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Several Properties of Fields. Field Theory

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Summary. The article includes a continuation of the paper [2]. Some simple theorems concerning basic properties of a field are proved.

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The articles [8], [7], [5], [6], [3], [1], [2], and [4] provide the terminology and notation for this paper. The following propositions are true:

(1) For every field F holds $-_F(\mathbf{0}_F) = \mathbf{0}_F$.

(2) For every field F holds $\overset{-1}{_F}(\mathbf{1}_F) = \mathbf{1}_F$.

(3) For every field F and for all elements a, b of the support of F holds $-_F(+_F(\langle a, -_F(b)\rangle)) = +_F(\langle b, -_F(a)\rangle)$.

(4) For every field F and for all elements a, b of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\overset{-1}{_F}(\cdot_F(\langle a, \overset{-1}{_F}(b)\rangle)) = \cdot_F(\langle b, \overset{-1}{_F}(a)\rangle)$.

(5) For every field F and for all elements a, b of the support of F holds $-_F(+_F(\langle a, b\rangle)) = +_F(\langle -_F(a), -_F(b)\rangle)$.

(6) For every field F and for all elements a, b of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\overset{-1}{_F}(\cdot_F(\langle a, b\rangle)) = \cdot_F(\langle \overset{-1}{_F}(a), \overset{-1}{_F}(b)\rangle)$.

(7) For every field F and for all elements a, b, c, d of the support of F holds $+_F(\langle a, -_F(b)\rangle) = +_F(\langle c, -_F(d)\rangle)$ if and only if $+_F(\langle a, d\rangle) = +_F(\langle b, c\rangle)$.

(8) Let F be a field. Then for all elements a, c of the support of F and for all elements b, d of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\cdot_F(\langle a, \overset{-1}{_F}(b)\rangle) = \cdot_F(\langle c, \overset{-1}{_F}(d)\rangle)$ if and only if $\cdot_F(\langle a, d\rangle) = \cdot_F(\langle b, c\rangle)$.

(9) For every field F and for all elements a, b of the support of F holds $\cdot_F(\langle a, b\rangle) = \mathbf{0}_F$ if and only if $a = \mathbf{0}_F$ or $b = \mathbf{0}_F$.

(10) Let F be a field. Let a, b be elements of the support of F. Let c, d be elements of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$. Then $\cdot_F(\langle \cdot_F(\langle a, \overset{-1}{_F}(c)\rangle), \cdot_F(\langle b, \overset{-1}{_F}(d)\rangle)\rangle) = \cdot_F(\langle \cdot_F(\langle a, b\rangle), \overset{-1}{_F}(\cdot_F(\langle c, d\rangle))\rangle)$.

(11) Let F be a field. Let a, b be elements of the support of F. Let c, d be elements of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$.

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Then $+_F(\langle \cdot_F(\langle a, \overset{-1}{\underset{F}{}}(c)\rangle), \cdot_F(\langle b, \overset{-1}{\underset{F}{}}(d)\rangle)\rangle) =$
$\cdot_F(\langle +_F(\langle \cdot_F(\langle a, d\rangle), \cdot_F(\langle b, c\rangle)\rangle), \overset{-1}{\underset{F}{}}(\cdot_F(\langle c, d\rangle))\rangle)$.

Let F be a field. The functor osf F yielding a binary operation of the support of F is defined as follows:

(Def.1) for all elements x, y of the support of F holds $(\text{osf } F)(\langle x, y\rangle) = +_F(\langle x, -_F(y)\rangle)$.

The following propositions are true:

(12) For every field F and for every binary operation S of the support of F holds $S = \text{osf } F$ if and only if for all elements x, y of the support of F holds $S(\langle x, y\rangle) = +_F(\langle x, -_F(y)\rangle)$.

(13) For every field F and for all elements x, y of the support of F holds $\text{osf } F(\langle x, y\rangle) = +_F(\langle x, -_F(y)\rangle)$.

(14) For every field F and for every element x of the support of F holds $\text{osf } F(\langle x, x\rangle) = \mathbf{0}_F$.

(15) For every field F and for all elements a, b, c of the support of F holds $\cdot_F(\langle a, \text{osf } F(\langle b, c\rangle)\rangle) = \text{osf } F(\langle \cdot_F(\langle a, b\rangle), \cdot_F(\langle a, c\rangle)\rangle)$.

(16) For every field F and for all elements a, b of the support of F holds $\text{osf } F(\langle a, b\rangle)$ is an element of the support of F.

(17) For every field F and for all elements a, b, c of the support of F holds $\cdot_F(\langle \text{osf } F(\langle a, b\rangle), c\rangle) = \text{osf } F(\langle \cdot_F(\langle a, c\rangle), \cdot_F(\langle b, c\rangle)\rangle)$.

(18) For every field F and for all elements a, b of the support of F holds $\text{osf } F(\langle a, b\rangle) = -_F(\text{osf } F(\langle b, a\rangle))$.

(19) For every field F and for all elements a, b of the support of F holds $\text{osf } F(\langle -_F(a), b\rangle) = -_F(+_F(\langle a, b\rangle))$.

(20) For every field F and for all elements a, b, c, d of the support of F holds $\text{osf } F(\langle a, b\rangle) = \text{osf } F(\langle c, d\rangle)$ if and only if $+_F(\langle a, d\rangle) = +_F(\langle b, c\rangle)$.

