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Series of Positive Real Numbers. Measure Theory

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Summary. We introduce properties of a series of nonnegative $\overline{\mathbb{R}}$ numbers, where $\overline{\mathbb{R}}$ denotes the enlarged set of real numbers, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. The paper contains definitions of sup F and inf F, for F being function, and a definition of a sumable subset of $\overline{\mathbb{R}}$. We prove the basic theorems regarding the definitions mentioned above. The work is the second part of a series of articles concerning the Lebesgue measure theory.

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The notation and terminology used here are introduced in the following articles: [6], [5], [2], [3], [4], and [1]. Let x, y be *Real numbers*. Let us assume that neither $x = +\infty$ and $y = -\infty$ nor $x = -\infty$ and $y = +\infty$. The functor $x + y$ yielding a *Real number* is defined by:

(Def.1) there exist real numbers a, b such that $x = a$ and $y = b$ and $x+y = a+b$ or $x = +\infty$ and $x + y = +\infty$ or $y = +\infty$ and $x + y = +\infty$ or $x = -\infty$ and $x + y = -\infty$ or $y = -\infty$ and $x + y = -\infty$.

Next we state four propositions:

(1) Let x, y be *Real numbers*. Suppose neither $x = +\infty$ and $y = -\infty$ nor $x = -\infty$ and $y = +\infty$. Then
 (i) there exist real numbers a, b such that $x = a$ and $y = b$ and $x+y = a+b$, or
 (ii) $x = +\infty$ and $x + y = +\infty$, or
 (iii) $y = +\infty$ and $x + y = +\infty$, or
 (iv) $x = -\infty$ and $x + y = -\infty$, or
 (v) $y = -\infty$ and $x + y = -\infty$.

(2) For all *Real numbers* x, y and for all real numbers a, b such that $x = a$ and $y = b$ holds $x + y = a + b$.

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(3) For every *Real number* x such that $x \neq -\infty$ holds $+\infty + x = +\infty$ and $x + +\infty = +\infty$.

(4) For every *Real number* x such that $x \neq +\infty$ holds $-\infty + x = -\infty$ and $x + -\infty = -\infty$.

Let x, y be *Real numbers*. Let us assume that neither $x = +\infty$ and $y = +\infty$ nor $x = -\infty$ and $y = -\infty$. The functor $x - y$ yielding a *Real number* is defined by:

(Def.2) there exist real numbers a, b such that $x = a$ and $y = b$ and $x-y = a-b$ or $x = +\infty$ and $x - y = +\infty$ or $y = +\infty$ and $x - y = -\infty$ or $x = -\infty$ and $x - y = -\infty$ or $y = -\infty$ and $x - y = +\infty$.

We now state a number of propositions:

(5) Let x, y be *Real numbers*. Suppose neither $x = +\infty$ and $y = +\infty$ nor $x = -\infty$ and $y = -\infty$. Then

(i) there exist real numbers a, b such that $x = a$ and $y = b$ and $x-y = a-b$, or

(ii) $x = +\infty$ and $x - y = +\infty$, or

(iii) $y = +\infty$ and $x - y = -\infty$, or

(iv) $x = -\infty$ and $x - y = -\infty$, or

(v) $y = -\infty$ and $x - y = +\infty$.

(6) For all *Real numbers* x, y and for all real numbers a, b such that $x = a$ and $y = b$ holds $x - y = a - b$.

(7) For every *Real number* x such that $x \neq +\infty$ holds $+\infty - x = +\infty$ and $x - +\infty = -\infty$.

(8) For every *Real number* x such that $x \neq -\infty$ holds $-\infty - x = -\infty$ and $x - -\infty = +\infty$.

(9) For all *Real numbers* x, s such that $x + s = +\infty$ holds $x = +\infty$ or $s = +\infty$.

(10) For all *Real numbers* x, s such that $x + s = -\infty$ holds $x = -\infty$ or $s = -\infty$.

(11) For all *Real numbers* x, s such that $x - s = +\infty$ holds $x = +\infty$ or $s = -\infty$.

