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Subspaces and Cosets of Subspaces in Vector Space

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Summary. We introduce the notions of subspace of vector space and coset of a subspace. We prove a number of theorems concerning those notions. Some theorems that belong rather to [1] are proved.

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The articles [3], [5], [2], [1], and [4] provide the terminology and notation for this paper. For simplicity we adopt the following rules: G_1 will denote a field, V, X, Y will denote vector spaces over G_1, u, v, v_1, v_2 will denote vectors of V, a, b, c will denote elements of G_1, and x will be arbitrary. Let us consider G_1, V. A subset of V is a subset of the carrier of the carrier of V.

In the sequel V_1, V_2, V_3 denote subsets of V. Let us consider G_1, V, V_1. We say that V_1 is linearly closed if and only if:

(Def.1) for all v, u such that $v \in V_1$ and $u \in V_1$ holds $v + u \in V_1$ and for all a, v such that $v \in V_1$ holds $a \cdot v \in V_1$.

The following propositions are true:

(1) If for all v, u such that $v \in V_1$ and $u \in V_1$ holds $v + u \in V_1$ and for all a, v such that $v \in V_1$ holds $a \cdot v \in V_1$, then V_1 is linearly closed.

(2) If V_1 is linearly closed, then for all v, u such that $v \in V_1$ and $u \in V_1$ holds $v + u \in V_1$.

(3) If V_1 is linearly closed, then for all a, v such that $v \in V_1$ holds $a \cdot v \in V_1$.

(4) If $V_1 \neq \emptyset$ and V_1 is linearly closed, then $\Theta_V \in V_1$.

(5) If V_1 is linearly closed, then for every v such that $v \in V_1$ holds $-v \in V_1$.

(6) If V_1 is linearly closed, then for all v, u such that $v \in V_1$ and $u \in V_1$ holds $v - u \in V_1$.

(7) $\{\Theta_V\}$ is linearly closed.

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(8) If the carrier of the carrier of $V = V_1$, then V_1 is linearly closed.

(9) If V_1 is linearly closed and V_2 is linearly closed and $V_3 = \{v + u : v \in V_1 \wedge u \in V_2\}$, then V_3 is linearly closed.

(10) If V_1 is linearly closed and V_2 is linearly closed, then $V_1 \cap V_2$ is linearly closed.

Let us consider G_1, V. A vector space over G_1 is said to be a subspace of V if:

(Def.2) the carrier of the carrier of it $\subseteq$ the carrier of the carrier of V and the zero of the carrier of it $=$ the zero of the carrier of V and the addition of the carrier of it $=$ (the addition of the carrier of V) $\upharpoonright [\!:$ the carrier of the carrier of it, the carrier of the carrier of it $:\!]$ and the multiplication of it $=$ (the multiplication of V) $\upharpoonright [\!:$ the carrier of G_1, the carrier of the carrier of it $:\!]$.

Next we state the proposition

(11) If the carrier of the carrier of $X \subseteq$ the carrier of the carrier of V and the zero of the carrier of $X =$ the zero of the carrier of V and the addition of the carrier of $X =$ (the addition of the carrier of V) $\upharpoonright [\!:$ the carrier of the carrier of X, the carrier of the carrier of $X :\!]$ and the multiplication of $X =$ (the multiplication of V) $\upharpoonright [\!:$ the carrier of G_1, the carrier of the carrier of $X :\!]$, then X is a subspace of V.

We adopt the following convention: W, W_1, W_2 will be subspaces of V and w, w_1, w_2 will be vectors of W. Next we state a number of propositions:

(12) The carrier of the carrier of $W \subseteq$ the carrier of the carrier of V.

(13) The zero of the carrier of $W =$ the zero of the carrier of V.

(14) The addition of the carrier of $W =$ (the addition of the carrier of V) $\upharpoonright [\!:$ the carrier of the carrier of W, the carrier of the carrier of $W :\!]$.

(15) The multiplication of $W =$ (the multiplication of V) $\upharpoonright [\!:$ the carrier of G_1, the carrier of the carrier of $W :\!]$.

