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Real Normed Space

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Summary. We construct a real normed space $\langle \mathrm{V}, \|.\| \rangle$, where V is a real vector space and $\|.\|$ is a norm. Auxillary properties of the norm are proved. Next, we introduce the notion of sequence in the real normed space. The basic operations on sequences (addition, substraction, multiplication by real number) are defined. We study some properties of sequences in the real normed space and the operations on them.

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The notation and terminology used in this paper have been introduced in the following papers: [5], [13], [16], [3], [4], [1], [2], [17], [11], [12], [9], [7], [8], [10], [15], [14], and [6]. We consider normed structures which are systems

$\langle$vectors, a norm$\rangle$,

where the vectors constitute a real linear space and the norm is a function from the vectors of the vectors into $\mathbb{R}$.

In the sequel X is a normed structure and a, b are real numbers. Let us consider X. A point of X is an element of the vectors of the vectors of X.

In the sequel x denotes a point of X. Let us consider X, x. The functor $\|x\|$ yields a real number and is defined as follows:

(Def.1) $\|x\| = (\text{the norm of } X)(x)$.

A normed structure is said to be a real normed space if:

(Def.2) for all points x, y of it and for every a holds $\|x\| = 0$ if and only if $x = 0_{\text{the vectors of it}}$ but $\|a \cdot x\| = |a| \cdot \|x\|$ and $\|x + y\| \leq \|x\| + \|y\|$.

We adopt the following rules: R_1 is a real normed space and x, y, z, g are points of R_1. The following propositions are true:

(2)[2] $\|x\| = 0$ if and only if $x = 0_{\text{the vectors of } R_1}$.

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[2]The proposition (1) was either repeated or obvious.

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(3) $\|a \cdot x\| = |a| \cdot \|x\|$.

(4) $\|x + y\| \leq \|x\| + \|y\|$.

(5) $\|0_{\text{the vectors of } R_1}\| = 0$.

(6) $\|-x\| = \|x\|$.

(7) $\|x - y\| \leq \|x\| + \|y\|$.

(8) $0 \leq \|x\|$.

(9) $\|a \cdot x + b \cdot y\| \leq |a| \cdot \|x\| + |b| \cdot \|y\|$.

(10) $\|x - y\| = 0$ if and only if $x = y$.

(11) $\|x - y\| = \|y - x\|$.

(12) $\|x\| - \|y\| \leq \|x - y\|$.

(13) $|\|x\| - \|y\|| \leq \|x - y\|$.

(14) $\|x - z\| \leq \|x - y\| + \|y - z\|$.

(15) If $x \neq y$, then $\|x - y\| \neq 0$.

Let us consider R_1. A subset of R_1 is a subset of the vectors of the vectors of R_1.

Let us consider R_1. A function is called a sequence of R_1 if:

(Def.3) dom it $= \mathbb{N}$ and rng it $\subseteq$ the vectors of the vectors of R_1.

For simplicity we adopt the following rules: S, S_1, S_2, T are sequences of R_1, k, n, m are natural numbers, r is a real number, f is a function, and d is arbitrary. We now state several propositions:

(17)[3] f is a sequence of R_1 if and only if dom $f = \mathbb{N}$ and for every d such that $d \in \mathbb{N}$ holds $f(d)$ is a point of R_1.

(18) For all S, T such that for every n holds $S(n) = T(n)$ holds $S = T$.

(19) For every x there exists S such that rng $S = \{x\}$.

(20) If there exists x such that for every n holds $S(n) = x$, then there exists x such that rng $S = \{x\}$.

(21) If there exists x such that rng $S = \{x\}$, then for every n holds $S(n) = S(n + 1)$.

(22) If for every n holds $S(n) = S(n + 1)$, then for all n, k holds $S(n) = S(n + k)$.

(23) If for all n, k holds $S(n) = S(n+k)$, then for all n, m holds $S(n) = S(m)$.

(24) If for all n, m holds $S(n) = S(m)$, then there exists x such that for every n holds $S(n) = x$.

(25) There exists S such that rng $S = \{0_{\text{the vectors of } R_1}\}$.