(12) For all *Real numbers* x, s such that $x - s = -\infty$ holds $x = -\infty$ or $s = +\infty$.

(13) For all *Real numbers* x, s such that neither $x = +\infty$ and $s = -\infty$ nor $x = -\infty$ and $s = +\infty$ and $x + s \in \mathbb{R}$ holds $x \in \mathbb{R}$ and $s \in \mathbb{R}$.

(14) For all *Real numbers* x, s such that neither $x = +\infty$ and $s = +\infty$ nor $x = -\infty$ and $s = -\infty$ and $x - s \in \mathbb{R}$ holds $x \in \mathbb{R}$ and $s \in \mathbb{R}$.

(15) Let x, y, s, t be *Real numbers*. Then if neither $x = +\infty$ and $s = -\infty$ nor $x = -\infty$ and $s = +\infty$ and neither $y = +\infty$ and $t = -\infty$ nor $y = -\infty$ and $t = +\infty$ and $x \leq y$ and $s \leq t$, then $x + s \leq y + t$.

(16) Let x, y, s, t be *Real numbers*. Then if neither $x = +\infty$ and $t = +\infty$ nor $x = -\infty$ and $t = -\infty$ and neither $y = +\infty$ and $s = +\infty$ nor $y = -\infty$

and $s = -\infty$ and $x \leq y$ and $s \leq t$, then $x - t \leq y - s$.

Let x be a *Real number*. The functor $-x$ yields a *Real number* and is defined by:

(Def.3) there exists a real number a such that $x = a$ and $-x = -a$ or $x = +\infty$ and $-x = -\infty$ or $x = -\infty$ and $-x = +\infty$.

We now state several propositions:

(17) For every *Real number* x and for every *Real number* z holds $z = -x$ if and only if there exists a real number a such that $x = a$ and $z = -a$ or $x = +\infty$ and $z = -\infty$ or $x = -\infty$ and $z = +\infty$.

(18) For every *Real number* x holds there exists a real number a such that $x = a$ and $-x = -a$ or $x = +\infty$ and $-x = -\infty$ or $x = -\infty$ and $-x = +\infty$.

(19) For every *Real number* x and for every real number a such that $x = a$ holds $-x = -a$.

(20) For every *Real number* x holds if $x = +\infty$, then $-x = -\infty$ but if $x = -\infty$, then $-x = +\infty$.

(21) For every *Real number* x holds $-(-x) = x$.

(22) For all *Real numbers* x, y holds $x \leq y$ if and only if $-y \leq -x$.

(23) For all *Real numbers* x, y holds $x < y$ if and only if $-y < -x$.

(24) For all *Real numbers* x, y such that $x = y$ holds $x \leq y$.

The *Real number* $0_{\overline{\mathbb{R}}}$ is defined by:

(Def.4) $0_{\overline{\mathbb{R}}} = 0$.

We now state several propositions:

(25) $0_{\overline{\mathbb{R}}} = 0$.

(26) For every *Real number* x holds $x + 0_{\overline{\mathbb{R}}} = x$ and $0_{\overline{\mathbb{R}}} + x = x$.

(27) $-\infty < 0_{\overline{\mathbb{R}}}$ and $0_{\overline{\mathbb{R}}} < +\infty$.

(28) For all *Real numbers* x, y, z such that $0_{\overline{\mathbb{R}}} \leq z$ and $0_{\overline{\mathbb{R}}} \leq x$ and $y = x + z$ holds $x \leq y$.

(29) For every real number x such that $x \in \mathbb{N}$ holds $0 \leq x$.

(30) For every *Real number* x such that $x \in \mathbb{N}$ holds $0_{\overline{\mathbb{R}}} \leq x$.

Let X, Y be non-empty subsets of $\overline{\mathbb{R}}$. Let us assume that neither $-\infty \in X$ and $+\infty \in Y$ nor $+\infty \in X$ and $-\infty \in Y$. The functor $X + Y$ yielding a non-empty subset of $\overline{\mathbb{R}}$ is defined as follows:

(Def.5) for every *Real number* z holds $z \in X + Y$ if and only if there exist *Real numbers* x, y such that $x \in X$ and $y \in Y$ and $z = x + y$.

