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Finite Join and Finite Meet, and Dual Lattices

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Summary. The concepts of finite join and finite meet in a lattice are introduced. Some properties of the finite join are proved. After introducing the concept of dual lattice in view of dualism we obtain analogous properties of the meet. We prove these properties of binary operations in a lattice, which are usually included in axioms of the lattice theory. We also introduce the concept of Heyting lattice (a bounded lattice with relative pseudo-complements).

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The papers [10], [3], [4], [5], [8], [2], [11], [6], [9], [7], and [1] provide the notation and terminology for this paper. For simplicity we adopt the following convention: A denotes a set, C denotes a non-empty set, B denotes a subset of A, x denotes an element of A, and f, g denote functions from A into C. The following propositions are true:

(1) $f \restriction B$ is a function from B into C.

(2) $\mathrm{dom}(g \restriction B) = B$.

(3) $f \circ B = (f \restriction B) \circ B$.

(4) If $x \in B$, then $(f \restriction B)(x) = f(x)$.

(5) $f \restriction B = g \restriction B$ if and only if for every x such that $x \in B$ holds $g(x) = f(x)$.

(6) For every set B holds $f +\cdot g \restriction B$ is a function from A into C.

(7) $g \restriction B +\cdot f = f$.

(8) For all functions f, g such that $g \leq f$ holds $f +\cdot g = f$.

(9) $f +\cdot f \restriction B = f$.

(10) If for every x such that $x \in B$ holds $g(x) = f(x)$, then $f +\cdot g \restriction B = f$.

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In the sequel B will denote a finite subset of A. We now state several propositions:

(11) For every set X holds X is a finite subset of A if and only if $X \subseteq A$ and X is finite.

(12) $g \upharpoonright B +\cdot f = f$.

(13) $\mathrm{dom}(g \upharpoonright B) = B$.

(14) If for every x such that $x \in B$ holds $g(x) = f(x)$, then $f +\cdot g \upharpoonright B = f$.

(15) $f \circ B = (f \upharpoonright B) \circ B$.

(16) If $f \upharpoonright B = g \upharpoonright B$, then $f \circ B = g \circ B$.

Let D be a non-empty set, and let o, o' be binary operations on D. We say that o absorbs o' if and only if:

(Def.1) for all elements x, y of D holds $o(x, o'(x, y)) = x$.

In the sequel L will be a lattice structure. The following proposition is true

(17) If the join operation of L is commutative and the join operation of L is associative and the meet operation of L is commutative and the meet operation of L is associative and the join operation of L absorbs the meet operation of L and the meet operation of L absorbs the join operation of L, then L is a lattice.

Let L be a lattice structure. The functor L° yields a lattice structure and is defined by:

(Def.2) $L^\circ = \langle$ the carrier of L, the meet operation of L, the join operation of $L\rangle$.

One can prove the following propositions:

(18) The carrier of $L =$ the carrier of L° and the join operation of $L =$ the meet operation of L° and the meet operation of $L =$ the join operation of L°.

(19) $(L^\circ)^\circ = L$.

We follow the rules: L will be a lattice and a, b, u, v will be elements of the carrier of L. We now state a number of propositions:

(20) If for every v holds $u \sqcap v = u$, then $u = \perp_L$.

(21) If for every v holds $u \sqcup v = v$, then $u = \perp_L$.

(22) If for every v holds (the join operation of $L)(u, v) = v$, then $u = \perp_L$.

(23) If for every v holds $u \sqcup v = u$, then $u = \top_L$.

(24) If for every v holds $u \sqcap v = v$, then $u = \top_L$.

(25) If for every v holds (the meet operation of $L)(u, v) = v$, then $u = \top_L$.

(26) The join operation of L is idempotent.

(27) The join operation of L is commutative.

(28) If the join operation of L has a unity, then $\perp_L = \mathbf{1}_{\text{the join operation of } L}$.

(29) The join operation of L is associative.

(30) The meet operation of L is idempotent.

(31) The meet operation of L is commutative.

(32) The meet operation of L is associative.

(33) If the meet operation of L has a unity, then $\top_L = \mathbf{1}_{\text{the meet operation of } L}$.

(34) The join operation of L is distributive w.r.t. the join operation of L.

(35) If L is a distributive lattice, then the join operation of L is distributive w.r.t. the meet operation of L.

(36) If the join operation of L is distributive w.r.t. the meet operation of L, then L is a distributive lattice.

(37) If L is a distributive lattice, then the meet operation of L is distributive w.r.t. the join operation of L.

(38) If the meet operation of L is distributive w.r.t. the join operation of L, then L is a distributive lattice.

(39) The meet operation of L is distributive w.r.t. the meet operation of L.

(40) The join operation of L absorbs the meet operation of L.