(21) For every field F and for every element a of the support of F holds $\text{osf } F(\langle \mathbf{0}_F, a\rangle) = -_F(a)$.

(22) For every field F and for every element a of the support of F holds $\text{osf } F(\langle a, \mathbf{0}_F\rangle) = a$.

(23) For every field F and for all elements a, b, c of the support of F holds $+_F(\langle a, b\rangle) = c$ if and only if $\text{osf } F(\langle c, a\rangle) = b$.

(24) For every field F and for all elements a, b, c of the support of F holds $+_F(\langle a, b\rangle) = c$ if and only if $\text{osf } F(\langle c, b\rangle) = a$.

(25) For every field F and for all elements a, b, c of the support of F holds $\text{osf } F(\langle a, \text{osf } F(\langle b, c\rangle)\rangle) = +_F(\langle \text{osf } F(\langle a, b\rangle), c\rangle)$.

(26) For every field F and for all elements a, b, c of the support of F holds $\text{osf } F(\langle a, +_F(\langle b, c\rangle)\rangle) = \text{osf } F(\langle \text{osf } F(\langle a, b\rangle), c\rangle)$.

Let F be a field. The functor ovf F yields a function from the support of F#(the support of $F \setminus \text{single}(\mathbf{0}_F)$) into the support of F and is defined as follows:

(Def.2) for every element x of the support of F and for every element y of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $(\mathrm{ovf}\ F)(\langle x, y\rangle) = \cdot_F(\langle x, \overset{-1}{_F}(y)\rangle)$.

Next we state a number of propositions:

(27) Let F be a field. Then for every function D from the support of F#(the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$) into the support of F holds $D = \mathrm{ovf}\ F$ if and only if for every element x of the support of F and for every element y of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $D(\langle x, y\rangle) = \cdot_F(\langle x, \overset{-1}{_F}(y)\rangle)$.

(28) For every field F and for every element x of the support of F and for every element y of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\mathrm{ovf}\ F(\langle x, y\rangle) = \cdot_F(\langle x, \overset{-1}{_F}(y)\rangle)$.

(29) For every field F and for every element x of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\mathrm{ovf}\ F(\langle x, x\rangle) = \mathbf{1}_F$.

(30) For every field F and for every element a of the support of F and for every element b of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\mathrm{ovf}\ F(\langle a, b\rangle)$ is an element of the support of F.

(31) For every field F and for all elements a, b of the support of F and for every element c of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\cdot_F(\langle a, \mathrm{ovf}\ F(\langle b, c\rangle)\rangle) = \mathrm{ovf}\ F(\langle \cdot_F(\langle a, b\rangle), c\rangle)$.

(32) For every field F and for every element a of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\cdot_F(\langle a, \mathrm{ovf}\ F(\langle \mathbf{1}_F, a\rangle)\rangle) = \mathbf{1}_F$ and $\cdot_F(\langle \mathrm{ovf}\ F(\langle \mathbf{1}_F, a\rangle), a\rangle) = \mathbf{1}_F$.

(34)[1] For every field F and for all elements a, b of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\cdot_F(\langle a, \overset{-1}{_F}(b)\rangle) = \overset{-1}{_F}(\cdot_F(\langle b, \overset{-1}{_F}(a)\rangle))$.

(35) For every field F and for all elements a, b of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\mathrm{ovf}\ F(\langle a, b\rangle) = \overset{-1}{_F}(\mathrm{ovf}\ F(\langle b, a\rangle))$.

(36) For every field F and for all elements a, b of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\mathrm{ovf}\ F(\langle \overset{-1}{_F}(a), b\rangle) = \overset{-1}{_F}(\cdot_F(\langle a, b\rangle))$.

(37) For every field F and for all elements a, c of the support of F and for all elements b, d of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\mathrm{ovf}\ F(\langle a, b\rangle) = \mathrm{ovf}\ F(\langle c, d\rangle)$ if and only if $\cdot_F(\langle a, d\rangle) = \cdot_F(\langle b, c\rangle)$.

(38) For every field F and for every element a of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\mathrm{ovf}\ F(\langle \mathbf{1}_F, a\rangle) = \overset{-1}{_F}(a)$.

(39) For every field F and for every element a of the support of F holds $\mathrm{ovf}\ F(\langle a, \mathbf{1}_F\rangle) = a$.

(40) For every field F and for every element a of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ and for all elements b, c of the support of F holds $\cdot_F(\langle a, b\rangle) = c$ if and only if $\mathrm{ovf}\ F(\langle c, a\rangle) = b$.

(41) For every field F and for all elements a, c of the support of F and for every element b of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\cdot_F(\langle a, b\rangle) = c$ if and only if $\mathrm{ovf}\ F(\langle c, b\rangle) = a$.

[1]The proposition (33) was either repeated or obvious.

(42) For every field F and for every element a of the support of F and for all elements b, c of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\mathrm{ovf}\, F(\langle a, \mathrm{ovf}\, F(\langle b, c\rangle)\rangle) = \cdot_F(\langle \mathrm{ovf}\, F(\langle a, b\rangle), c\rangle)$.

(43) For every field F and for every element a of the support of F and for all elements b, c of the support of $F \setminus \mathrm{single}(\mathbf{0}_F)$ holds $\mathrm{ovf}\, F(\langle a, \cdot_F(\langle b, c\rangle)\rangle) = \mathrm{ovf}\, F(\langle \mathrm{ovf}\, F(\langle a, b\rangle), c\rangle)$.

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