We now state two propositions:

(31) For all non-empty subsets X, Y of $\overline{\mathbb{R}}$ such that neither $-\infty \in X$ and $+\infty \in Y$ nor $+\infty \in X$ and $-\infty \in Y$ for every *Real number* z holds $z \in X + Y$ if and only if there exist *Real numbers* x, y such that $x \in X$ and $y \in Y$ and $z = x + y$.

(32) Let X, Y, Z be non-empty subsets of $\overline{\mathbb{R}}$. Then if neither $-\infty \in X$ and $+\infty \in Y$ nor $+\infty \in X$ and $-\infty \in Y$, then $Z = X + Y$ if and only if for every *Real number* z holds $z \in Z$ if and only if there exist *Real numbers* x, y such that $x \in X$ and $y \in Y$ and $z = x + y$.

Let X be a non-empty subset of $\overline{\mathbb{R}}$. The functor $-X$ yielding a non-empty subset of $\overline{\mathbb{R}}$ is defined as follows:

(Def.6) for every *Real number* z holds $z \in -X$ if and only if there exists a *Real number* x such that $x \in X$ and $z = -x$.

Next we state a number of propositions:

(33) For every non-empty subset X of $\overline{\mathbb{R}}$ and for every *Real number* z holds $z \in -X$ if and only if there exists a *Real number* x such that $x \in X$ and $z = -x$.

(34) For all non-empty subsets X, Z of $\overline{\mathbb{R}}$ holds $Z = -X$ if and only if for every *Real number* z holds $z \in Z$ if and only if there exists a *Real number* x such that $x \in X$ and $z = -x$.

(35) For every non-empty subset X of $\overline{\mathbb{R}}$ holds $-(-X) = X$.

(36) For every non-empty subset X of $\overline{\mathbb{R}}$ and for every *Real number* z holds $z \in X$ if and only if $-z \in -X$.

(37) For all non-empty subsets X, Y of $\overline{\mathbb{R}}$ holds $X \subseteq Y$ if and only if $-X \subseteq -Y$.

(38) For every *Real number* z holds $z \in \mathbb{R}$ if and only if $-z \in \mathbb{R}$.

(39) Let X, Y be non-empty subsets of $\overline{\mathbb{R}}$. Then if neither $-\infty \in X$ and $+\infty \in Y$ nor $+\infty \in X$ and $-\infty \in Y$ and neither $\sup X = +\infty$ and $\sup Y = -\infty$ nor $\sup X = -\infty$ and $\sup Y = +\infty$, then $\sup(X + Y) \leq \sup X + \sup Y$.

(40) Let X, Y be non-empty subsets of $\overline{\mathbb{R}}$. Then if neither $-\infty \in X$ and $+\infty \in Y$ nor $+\infty \in X$ and $-\infty \in Y$ and neither $\inf X = +\infty$ and $\inf Y = -\infty$ nor $\inf X = -\infty$ and $\inf Y = +\infty$, then $\inf X + \inf Y \leq \inf(X + Y)$.

(41) For all non-empty subsets X, Y of $\overline{\mathbb{R}}$ such that X is upper bounded and Y is upper bounded holds $\sup(X + Y) \leq \sup X + \sup Y$.

(42) For all non-empty subsets X, Y of $\overline{\mathbb{R}}$ such that X is lower bounded and Y is lower bounded holds $\inf X + \inf Y \leq \inf(X + Y)$.

(43) For every non-empty subset X of $\overline{\mathbb{R}}$ and for every *Real number* a holds a is a majorant of X if and only if $-a$ is a minorant of $-X$.

(44) For every non-empty subset X of $\overline{\mathbb{R}}$ and for every *Real number* a holds a is a minorant of X if and only if $-a$ is a majorant of $-X$.

(45) For every non-empty subset X of $\overline{\mathbb{R}}$ holds $\inf(-X) = -\sup X$.

(46) For every non-empty subset X of $\overline{\mathbb{R}}$ holds $\sup(-X) = -\inf X$.

