FORMALIZED MATHEMATICS
Vol.1, No.5, November–December 1990
Université Catholique de Louvain

Basic Properties of Rational Numbers

Andrzej Kondracki[1]
Warsaw University

Summary. A definition of rational numbers and some basic properties of them. Operations of addition, substraction, multiplication are redefined for rational numbers. Functors numerator (num p) and denominator (den p) (p is rational) are defined and some properties of them are presented. Density of rational numbers is also given.

MML Identifier: `RAT_1`.

The notation and terminology used here are introduced in the following papers: [4], [2], [1], [3], and [5]. For simplicity we follow the rules: x is arbitrary, a, b are real numbers, k, k_1, l, l_1 are natural numbers, m, m_1, n, n_1 are integers, and D is a non-empty set. Let us consider m. Then $|m|$ is a natural number.

Let us consider k. Then $|k|$ is a natural number.

The non-empty set $\mathbb{Q}$ is defined by:

(Def.1) $x \in \mathbb{Q}$ if and only if there exist m, n such that $n \neq 0$ and $x = \frac{m}{n}$.

One can prove the following proposition

(1) $D = \mathbb{Q}$ if and only if for every x holds $x \in D$ if and only if there exist m, n such that $n \neq 0$ and $x = \frac{m}{n}$.

A real number is said to be a rational number if:

(Def.2) it is an element of $\mathbb{Q}$.

We now state a number of propositions:

(2) For every real number x holds x is a rational number if and only if x is an element of $\mathbb{Q}$.

(4)[2] If $x \in \mathbb{Q}$, then $x \in \mathbb{R}$.

(5) x is a rational number if and only if $x \in \mathbb{Q}$.

(6) x is a rational number if and only if there exist m, n such that $n \neq 0$ and $x = \frac{m}{n}$.

[1]Supported by RPBP.III-24.C1.

[2]The proposition (3) became obvious.

ISSN 0777–4028

(7) For every integer x holds x is a rational number.

(8) For every natural number x holds x is a rational number.

(9) 1 is a rational number and 0 is a rational number.

(10) $\mathbb{Q} \subseteq \mathbb{R}$.

(11) $\mathbb{Z} \subseteq \mathbb{Q}$.

(12) $\mathbb{N} \subseteq \mathbb{Q}$.

In the sequel p, q denote rational numbers. Next we state three propositions:

(13) If $x = \frac{k}{l}$ and $l \neq 0$, then x is a rational number.

(14) If $x = \frac{m}{k}$ and $k \neq 0$, then x is a rational number.

(15) If $x = \frac{k}{m}$ and $m \neq 0$, then x is a rational number.

Let us consider p, q. Then $p \cdot q$ is a rational number. Then $p + q$ is a rational number. Then $p - q$ is a rational number.

Let us consider p, m. Then $p + m$ is a rational number. Then $p - m$ is a rational number. Then $p \cdot m$ is a rational number.

Let us consider m, p. Then $m + p$ is a rational number. Then $m - p$ is a rational number. Then $m \cdot p$ is a rational number.

Let us consider p, k. Then $p + k$ is a rational number. Then $p - k$ is a rational number. Then $p \cdot k$ is a rational number.

Let us consider k, p. Then $k + p$ is a rational number. Then $k - p$ is a rational number. Then $k \cdot p$ is a rational number.

Let us consider p. Then $-p$ is a rational number. Then $|p|$ is a rational number.

One can prove the following propositions:

(16) For all p, q such that $q \neq 0$ holds $\frac{p}{q}$ is a rational number.

(17) If $k \neq 0$, then $\frac{p}{k}$ is a rational number.

(18) If $m \neq 0$, then $\frac{p}{m}$ is a rational number.

(19) If $p \neq 0$, then $\frac{k}{p}$ is a rational number and $\frac{m}{p}$ is a rational number.

(20) For every p such that $p \neq 0$ holds $\frac{1}{p}$ is a rational number.