(16) If $x \in W_1$ and W_1 is a subspace of W_2, then $x \in W_2$.

(17) If $x \in W$, then $x \in V$.

(18) w is a vector of V.

(19) $\Theta_W = \Theta_V$.

(20) $\Theta_{W_1} = \Theta_{W_2}$.

(21) If $w_1 = v$ and $w_2 = u$, then $w_1 + w_2 = v + u$.

(22) If $w = v$, then $a \cdot w = a \cdot v$.

(23) If $w = v$, then $-v = -w$.

(24) If $w_1 = v$ and $w_2 = u$, then $w_1 - w_2 = v - u$.

(25) $\Theta_V \in W$.

(26) $\Theta_{W_1} \in W_2$.

(27) $\Theta_W \in V$.

(28) If $u \in W$ and $v \in W$, then $u + v \in W$.

(29) If $v \in W$, then $a \cdot v \in W$.
(30) If $v \in W$, then $-v \in W$.
(31) If $u \in W$ and $v \in W$, then $u - v \in W$.
(32) V is a subspace of V.
(33) If V is a subspace of X and X is a subspace of V, then $V = X$.
(34) If V is a subspace of X and X is a subspace of Y, then V is a subspace of Y.
(35) If the carrier of the carrier of $W_1 \subseteq$ the carrier of the carrier of W_2, then W_1 is a subspace of W_2.
(36) If for every v such that $v \in W_1$ holds $v \in W_2$, then W_1 is a subspace of W_2.
(37) If the carrier of the carrier of $W_1 =$ the carrier of the carrier of W_2, then $W_1 = W_2$.
(38) If for every v holds $v \in W_1$ if and only if $v \in W_2$, then $W_1 = W_2$.
(39) If the carrier of the carrier of $W =$ the carrier of the carrier of V, then $W = V$.
(40) If for every v holds $v \in W$, then $W = V$.
(41) If the carrier of the carrier of $W = V_1$, then V_1 is linearly closed.
(42) If $V_1 \neq \emptyset$ and V_1 is linearly closed, then there exists W such that $V_1 =$ the carrier of the carrier of W.

Let us consider G_1, V. The functor $\mathbf{0}_V$ yielding a subspace of V is defined by:

(Def.3) the carrier of the carrier of $\mathbf{0}_V = \{\Theta_V\}$.

Let us consider G_1, V. The functor Ω_V yields a subspace of V and is defined by:

(Def.4) $\Omega_V = V$.

The following propositions are true:

(43) The carrier of the carrier of $\mathbf{0}_V = \{\Theta_V\}$.
(44) If the carrier of the carrier of $W = \{\Theta_V\}$, then $W = \mathbf{0}_V$.
(45) $\Omega_V = V$.
(46) $x \in \mathbf{0}_V$ if and only if $x = \Theta_V$.
(47) $\mathbf{0}_W = \mathbf{0}_V$.
(48) $\mathbf{0}_{W_1} = \mathbf{0}_{W_2}$.
(49) $\mathbf{0}_W$ is a subspace of V.
(50) $\mathbf{0}_V$ is a subspace of W.
(51) $\mathbf{0}_{W_1}$ is a subspace of W_2.
(52) W is a subspace of Ω_V.
(53) V is a subspace of Ω_V.

Let us consider G_1, V, v, W. The functor $v + W$ yielding a subset of V is defined by:

(Def.5) $v + W = \{v + u : u \in W\}$.

Let us consider G_1, V, W. A subset of V is said to be a coset of W if:

(Def.6) there exists v such that it $= v + W$.

In the sequel B, C will denote cosets of W. The following propositions are true:

(54) $v + W = \{v + u : u \in W\}$.

(55) There exists v such that $C = v + W$.

(56) If $V_1 = v + W$, then V_1 is a coset of W.

(57) $x \in v + W$ if and only if there exists u such that $u \in W$ and $x = v + u$.

(58) $\Theta_V \in v + W$ if and only if $v \in W$.

(59) $v \in v + W$.