(41) The meet operation of L absorbs the join operation of L.

We now define two new functors. Let A be a non-empty set, and let L be a lattice, and let B be a finite subset of A, and let f be a function from A into the carrier of L. The functor $\bigsqcup^{\mathsf{f}}_B f$ yields an element of the carrier of L and is defined as follows:

(Def.3) $\bigsqcup^{\mathsf{f}}_B f =$ (the join operation of L)-$\sum_B f$.

The functor $\bigsqcap^{\mathsf{f}}_B f$ yields an element of the carrier of L and is defined by:

(Def.4) $\bigsqcap^{\mathsf{f}}_B f =$ (the meet operation of L)-$\sum_B f$.

We now state the proposition

(42) For every non-empty set A and for every lattice L and for every finite subset B of A and for every function f from A into the carrier of L holds $\bigsqcup^{\mathsf{f}}_B f =$ (the join operation of L)-$\sum_B f$.

For simplicity we adopt the following convention: A will be a non-empty set, x will be an element of A, B will be a finite subset of A, and f, g will be functions from A into the carrier of L. Next we state several propositions:

(43) If $x \in B$, then $f(x) \sqsubseteq \bigsqcup^{\mathsf{f}}_B f$.

(44) If there exists x such that $x \in B$ and $u \sqsubseteq f(x)$, then $u \sqsubseteq \bigsqcup^{\mathsf{f}}_B f$.

(45) If for every x such that $x \in B$ holds $f(x) = u$ and $B \neq \emptyset$, then $\bigsqcup^{\mathsf{f}}_B f = u$.

(46) If $\bigsqcup^{\mathsf{f}}_B f \sqsubseteq u$, then for every x such that $x \in B$ holds $f(x) \sqsubseteq u$.

(47) If $B \neq \emptyset$ and for every x such that $x \in B$ holds $f(x) \sqsubseteq u$, then $\bigsqcup^{\mathsf{f}}_B f \sqsubseteq u$.

(48) If $B \neq \emptyset$ and for every x such that $x \in B$ holds $f(x) \sqsubseteq g(x)$, then $\bigsqcup^{\mathsf{f}}_B f \sqsubseteq \bigsqcup^{\mathsf{f}}_B g$.

(49) If $B \neq \emptyset$ and $f \restriction B = g \restriction B$, then $\bigsqcup^{\mathsf{f}}_B f = \bigsqcup^{\mathsf{f}}_B g$.

(50) If $B \neq \emptyset$, then $v \sqcup \bigsqcup^{\mathsf{f}}_B f = \bigsqcup^{\mathsf{f}}_B($ (the join operation of $L)^{\circ}(v, f))$.

Let L be a lattice. Then L° is a lattice.

We now state a number of propositions:

(51) For every lattice L and for every finite subset B of A and for every function f from A into the carrier of L and for every function f' from A into the carrier of L° such that $f = f'$ holds $\bigsqcup^{\mathsf{f}}_{B} f = \bigsqcap^{\mathsf{f}}_{B} f'$ and $\bigsqcap^{\mathsf{f}}_{B} f = \bigsqcup^{\mathsf{f}}_{B} f'$.

(52) For all elements a', b' of the carrier of L° such that $a = a'$ and $b = b'$ holds $a \sqcap b = a' \sqcup b'$ and $a \sqcup b = a' \sqcap b'$.

(53) If $a \sqsubseteq b$, then for all elements a', b' of the carrier of L° such that $a = a'$ and $b = b'$ holds $b' \sqsubseteq a'$.

(54) For all elements a', b' of the carrier of L° such that $a' \sqsubseteq b'$ and $a = a'$ and $b = b'$ holds $b \sqsubseteq a$.

(55) If $x \in B$, then $\bigsqcap^{\mathsf{f}}_{B} f \sqsubseteq f(x)$.

(56) If there exists x such that $x \in B$ and $f(x) \sqsubseteq u$, then $\bigsqcap^{\mathsf{f}}_{B} f \sqsubseteq u$.

(57) If for every x such that $x \in B$ holds $f(x) = u$ and $B \neq \emptyset$, then $\bigsqcap^{\mathsf{f}}_{B} f = u$.

(58) If $B \neq \emptyset$, then $v \sqcap \bigsqcap^{\mathsf{f}}_{B} f = \bigsqcap^{\mathsf{f}}_{B}($ (the meet operation of $L)^{\circ}(v, f))$.

(59) If $u \sqsubseteq \bigsqcap^{\mathsf{f}}_{B} f$, then for every x such that $x \in B$ holds $u \sqsubseteq f(x)$.

(60) If $B \neq \emptyset$ and $f \restriction B = g \restriction B$, then $\bigsqcap^{\mathsf{f}}_{B} f = \bigsqcap^{\mathsf{f}}_{B} g$.