(21) For every p such that $p \neq 0$ holds p^{-1} is a rational number.

(22) For all a, b such that $a < b$ there exists p such that $a < p$ and $p < b$.

(23) $a < b$ if and only if there exists p such that $a < p$ and $p < b$.

(24) For every p there exist m, k such that $k \neq 0$ and $p = \frac{m}{k}$.

(25) For every p there exist m, k such that $k \neq 0$ and $p = \frac{m}{k}$ and for all n, l such that $l \neq 0$ and $p = \frac{n}{l}$ holds $k \leq l$.

Let us consider p. The functor $\operatorname{den} p$ yielding a natural number is defined by:

(Def.3) $\operatorname{den} p \neq 0$ and there exists m such that $p = \frac{m}{\operatorname{den} p}$ and for all n, k such that $k \neq 0$ and $p = \frac{n}{k}$ holds $\operatorname{den} p \leq k$.

We now state the proposition

(26) $\operatorname{den} p \neq 0$ and there exists m such that $p = \frac{m}{\operatorname{den} p}$ and for all n, k such that $k \neq 0$ and $p = \frac{n}{k}$ holds $\operatorname{den} p \leq k$.

Let us consider p. The functor $\operatorname{num} p$ yields an integer and is defined by:

(Def.4) $\operatorname{num} p = \operatorname{den} p \cdot p$.

One can prove the following propositions:

(27) $0 < \operatorname{den} p$.

(28) $0 \neq \operatorname{den} p$.

(29) $1 \leq \operatorname{den} p$.

(30) $0 < \operatorname{den} p^{-1}$.

(31) $0 \leq \operatorname{den} p$.

(32) $0 \leq \operatorname{den} p^{-1}$.

(33) $0 \neq \operatorname{den} p^{-1}$.

(34) $1 \geq \operatorname{den} p^{-1}$.

(35) $\operatorname{num} p = \operatorname{den} p \cdot p$ and $\operatorname{num} p = p \cdot \operatorname{den} p$.

(36) $\operatorname{num} p = 0$ if and only if $p = 0$.

(37) $p = \frac{\operatorname{num} p}{\operatorname{den} p}$ and $p = \operatorname{num} p \cdot \operatorname{den} p^{-1}$ and $p = \operatorname{den} p^{-1} \cdot \operatorname{num} p$.

(38) If $p \neq 0$, then $\operatorname{den} p = \frac{\operatorname{num} p}{p}$.

(39) If $p = \frac{m}{k}$ and $k \neq 0$, then $\operatorname{den} p \leq k$.

(40) If p is an integer, then $\operatorname{den} p = 1$ and $\operatorname{num} p = p$.

(41) If $\operatorname{num} p = p$ or $\operatorname{den} p = 1$, then p is an integer.

(42) $\operatorname{num} p = p$ if and only if $\operatorname{den} p = 1$.

(43) If p is a natural number, then $\operatorname{den} p = 1$ and $\operatorname{num} p = p$.

(44) If $\operatorname{num} p = p$ or $\operatorname{den} p = 1$ but $0 \leq p$, then p is a natural number.

(45) $1 < \operatorname{den} p$ if and only if p is not an integer.

(46) $1 > \operatorname{den} p^{-1}$ if and only if p is not an integer.

(47) $\operatorname{num} p = \operatorname{den} p$ if and only if $p = 1$.

(48) $\operatorname{num} p = -\operatorname{den} p$ if and only if $p = -1$.

(49) $-\operatorname{num} p = \operatorname{den} p$ if and only if $p = -1$.

(50) Suppose $m \neq 0$. Then $p = \frac{\operatorname{num} p \cdot m}{\operatorname{den} p \cdot m}$ and $p = \frac{m \cdot \operatorname{num} p}{\operatorname{den} p \cdot m}$ and $p = \frac{m \cdot \operatorname{num} p}{m \cdot \operatorname{den} p}$ and $p = \frac{\operatorname{num} p \cdot m}{m \cdot \operatorname{den} p}$.

