equiv_{F_2} i$ and $a \equiv_{F_1} a$.

(89) If $p \equiv_F q$, then $q \equiv_F p$.

(90) If $i \equiv_{F_2} j$ and $j \equiv_{F_2} k$, then $i \equiv_{F_2} k$ but if $a \equiv_{F_1} b$ and $b \equiv_{F_1} c$, then $a \equiv_{F_1} c$.

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