Let X be a non-empty set, and let Y be a non-empty subset of $\overline{\mathbb{R}}$, and let F be a function from X into Y. Then $\operatorname{rng} F$ is a non-empty subset of $\overline{\mathbb{R}}$.

Let X be a non-empty set, and let Y be a non-empty subset of $\overline{\mathbb{R}}$, and let F be a function from X into Y. The functor $\sup F$ yielding a *Real number* is defined by:

(Def.7) $\sup F = \sup(\operatorname{rng} F)$.

The following proposition is true

(47) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y holds $\sup F = \sup(\operatorname{rng} F)$.

Let X be a non-empty set, and let Y be a non-empty subset of $\overline{\mathbb{R}}$, and let F be a function from X into Y. The functor $\inf F$ yields a *Real number* and is defined by:

(Def.8) $\inf F = \inf(\operatorname{rng} F)$.

Next we state the proposition

(48) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y holds $\inf F = \inf(\operatorname{rng} F)$.

Let X be a non-empty set, and let Y be a non-empty subset of $\overline{\mathbb{R}}$, and let F be a function from X into Y, and let x be an element of X. Then $F(x)$ is a *Real number*.

The scheme *FunctR_ealEx* concerns a non-empty set $\mathcal{A}$, a set $\mathcal{B}$, and a unary functor $\mathcal{F}$ and states that:

there exists a function f from $\mathcal{A}$ into $\mathcal{B}$ such that for every element x of $\mathcal{A}$ holds $f(x) = \mathcal{F}(x)$

provided the parameters have the following property:

- for every element x of $\mathcal{A}$ holds $\mathcal{F}(x) \in \mathcal{B}$.

Let X be a non-empty set, and let Y, Z be non-empty subsets of $\overline{\mathbb{R}}$, and let F be a function from X into Y, and let G be a function from X into Z. Let us assume that neither $-\infty \in Y$ and $+\infty \in Z$ nor $+\infty \in Y$ and $-\infty \in Z$. The functor $F + G$ yields a function from X into $Y + Z$ and is defined by:

(Def.9) for every element x of X holds $(F + G)(x) = F(x) + G(x)$.

Next we state several propositions:

(49) Let X be a non-empty set. Let Y, Z be non-empty subsets of $\overline{\mathbb{R}}$. Suppose neither $-\infty \in Y$ and $+\infty \in Z$ nor $+\infty \in Y$ and $-\infty \in Z$. Then for every function F from X into Y and for every function G from X into Z and for every function H from X into $Y + Z$ holds $H = F + G$ if and only if for every element x of X holds $H(x) = F(x) + G(x)$.

(50) Let X be a non-empty set. Then for all non-empty subsets Y, Z of $\overline{\mathbb{R}}$ such that neither $-\infty \in Y$ and $+\infty \in Z$ nor $+\infty \in Y$ and $-\infty \in Z$ for every function F from X into Y and for every function G from X into Z and for every element x of X holds $(F + G)(x) = F(x) + G(x)$.

(51) For every non-empty set X and for all non-empty subsets Y, Z of $\overline{\mathbb{R}}$ such that neither $-\infty \in Y$ and $+\infty \in Z$ nor $+\infty \in Y$ and $-\infty \in Z$ for every function F from X into Y and for every function G from X into Z holds $\operatorname{rng}(F + G) \subseteq \operatorname{rng} F + \operatorname{rng} G$.

(52) Let X be a non-empty set. Let Y, Z be non-empty subsets of $\overline{\mathbb{R}}$. Suppose neither $-\infty \in Y$ and $+\infty \in Z$ nor $+\infty \in Y$ and $-\infty \in Z$. Then for

every function F from X into Y and for every function G from X into Z such that neither $\sup F = +\infty$ and $\sup G = -\infty$ nor $\sup F = -\infty$ and $\sup G = +\infty$ holds $\sup(F + G) \leq \sup F + \sup G$.