(51) Suppose $k \neq 0$. Then $p = \frac{\operatorname{num} p \cdot k}{\operatorname{den} p \cdot k}$ and $p = \frac{k \cdot \operatorname{num} p}{\operatorname{den} p \cdot k}$ and $p = \frac{k \cdot \operatorname{num} p}{k \cdot \operatorname{den} p}$ and $p = \frac{\operatorname{num} p \cdot k}{k \cdot \operatorname{den} p}$.

(52) Suppose $p = \frac{m}{n}$ and $n \neq 0$ and $m_1 \neq 0$. Then $p = \frac{m \cdot m_1}{n \cdot m_1}$ and $p = \frac{m_1 \cdot m}{n \cdot m_1}$ and $p = \frac{m_1 \cdot m}{m_1 \cdot n}$ and $p = \frac{m \cdot m_1}{m_1 \cdot n}$.

(53) Suppose $p = \frac{m}{l}$ and $l \neq 0$ and $m_1 \neq 0$. Then $p = \frac{m \cdot m_1}{l \cdot m_1}$ and $p = \frac{m_1 \cdot m}{l \cdot m_1}$ and $p = \frac{m_1 \cdot m}{m_1 \cdot l}$ and $p = \frac{m \cdot m_1}{m_1 \cdot l}$.

(54) Suppose $p = \frac{l}{n}$ and $n \neq 0$ and $m_1 \neq 0$. Then $p = \frac{l \cdot m_1}{n \cdot m_1}$ and $p = \frac{m_1 \cdot l}{n \cdot m_1}$ and $p = \frac{m_1 \cdot l}{m_1 \cdot n}$ and $p = \frac{l \cdot m_1}{m_1 \cdot n}$.

(55) Suppose $p = \frac{l}{l_1}$ and $l_1 \neq 0$ and $m_1 \neq 0$. Then $p = \frac{l \cdot m_1}{l_1 \cdot m_1}$ and $p = \frac{m_1 \cdot l}{l_1 \cdot m_1}$ and $p = \frac{m_1 \cdot l}{m_1 \cdot l_1}$ and $p = \frac{l \cdot m_1}{m_1 \cdot l_1}$.

(56) Suppose $p = \frac{m}{n}$ and $n \neq 0$ and $k \neq 0$. Then $p = \frac{m \cdot k}{n \cdot k}$ and $p = \frac{k \cdot m}{n \cdot k}$ and $p = \frac{k \cdot m}{k \cdot n}$ and $p = \frac{m \cdot k}{k \cdot n}$.

(57) Suppose $p = \frac{m}{l}$ and $l \neq 0$ and $k \neq 0$. Then $p = \frac{m \cdot k}{l \cdot k}$ and $p = \frac{k \cdot m}{l \cdot k}$ and $p = \frac{k \cdot m}{k \cdot l}$ and $p = \frac{m \cdot k}{k \cdot l}$.

(58) Suppose $p = \frac{l}{n}$ and $n \neq 0$ and $k \neq 0$. Then $p = \frac{l \cdot k}{n \cdot k}$ and $p = \frac{k \cdot l}{n \cdot k}$ and $p = \frac{k \cdot l}{k \cdot n}$ and $p = \frac{l \cdot k}{k \cdot n}$.

(59) Suppose $p = \frac{l}{l_1}$ and $l_1 \neq 0$ and $k \neq 0$. Then $p = \frac{l \cdot k}{l_1 \cdot k}$ and $p = \frac{k \cdot l}{l_1 \cdot k}$ and $p = \frac{k \cdot l}{k \cdot l_1}$ and $p = \frac{l \cdot k}{k \cdot l_1}$.

(60) If $k \neq 0$ and $p = \frac{m}{k}$, then there exists l such that $m = \operatorname{num} p \cdot l$ and $k = \operatorname{den} p \cdot l$.

