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The Reflection Theorem

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Summary. The goal is show that the reflection theorem holds. The theorem is as usual in the Morse-Kelley theory of classes (MK). That theory works with universal class which consists of all sets and every class is a subclass of it. In this paper (and in another Mizar articles) we work in Tarski-Grothendieck (TG) theory (see [16]) which ensures the existence of sets that have properties like universal class (i.e. this theory is stronger than MK). The sets are introduced in [14] and some concepts of MK are modeled. The concepts are: the class *On* of all ordinal numbers belonging to the universe, subclasses, transfinite sequences of non-empty elements of universe, etc. The reflection theorem states that if A_ξ is an increasing and continuous transfinite sequence of non-empty sets and class $A = \bigcup_{\xi \in On} A_\xi$, then for every formula H there is a strictly increasing continuous mapping $F : On \to On$ such that if $\varkappa$ is a critical number of F (i.e. $F(\varkappa) = \varkappa > 0$) and $f \in A_\varkappa^{\mathbf{VAR}}$, then $A, f \models H \equiv A_\varkappa, f \models H$. The proof is based on [13]. Besides, in the article it is shown that every universal class is a model of ZF set theory if ω (the first infinite ordinal number) belongs to it. Some propositions concerning ordinal numbers and sequences of them are also present.

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The notation and terminology used in this paper have been introduced in the following articles: [16], [15], [11], [12], [4], [5], [6], [10], [8], [1], [3], [9], [14], [2], and [7]. In the sequel W is a universal class, H is a ZF-formula, x is arbitrary, and X is a set. We now state several propositions:

(1) $W \models$ the axiom of extensionality.

(2) $W \models$ the axiom of pairs.

(3) $W \models$ the axiom of unions.

(4) If $\omega \in W$, then $W \models$ the axiom of infinity.

(5) $W \models$ the axiom of power sets.

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(6) For every H such that $\{x_0, x_1, x_2\}$ misses $\mathrm{Free}\, H$ holds $W \models$ the axiom of substitution for H.

(7) If $\omega \in W$, then W is a model of ZF.

For simplicity we follow the rules: E denotes a non-empty family of sets, F denotes a function, f denotes a function from VAR into E, A, B, C denote ordinal numbers, a, b denote ordinals of W, p_1 denotes a transfinite sequence of ordinals of W, and H denotes a ZF-formula. Let us consider A, B. Let us note that one can characterize the predicate $A \subseteq B$ by the following (equivalent) condition:

(Def.1) for every C such that $C \in A$ holds $C \in B$.

In this article we present several logical schemes. The scheme *ALFA* deals with a non-empty set $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists F such that $\mathrm{dom}\, F = \mathcal{A}$ and for every element d of $\mathcal{A}$ there exists A such that $A = F(d)$ and $\mathcal{P}[d, A]$ and for every B such that $\mathcal{P}[d, B]$ holds $A \subseteq B$

provided the parameters meet the following condition:

- for every element d of $\mathcal{A}$ there exists A such that $\mathcal{P}[d, A]$.

The scheme *ALFA'Universe* deals with a universal class $\mathcal{A}$, a non-empty set $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

there exists F such that $\mathrm{dom}\, F = \mathcal{B}$ and for every element d of $\mathcal{B}$ there exists an ordinal a of $\mathcal{A}$ such that $a = F(d)$ and $\mathcal{P}[d, a]$ and for every ordinal b of $\mathcal{A}$ such that $\mathcal{P}[d, b]$ holds $a \subseteq b$

provided the following condition is met:

- for every element d of $\mathcal{B}$ there exists an ordinal a of $\mathcal{A}$ such that $\mathcal{P}[d, a]$.

One can prove the following proposition

(8) x is an ordinal of W if and only if $x \in \mathrm{On}\, W$.

In the sequel p_2 is a sequence of ordinal numbers. Now we present three schemes. The scheme *OrdSeqOfUnivEx* deals with a universal class $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a transfinite sequence p_1 of ordinals of $\mathcal{A}$ such that for every ordinal a of $\mathcal{A}$ holds $\mathcal{P}[a, p_1(a)]$

provided the following conditions are satisfied:

- for all ordinals a, b_1, b_2 of $\mathcal{A}$ such that $\mathcal{P}[a, b_1]$ and $\mathcal{P}[a, b_2]$ holds $b_1 = b_2$,
- for every ordinal a of $\mathcal{A}$ there exists an ordinal b of $\mathcal{A}$ such that $\mathcal{P}[a, b]$.

The scheme *UOS_Exist* concerns a universal class $\mathcal{A}$, an ordinal $\mathcal{B}$ of $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding an ordinal of $\mathcal{A}$, and a binary functor $\mathcal{G}$ yielding an ordinal of $\mathcal{A}$ and states that:

there exists a transfinite sequence p_1 of ordinals of $\mathcal{A}$ such that $p_1(\mathbf{0}_{\mathcal{A}}) = \mathcal{B}$ and for all ordinals a, b of $\mathcal{A}$ such that $b = p_1(a)$ holds $p_1(\mathrm{succ}\, a) = \mathcal{F}(a, b)$ and for every ordinal a of $\mathcal{A}$ and for every sequence p_2 of ordinal numbers such that $a \neq \mathbf{0}_{\mathcal{A}}$ and a is a limit ordinal number and $p_2 = p_1 \restriction a$ holds $p_1(a) = \mathcal{G}(a, p_2)$

for all values of the parameters.