(53) Let X be a non-empty set. Let Y, Z be non-empty subsets of $\overline{\mathbb{R}}$. Suppose neither $-\infty \in Y$ and $+\infty \in Z$ nor $+\infty \in Y$ and $-\infty \in Z$. Then for every function F from X into Y and for every function G from X into Z such that neither $\inf F = +\infty$ and $\inf G = -\infty$ nor $\inf F = -\infty$ and $\inf G = +\infty$ holds $\inf F + \inf G \leq \inf(F + G)$.

Let X be a non-empty set, and let Y be a non-empty subset of $\overline{\mathbb{R}}$, and let F be a function from X into Y. The functor $-F$ yielding a function from X into $-Y$ is defined by:

(Def.10) for every element x of X holds $(-F)(x) = -F(x)$.

One can prove the following three propositions:

(54) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y and for every function G from X into $-Y$ holds $G = -F$ if and only if for every element x of X holds $G(x) = -F(x)$.

(55) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y holds $\operatorname{rng}(-F) = -\operatorname{rng} F$.

(56) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y holds $\inf(-F) = -\sup F$ and $\sup(-F) = -\inf F$.

Let X be a non-empty set, and let Y be a non-empty subset of $\overline{\mathbb{R}}$, and let F be a function from X into Y. We say that F is upper bounded if and only if:

(Def.11) $\sup F < +\infty$.

Let X be a non-empty set, and let Y be a non-empty subset of $\overline{\mathbb{R}}$, and let F be a function from X into Y. We say that F is lower bounded if and only if:

(Def.12) $-\infty < \inf F$.

Let X be a non-empty set, and let Y be a non-empty subset of $\overline{\mathbb{R}}$, and let F be a function from X into Y. We say that F is bounded if and only if:

(Def.13) F is upper bounded and F is lower bounded.

We now state a number of propositions:

(60)[1] For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y holds F is bounded if and only if $\sup F < +\infty$ and $-\infty < \inf F$.

(61) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y holds F is upper bounded if and only if $-F$ is lower bounded.

[1]The propositions (57)–(59) were either repeated or obvious.

(62) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y holds F is lower bounded if and only if $-F$ is upper bounded.

(63) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y holds F is bounded if and only if $-F$ is bounded.

(64) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y and for every element x of X holds $-\infty \leq F(x)$ and $F(x) \leq +\infty$.

(65) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y and for every element x of X such that $Y \subseteq \mathbb{R}$ holds $-\infty < F(x)$ and $F(x) < +\infty$.

(66) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y and for every element x of X holds $\inf F \leq F(x)$ and $F(x) \leq \sup F$.

(67) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y such that $Y \subseteq \mathbb{R}$ holds F is upper bounded if and only if $\sup F \in \mathbb{R}$.

(68) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y such that $Y \subseteq \mathbb{R}$ holds F is lower bounded if and only if $\inf F \in \mathbb{R}$.

(69) For every non-empty set X and for every non-empty subset Y of $\overline{\mathbb{R}}$ and for every function F from X into Y such that $Y \subseteq \mathbb{R}$ holds F is bounded if and only if $\inf F \in \mathbb{R}$ and $\sup F \in \mathbb{R}$.

(70) For every non-empty set X and for all non-empty subsets Y, Z of $\overline{\mathbb{R}}$ such that $Y \subseteq \mathbb{R}$ and $Z \subseteq \mathbb{R}$ for every function F_1 from X into Y and for every function F_2 from X into Z such that F_1 is upper bounded and F_2 is upper bounded holds $F_1 + F_2$ is upper bounded.

(71) For every non-empty set X and for all non-empty subsets Y, Z of $\overline{\mathbb{R}}$ such that $Y \subseteq \mathbb{R}$ and $Z \subseteq \mathbb{R}$ for every function F_1 from X into Y and for every function F_2 from X into Z such that F_1 is lower bounded and F_2 is lower bounded holds $F_1 + F_2$ is lower bounded.

(72) For every non-empty set X and for all non-empty subsets Y, Z of $\overline{\mathbb{R}}$ such that $Y \subseteq \mathbb{R}$ and $Z \subseteq \mathbb{R}$ for every function F_1 from X into Y and for every function F_2 from X into Z such that F_1 is bounded and F_2 is bounded holds $F_1 + F_2$ is bounded.