(60) $\Theta_V + W =$ the carrier of the carrier of W.

(61) $v + \mathbf{0}_V = \{v\}$.

(62) $v + \Omega_V =$ the carrier of the carrier of V.

(63) $\Theta_V \in v + W$ if and only if $v + W =$ the carrier of the carrier of W.

(64) $v \in W$ if and only if $v + W =$ the carrier of the carrier of W.

(65) If $v \in W$, then $a \cdot v + W =$ the carrier of the carrier of W.

(66) If $a \neq 0_{G_1}$ and $a \cdot v + W =$ the carrier of the carrier of W, then $v \in W$.

(67) $v \in W$ if and only if $(-v) + W =$ the carrier of the carrier of W.

(68) $u \in W$ if and only if $v + W = (v + u) + W$.

(69) $u \in W$ if and only if $v + W = (v - u) + W$.

(70) $v \in u + W$ if and only if $u + W = v + W$.

(71) If $u \in v_1 + W$ and $u \in v_2 + W$, then $v_1 + W = v_2 + W$.

(72) If $a \neq 1_{G_1}$ and $a \cdot v \in v + W$, then $v \in W$.

(73) If $v \in W$, then $a \cdot v \in v + W$.

(74) If $v \in W$, then $-v \in v + W$.

(75) $u + v \in v + W$ if and only if $u \in W$.

(76) $v - u \in v + W$ if and only if $u \in W$.

(77) $u \in v + W$ if and only if there exists v_1 such that $v_1 \in W$ and $u = v + v_1$.

(78) $u \in v + W$ if and only if there exists v_1 such that $v_1 \in W$ and $u = v - v_1$.

(79) There exists v such that $v_1 \in v + W$ and $v_2 \in v + W$ if and only if $v_1 - v_2 \in W$.

(80) If $v + W = u + W$, then there exists v_1 such that $v_1 \in W$ and $v + v_1 = u$.

(81) If $v + W = u + W$, then there exists v_1 such that $v_1 \in W$ and $v - v_1 = u$.

(82) $v + W_1 = v + W_2$ if and only if $W_1 = W_2$.

(83) If $v + W_1 = u + W_2$, then $W_1 = W_2$.

In the sequel C_1 denotes a coset of W_1 and C_2 denotes a coset of W_2. One can prove the following propositions:

(84) There exists C such that $v \in C$.

(85) C is linearly closed if and only if $C =$ the carrier of the carrier of W.
(86) If $C_1 = C_2$, then $W_1 = W_2$.
(87) $\{v\}$ is a coset of $\mathbf{0}_V$.
(88) If V_1 is a coset of $\mathbf{0}_V$, then there exists v such that $V_1 = \{v\}$.
(89) The carrier of the carrier of W is a coset of W.
(90) The carrier of the carrier of V is a coset of Ω_V.
(91) If V_1 is a coset of Ω_V, then $V_1 =$ the carrier of the carrier of V.
(92) $\Theta_V \in C$ if and only if $C =$ the carrier of the carrier of W.
(93) $u \in C$ if and only if $C = u + W$.
(94) If $u \in C$ and $v \in C$, then there exists v_1 such that $v_1 \in W$ and $u+v_1 = v$.
(95) If $u \in C$ and $v \in C$, then there exists v_1 such that $v_1 \in W$ and $u-v_1 = v$.
(96) There exists C such that $v_1 \in C$ and $v_2 \in C$ if and only if $v_1 - v_2 \in W$.
(97) If $u \in B$ and $u \in C$, then $B = C$.

In the sequel w will denote a vector of V. One can prove the following propositions:

(99)[2] $(u + v) - w = u + (v - w)$.
(100) $-(-v) = v$.
(101) $v - (u - w) = (v - u) + w$.
(102) If $v + u = v$ or $u + v = v$, then $u = \Theta_V$.
(103) $(a - b) \cdot v = a \cdot v - b \cdot v$.
(104) $a - 0_{G_1} = a$.
(105) $a - a = 0_{G_1}$.
(106) $a - (b - c) = (a - b) + c$.

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