(61) If $p = \frac{m}{n}$ and $n \neq 0$, then there exists m_1 such that $m = \operatorname{num} p \cdot m_1$ and $n = \operatorname{den} p \cdot m_1$.

(62) For no l holds $1 < l$ and there exist m, k such that $\operatorname{num} p = m \cdot l$ and $\operatorname{den} p = k \cdot l$.

(63) If $p = \frac{m}{k}$ and $k \neq 0$ and for no l holds $1 < l$ and there exist m_1, k_1 such that $m = m_1 \cdot l$ and $k = k_1 \cdot l$, then $k = \operatorname{den} p$ and $m = \operatorname{num} p$.

(64) $p < -1$ if and only if $\operatorname{num} p < -\operatorname{den} p$.

(65) $p \leq -1$ if and only if $\operatorname{num} p \leq -\operatorname{den} p$.

(66) $p < -1$ if and only if $\operatorname{den} p < -\operatorname{num} p$.

(67) $p \leq -1$ if and only if $\operatorname{den} p \leq -\operatorname{num} p$.

(68) $-1 < p$ if and only if $-\operatorname{den} p < \operatorname{num} p$.

(69) $p \geq -1$ if and only if $\operatorname{num} p \geq -\operatorname{den} p$.

(70) $-1 < p$ if and only if $-\operatorname{num} p < \operatorname{den} p$.

(71) $p \geq -1$ if and only if $\operatorname{den} p \geq -\operatorname{num} p$.

(72) $p < 1$ if and only if $\operatorname{num} p < \operatorname{den} p$.

(73) $p \leq 1$ if and only if $\operatorname{num} p \leq \operatorname{den} p$.

(74) $1 < p$ if and only if $\operatorname{den} p < \operatorname{num} p$.

(75) $p \geq 1$ if and only if $\operatorname{num} p \geq \operatorname{den} p$.

(76) $p < 0$ if and only if $\operatorname{num} p < 0$.

(77) $p \leq 0$ if and only if $\operatorname{num} p \leq 0$.

(78) $0 < p$ if and only if $0 < \operatorname{num} p$.

(79) $p \geq 0$ if and only if $\operatorname{num} p \geq 0$.

(80) $a < p$ if and only if $a \cdot \operatorname{den} p < \operatorname{num} p$.

(81) $a \leq p$ if and only if $a \cdot \operatorname{den} p \leq \operatorname{num} p$.

(82) $p < a$ if and only if $\operatorname{num} p < a \cdot \operatorname{den} p$.

(83) $a \geq p$ if and only if $a \cdot \operatorname{den} p \geq \operatorname{num} p$.

(84) $p = q$ if and only if $\operatorname{den} p = \operatorname{den} q$ and $\operatorname{num} p = \operatorname{num} q$.

(85) If $p = \frac{m}{n}$ and $n \neq 0$ and $q = \frac{m_1}{n_1}$ and $n_1 \neq 0$, then $p = q$ if and only if $m \cdot n_1 = m_1 \cdot n$.

(86) $p < q$ if and only if $\mathrm{num}\, p \cdot \mathrm{den}\, q < \mathrm{num}\, q \cdot \mathrm{den}\, p$.

(87) $\mathrm{den}(-p) = \mathrm{den}\, p$ and $\mathrm{num}(-p) = -\mathrm{num}\, p$.

(88) $0 < p$ and $q = \frac{1}{p}$ if and only if $\mathrm{num}\, q = \mathrm{den}\, p$ and $\mathrm{den}\, q = \mathrm{num}\, p$.

(89) $p < 0$ and $q = \frac{1}{p}$ if and only if $\mathrm{num}\, q = -\mathrm{den}\, p$ and $\mathrm{den}\, q = -\mathrm{num}\, p$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[3] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[4] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[5] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.

Received July 10, 1990