(73) There exists a function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F is one-to-one and $\mathbb{N} = \operatorname{rng} F$ and $\operatorname{rng} F$ is a non-empty subset of $\overline{\mathbb{R}}$.

A non-empty subset of $\overline{\mathbb{R}}$ is called a denumerable set of larged real if:

(Def.14) there exists a function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that it $= \operatorname{rng} F$.

Next we state the proposition

(75)[2] $\mathbb{N}$ is a denumerable set of larged real.

A denumerable set of larged real is said to be a denumerable set of positive larged real if:

(Def.15) for every *Real number* x such that $x \in$ it holds $0_{\overline{\mathbb{R}}} \leq x$.

Let D be a denumerable set of larged real. A function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ is said to be a numeration of D if:

(Def.16) $D = \mathrm{rng}\,\mathrm{it}$.

One can prove the following proposition

(78)[3] For every denumerable set D of positive larged real and for every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ holds F is a numeration of D if and only if $D = \mathrm{rng}\, F$.

Let N be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$, and let n be a natural number. Then $N(n)$ is a *Real number*.

We see that the *Real number* is an element of $\overline{\mathbb{R}}$.

The scheme *RecFuncExR_eal* concerns a *Real number* $\mathcal{A}$ and a binary functor $\mathcal{F}$ yielding a *Real number* and states that:

there exists a function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that $F(0) = \mathcal{A}$ and for every natural number n and for every *Real number* x such that $x = F(n)$ holds $F(n+1) = \mathcal{F}(n, x)$

for all values of the parameters.

We now state the proposition

(79) For every denumerable set D of larged real and for every numeration N of D there exists a function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that $F(0) = N(0)$ and for every natural number n and for every *Real number* y such that $y = F(n)$ holds $F(n+1) = y + N(n+1)$.

Let D be a denumerable set of larged real, and let N be a numeration of D. The functor $\mathrm{Ser}(D, N)$ yields a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ and is defined by:

(Def.17) $\mathrm{Ser}(D, N)(0) = N(0)$ and for every natural number n and for every *Real number* y such that $y = \mathrm{Ser}(D, N)(n)$ holds $\mathrm{Ser}(D, N)(n+1) = y + N(n+1)$.

The following propositions are true:

(80) Let D be a denumerable set of larged real. Then for every numeration N of D and for every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ holds $F = \mathrm{Ser}(D, N)$ if and only if $F(0) = N(0)$ and for every natural number n and for every *Real number* y such that $y = F(n)$ holds $F(n+1) = y + N(n+1)$.

(81) For every denumerable set D of larged real and for every numeration N of D holds $\mathrm{Ser}(D, N)(0) = N(0)$ and for every natural number n and for every *Real number* y such that $y = \mathrm{Ser}(D, N)(n)$ holds $\mathrm{Ser}(D, N)(n+1) = y + N(n+1)$.

(82) For every denumerable set D of positive larged real and for every numeration N of D and for every natural number n holds $0_{\overline{\mathbb{R}}} \leq N(n)$.

[2]The proposition (74) was either repeated or obvious.

[3]The propositions (76)–(77) were either repeated or obvious.

(83) For every denumerable set D of positive larged real and for every numeration N of D and for every natural number n holds $\mathrm{Ser}(D, N)(n) \leq \mathrm{Ser}(D, N)(n+1)$ and $0_{\overline{\mathbb{R}}} \leq \mathrm{Ser}(D, N)(n)$.

(84) For every denumerable set D of positive larged real and for every numeration N of D and for all natural numbers n, m holds $\mathrm{Ser}(D, N)(n) \leq \mathrm{Ser}(D, N)(n+m)$.

Let D be a denumerable set of larged real. A non-empty subset of $\overline{\mathbb{R}}$ is called a set of series of D if:

(Def.18) there exists a numeration N of D such that it $= \mathrm{rng}\,\mathrm{Ser}(D, N)$.