The scheme *Universe_Ind* concerns a universal class $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

for every ordinal a of $\mathcal{A}$ holds $\mathcal{P}[a]$

provided the parameters have the following properties:

- $\mathcal{P}[\mathbf{0}_{\mathcal{A}}]$,
- for every ordinal a of $\mathcal{A}$ such that $\mathcal{P}[a]$ holds $\mathcal{P}[\operatorname{succ} a]$,
- for every ordinal a of $\mathcal{A}$ such that $a \neq \mathbf{0}_{\mathcal{A}}$ and a is a limit ordinal number and for every ordinal b of $\mathcal{A}$ such that $b \in a$ holds $\mathcal{P}[b]$ holds $\mathcal{P}[a]$.

Let f be a function, and let W be a universal class, and let a be an ordinal of W. The functor $\bigcup_a f$ yields a set and is defined as follows:

(Def.2) $\bigcup_a f = \bigcup(W \restriction (f \restriction \mathbf{R}_a))$.

We now state several propositions:

(9) $\bigcup_a f = \bigcup(W \restriction (f \restriction \mathbf{R}_a))$.

(10) For every transfinite sequence L and for every A holds $L \restriction \mathbf{R}_A$ is a transfinite sequence.

(11) For every sequence L of ordinal numbers and for every A holds $L \restriction \mathbf{R}_A$ is a sequence of ordinal numbers.

(12) $\bigcup p_2$ is an ordinal number.

(13) $\bigcup(X \restriction p_2)$ is an ordinal number.

(14) $\mathrm{On}\,\mathbf{R}_A = A$.

(15) $p_2 \restriction \mathbf{R}_A = p_2 \restriction A$.

Let p_1 be a sequence of ordinal numbers, and let W be a universal class, and let a be an ordinal of W. Then $\bigcup_a p_1$ is an ordinal of W.

Next we state the proposition

(17)[2] For every transfinite sequence p_1 of ordinals of W holds $\bigcup_a p_1 = \bigcup(p_1 \restriction a)$ and $\bigcup_a(p_1 \restriction a) = \bigcup(p_1 \restriction a)$.

Let W be a universal class, and let a, b be ordinals of W. Then $a \cup b$ is an ordinal of W.

Let us consider W. A non-empty family of sets is said to be a non-empty set from W if:

(Def.3) it $\in W$.

Let us consider W. A non-empty family of sets is said to be a subclass of W if:

(Def.4) it $\subseteq W$.

Let us consider W. A transfinite sequence of elements of W is called a transfinite sequence of non-empty sets from W if:

(Def.5) $\operatorname{dom} \mathrm{it} = \mathrm{On}\, W$ and $\emptyset \notin \operatorname{rng} \mathrm{it}$.

[2]The proposition (16) became obvious.

We now state four propositions:

(18) E is a non-empty set from W if and only if $E \in W$.

(19) E is a subclass of W if and only if $E \subseteq W$.

(20) For every transfinite sequence T of elements of W holds T is a transfinite sequence of non-empty sets from W if and only if $\mathrm{dom}\,T = \mathrm{On}\,W$ and $\emptyset \notin \mathrm{rng}\,T$.

(21) For every non-empty set D from W holds D is a subclass of W.

Let us consider W, and let L be a transfinite sequence of non-empty sets from W. Then $\bigcup L$ is a subclass of W. Let us consider a. Then $L(a)$ is a non-empty set from W.

In the sequel L is a transfinite sequence of non-empty sets from W and f is a function from VAR into $L(a)$. Next we state several propositions:

(22) If $X \in W$, then $\overline{\overline{X}} < \overline{\overline{W}}$.

(23) $a \in \mathrm{dom}\,L$.

(24) $L(a) \subseteq \bigcup L$.

(25) $\mathbb{N} \approx \mathrm{VAR}$ and $\overline{\overline{\mathrm{VAR}}} = \overline{\overline{\mathbb{N}}}$.

(26) $\bigcup(\mathrm{On}\,X)$ is an ordinal number.

(27) $\sup X \subseteq \mathrm{succ}(\bigcup(\mathrm{On}\,X))$.

(28) If $X \in W$, then $\sup X \in W$.

The following proposition is true

(29) Suppose $\omega \in W$ and for all a, b such that $a \in b$ holds $L(a) \subseteq L(b)$ and for every a such that $a \neq \mathbf{0}$ and a is a limit ordinal number holds $L(a) = \bigcup(L \upharpoonright a)$. Then for every H there exists p_1 such that p_1 is increasing and p_1 is continuous and for every a such that $p_1(a) = a$ and $\mathbf{0} \neq a$ for every f holds $\bigcup L, \bigcup L[f] \models H$ if and only if $L(a), f \models H$.

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