(61) If $B \neq \emptyset$ and for every x such that $x \in B$ holds $u \sqsubseteq f(x)$, then $u \sqsubseteq \bigsqcap^{\mathsf{f}}_{B} f$.

(62) If $B \neq \emptyset$ and for every x such that $x \in B$ holds $f(x) \sqsubseteq g(x)$, then $\bigsqcap^{\mathsf{f}}_{B} f \sqsubseteq \bigsqcap^{\mathsf{f}}_{B} g$.

(63) For every lattice L holds L is a lower bound lattice if and only if L° is an upper bound lattice.

(64) For every lattice L holds L is an upper bound lattice if and only if L° is a lower bound lattice.

(65) L is a distributive lattice if and only if L° is a distributive lattice.

In the sequel L denotes a lower bound lattice, f, g denote functions from A into the carrier of L, and u denotes an element of the carrier of L. The following propositions are true:

(66) $\perp_L$ is a unity w.r.t. the join operation of L.

(67) The join operation of L has a unity.

(68) $\perp_L = \mathbf{1}_{\text{the join operation of } L}$.

(69) If $f \restriction B = g \restriction B$, then $\bigsqcup^{\mathsf{f}}_{B} f = \bigsqcup^{\mathsf{f}}_{B} g$.

(70) If for every x such that $x \in B$ holds $f(x) \sqsubseteq u$, then $\bigsqcup^{\mathsf{f}}_{B} f \sqsubseteq u$.

(71) If for every x such that $x \in B$ holds $f(x) \sqsubseteq g(x)$, then $\bigsqcup^{\mathsf{f}}_{B} f \sqsubseteq \bigsqcup^{\mathsf{f}}_{B} g$.

In the sequel L will denote an upper bound lattice, f, g will denote functions from A into the carrier of L, and u will denote an element of the carrier of L. The following propositions are true:

(72) $\top_L$ is a unity w.r.t. the meet operation of L.

(73) The meet operation of L has a unity.

(74) $\top_L = \mathbf{1}_{\text{the meet operation of } L}$.

(75) If $f \restriction B = g \restriction B$, then $\bigsqcap^{\mathsf{f}}_{B} f = \bigsqcap^{\mathsf{f}}_{B} g$.

(76) If for every x such that $x \in B$ holds $u \sqsubseteq f(x)$, then $u \sqsubseteq \bigsqcap^{\mathrm{f}}_{B} f$.

(77) If for every x such that $x \in B$ holds $f(x) \sqsubseteq g(x)$, then $\bigsqcap^{\mathrm{f}}_{B} f \sqsubseteq \bigsqcap^{\mathrm{f}}_{B} g$.

(78) For every lower bound lattice L holds $\bot_L = \top_{L^\circ}$.

(79) For every upper bound lattice L holds $\top_L = \bot_{L^\circ}$.

A lower bound lattice is called a distributive lower bounded lattice if:

(Def.5) it is a distributive lattice.

In the sequel L will denote a distributive lower bounded lattice, f, g will denote functions from A into the carrier of L, and u will denote an element of the carrier of L. We now state four propositions:

(80) The meet operation of L is distributive w.r.t. the join operation of L.

(81) (the meet operation of L)$(u, \bigsqcup^{\mathrm{f}}_{B} f) = \bigsqcup^{\mathrm{f}}_{B}($ (the meet operation of $L)^\circ(u, f))$.

(82) If for every x such that $x \in B$ holds $g(x) = u \sqcap f(x)$, then $u \sqcap \bigsqcup^{\mathrm{f}}_{B} f = \bigsqcup^{\mathrm{f}}_{B} g$.

(83) $u \sqcap \bigsqcup^{\mathrm{f}}_{B} f = \bigsqcup^{\mathrm{f}}_{B}($ (the meet operation of $L)^\circ(u, f))$.

A lower bound lattice is said to be a Heyting lattice if:

(Def.6) it is a implicative lattice.

Next we state the proposition

(84) For every lower bound lattice L holds L is a Heyting lattice if and only if for every elements x, z of the carrier of L there exists an element y of the carrier of L such that $x \sqcap y \sqsubseteq z$ and for every element v of the carrier of L such that $x \sqcap v \sqsubseteq z$ holds $v \sqsubseteq y$.

Let L be a lattice. We say that L is finite if and only if:

(Def.7) the carrier of L is finite.

We now state several propositions:

(85) For every lattice L holds L is finite if and only if L° is finite.

(86) For every lattice L such that L is finite holds L is a lower bound lattice.

(87) For every lattice L such that L is finite holds L is an upper bound lattice.

(88) For every lattice L such that L is finite holds L is a bound lattice.

(89) For every distributive lattice L such that L is finite holds L is a Heyting lattice.

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