Let us consider R_1, S. We say that S is constant if and only if:

(Def.4) there exists x such that for every n holds $S(n) = x$.

The following propositions are true:

(27)[4] S is constant if and only if there exists x such that rng $S = \{x\}$.

[3]The proposition (16) was either repeated or obvious.
[4]The proposition (26) was either repeated or obvious.

(28) For every n holds $S(n)$ is a point of R_1.

Let us consider R_1, S, n. Then $S(n)$ is a point of R_1.

The scheme *ExRNSSeq* concerns a real normed space $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a point of $\mathcal{A}$ and states that:

there exists a sequence S of $\mathcal{A}$ such that for every n holds $S(n) = \mathcal{F}(n)$

for all values of the parameters.

Let us consider R_1, S_1, S_2. The functor $S_1 + S_2$ yielding a sequence of R_1 is defined as follows:

(Def.5) for every n holds $(S_1 + S_2)(n) = S_1(n) + S_2(n)$.

One can prove the following proposition

(29) $S = S_1 + S_2$ if and only if for every n holds $S(n) = S_1(n) + S_2(n)$.

Let us consider R_1, S_1, S_2. The functor $S_1 - S_2$ yielding a sequence of R_1 is defined as follows:

(Def.6) for every n holds $(S_1 - S_2)(n) = S_1(n) - S_2(n)$.

The following proposition is true

(30) $S = S_1 - S_2$ if and only if for every n holds $S(n) = S_1(n) - S_2(n)$.

Let us consider R_1, S, x. The functor $S - x$ yields a sequence of R_1 and is defined by:

(Def.7) for every n holds $(S - x)(n) = S(n) - x$.

Next we state the proposition

(31) $T = S - x$ if and only if for every n holds $T(n) = S(n) - x$.

Let us consider R_1, S, a. The functor $a \cdot S$ yields a sequence of R_1 and is defined by:

(Def.8) for every n holds $(a \cdot S)(n) = a \cdot S(n)$.

We now state the proposition

(32) $T = a \cdot S$ if and only if for every n holds $T(n) = a \cdot S(n)$.

Let us consider R_1, S. We say that S is convergent if and only if:

(Def.9) there exists g such that for every r such that $0 < r$ there exists m such that for every n such that $m \leq n$ holds $\|S(n) - g\| < r$.

One can prove the following propositions:

(34)[5] If S_1 is convergent and S_2 is convergent, then $S_1 + S_2$ is convergent.

(35) If S_1 is convergent and S_2 is convergent, then $S_1 - S_2$ is convergent.

(36) If S is convergent, then $S - x$ is convergent.

(37) If S is convergent, then $a \cdot S$ is convergent.

Let us consider R_1, S. The functor $\|S\|$ yielding a sequence of real numbers is defined by:

(Def.10) for every n holds $\|S\|(n) = \|S(n)\|$.

Next we state two propositions:

[5]The proposition (33) was either repeated or obvious.

(38) $\|S\|$ is a sequence of real numbers if and only if for every n holds $\|S\|(n) = \|S(n)\|$.

(39) If S is convergent, then $\|S\|$ is convergent.

Let us consider R_1, S. Let us assume that S is convergent. The functor $\lim S$ yielding a point of R_1 is defined by:

(Def.11) for every r such that $0 < r$ there exists m such that for every n such that $m \leq n$ holds $\|S(n) - (\lim S)\| < r$.

The following propositions are true:

(40) If S is convergent, then $\lim S = g$ if and only if for every r such that $0 < r$ there exists m such that for every n such that $m \leq n$ holds $\|S(n) - g\| < r$.

(41) If S is convergent and $\lim S = g$, then $\|S - g\|$ is convergent and $\lim \|S - g\| = 0$.

(42) If S_1 is convergent and S_2 is convergent, then $\lim(S_1 + S_2) = \lim S_1 + \lim S_2$.

(43) If S_1 is convergent and S_2 is convergent, then $\lim(S_1 - S_2) = \lim S_1 - \lim S_2$.

(44) If S is convergent, then $\lim(S - x) = \lim S - x$.

(45) If S is convergent, then $\lim(a \cdot S) = a \cdot (\lim S)$.

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