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Basis of Real Linear Space

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Summary. Notions of linear independence and dependence of set of vectors, the subspace generated by a set of vectors and basis of real linear space are introduced. Some theorems concerning those notion, are proved.

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The papers [6], [2], [1], [3], [11], [4], [10], [9], [5], [8], and [7] provide the notation and terminology for this paper. For simplicity we follow a convention: x is arbitrary, a, b are real numbers, V is a real linear space, W, W_1, W_2, W_3 are subspaces of V, v, v_1, v_2 are vectors of V, A, B are subsets of the vectors of V, L, L_1, L_2 are linear combinations of V, l is a linear combination of A, F, G are finite sequences of elements of the vectors of V, f is a function from the vectors of V into $\mathbb{R}$, X, Y, Z are sets, M is a non-empty family of sets, and C_1 is a choice function of M. One can prove the following four propositions:

(1) $\sum(L_1 + L_2) = \sum L_1 + \sum L_2$.

(2) $\sum(a \cdot L) = a \cdot \sum L$.

(3) $\sum(-L) = -\sum L$.

(4) $\sum(L_1 - L_2) = \sum L_1 - \sum L_2$.

We now define two new predicates. Let us consider V, A. We say that A is linearly independent if and only if:

(Def.1) for every l such that $\sum l = 0_V$ holds support $l = \emptyset$.

We say that A is linearly dependent if and only if A is not linearly independent.

One can prove the following propositions:

(5) A is linearly independent if and only if for every l such that $\sum l = 0_V$ holds support $l = \emptyset$.

(6) If $A \subseteq B$ and B is linearly independent, then A is linearly independent.

(7) If A is linearly independent, then $0_V \notin A$.

(8) $\emptyset_{\text{the vectors of } V}$ is linearly independent.

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(9) $\{v\}$ is linearly independent if and only if $v \neq 0_V$.

(10) $\{0_V\}$ is linearly dependent.

(11) If $\{v_1, v_2\}$ is linearly independent, then $v_1 \neq 0_V$ and $v_2 \neq 0_V$.

(12) $\{v, 0_V\}$ is linearly dependent and $\{0_V, v\}$ is linearly dependent.

(13) $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent if and only if $v_2 \neq 0_V$ and for every a holds $v_1 \neq a \cdot v_2$.

(14) $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent if and only if for all a, b such that $a \cdot v_1 + b \cdot v_2 = 0_V$ holds $a = 0$ and $b = 0$.

Let us consider V, A. The functor $\mathrm{Lin}(A)$ yields a subspace of V and is defined by:

(Def.2) the vectors of $\mathrm{Lin}(A) = \{\sum l\}$.

We now state four propositions:

(15) If the vectors of $W = \{\sum l\}$, then $W = \mathrm{Lin}(A)$.

(16) The vectors of $\mathrm{Lin}(A) = \{\sum l\}$.

(17) $x \in \mathrm{Lin}(A)$ if and only if there exists l such that $x = \sum l$.

(18) If $x \in A$, then $x \in \mathrm{Lin}(A)$.

The following propositions are true:

(19) $\mathrm{Lin}(\emptyset_{\text{the vectors of } V}) = \mathbf{0}_V$.

(20) If $\mathrm{Lin}(A) = \mathbf{0}_V$, then $A = \emptyset$ or $A = \{0_V\}$.

(21) If $A =$ the vectors of W, then $\mathrm{Lin}(A) = W$.

(22) If $A =$ the vectors of V, then $\mathrm{Lin}(A) = V$.

(23) If $A \subseteq B$, then $\mathrm{Lin}(A)$ is a subspace of $\mathrm{Lin}(B)$.

(24) If $\mathrm{Lin}(A) = V$ and $A \subseteq B$, then $\mathrm{Lin}(B) = V$.

(25) $\mathrm{Lin}(A \cup B) = \mathrm{Lin}(A) + \mathrm{Lin}(B)$.

(26) $\mathrm{Lin}(A \cap B)$ is a subspace of $\mathrm{Lin}(A) \cap \mathrm{Lin}(B)$.

(27) If A is linearly independent, then there exists B such that $A \subseteq B$ and B is linearly independent and $\mathrm{Lin}(B) = V$.

(28) If $\mathrm{Lin}(A) = V$, then there exists B such that $B \subseteq A$ and B is linearly independent and $\mathrm{Lin}(B) = V$.

Let us consider V. A subset of the vectors of V is called a basis of V if:

(Def.3) it is linearly independent and $\mathrm{Lin}(\mathrm{it}) = V$.

The following proposition is true

(29) If A is linearly independent and $\mathrm{Lin}(A) = V$, then A is a basis of V.

In the sequel I is a basis of V. Next we state a number of propositions:

(30) I is linearly independent.

(31) $\mathrm{Lin}(I) = V$.

(32) If A is linearly independent, then there exists I such that $A \subseteq I$.

(33) If $\mathrm{Lin}(A) = V$, then there exists I such that $I \subseteq A$.

(34) If $Z \neq \emptyset$ and Z is finite and for all X, Y such that $X \in Z$ and $Y \in Z$ holds $X \subseteq Y$ or $Y \subseteq X$, then $\bigcup Z \in Z$.

(35) If $\emptyset \notin M$, then $\operatorname{dom} C_1 = M$ and $\operatorname{rng} C_1 \subseteq \bigcup M$.

(36) $x \in \mathbf{0}_V$ if and only if $x = 0_V$.

(37) If W_1 is a subspace of W_3, then $W_1 \cap W_2$ is a subspace of W_3.

(38) If W_1 is a subspace of W_2 and W_1 is a subspace of W_3, then W_1 is a subspace of $W_2 \cap W_3$.

(39) If W_1 is a subspace of W_3 and W_2 is a subspace of W_3, then $W_1 + W_2$ is a subspace of W_3.

(40) If W_1 is a subspace of W_2, then W_1 is a subspace of $W_2 + W_3$.

(41) $f \cdot (F \frown G) = (f \cdot F) \frown (f \cdot G)$.

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