Let F be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Then $\mathrm{rng}\,F$ is a non-empty subset of $\overline{\mathbb{R}}$.

Let D be a denumerable set of positive larged real, and let N be a numeration of D. The functor $\sum_D N$ yields a *Real number* and is defined as follows:

(Def.19) $\sum_D N = \sup(\mathrm{rng}\,\mathrm{Ser}(D, N))$.

One can prove the following propositions:

(86)[4] For every denumerable set D of positive larged real and for every numeration N of D and for every *Real number* s holds $s = \sum_D N$ if and only if $s = \sup(\mathrm{rng}\,\mathrm{Ser}(D, N))$.

(87) For every denumerable set D of positive larged real and for every numeration N of D holds $\sum_D N = \sup(\mathrm{rng}\,\mathrm{Ser}(D, N))$.

Let D be a denumerable set of positive larged real, and let N be a numeration of D. We say that D is N sumable if and only if:

(Def.20) $\sum_D N \in \mathbb{R}$.

One can prove the following proposition

(89)[5] For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ holds $\mathrm{rng}\,F$ is a denumerable set of larged real.

Let F be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Then $\mathrm{rng}\,F$ is a denumerable set of larged real.

Next we state the proposition

(90) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ holds F is a numeration of $\mathrm{rng}\,F$.

Let F be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. The functor $\mathrm{Ser}\,F$ yields a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ and is defined by:

(Def.21) for every numeration N of $\mathrm{rng}\,F$ such that $N = F$ holds $\mathrm{Ser}\,F = \mathrm{Ser}(\mathrm{rng}\,F, N)$.

We now state the proposition

(91) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ and for every numeration N of $\mathrm{rng}\,F$ such that $N = F$ holds $\mathrm{Ser}\,F = \mathrm{Ser}(\mathrm{rng}\,F, N)$.

Let F be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. We say that F is non-negative if and only if:

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[4]The proposition (85) was either repeated or obvious.

[5]The proposition (88) was either repeated or obvious.

(Def.22) $\operatorname{rng} F$ is a denumerable set of positive larged real.

Let F be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Let us assume that F is non-negative. The functor $\sum F$ yields a *Real number* and is defined by:

(Def.23) $\sum F = \sup(\operatorname{rng} \operatorname{Ser} F)$.

The following propositions are true:

(93)[6] For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F is non-negative holds $\sum F = \sup(\operatorname{rng} \operatorname{Ser} F)$.

(94) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ holds F is non-negative if and only if for every natural number n holds $0_{\overline{\mathbb{R}}} \leq F(n)$.

(95) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ and for every natural number n such that F is non-negative holds $\operatorname{Ser} F(n) \leq \operatorname{Ser} F(n+1)$ and $0_{\overline{\mathbb{R}}} \leq \operatorname{Ser} F(n)$.

(96) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F is non-negative for all natural numbers n, m holds $\operatorname{Ser} F(n) \leq \operatorname{Ser} F(n+m)$.

(97) For all functions F_1, F_2 from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F_1 is non-negative holds if for every natural number n holds $F_1(n) \leq F_2(n)$, then for every natural number n holds $\operatorname{Ser} F_1(n) \leq \operatorname{Ser} F_2(n)$.

(98) For all functions F_1, F_2 from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F_1 is non-negative holds if for every natural number n holds $F_1(n) \leq F_2(n)$, then $\sum F_1 \leq \sum F_2$.

(99) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ holds $\operatorname{Ser} F(0) = F(0)$ and for every natural number n and for every *Real number* y such that $y = \operatorname{Ser} F(n)$ holds $\operatorname{Ser} F(n+1) = y + F(n+1)$.

(100) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F is non-negative holds if there exists a natural number n such that $F(n) = +\infty$, then $\sum F = +\infty$.

Let F be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Let us assume that F is non-negative. We say that F is sumable if and only if:

(Def.24) $\sum F \in \mathbb{R}$.

One can prove the following propositions:

(102)[7] For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F is non-negative holds if there exists a natural number n such that $F(n) = +\infty$, then F is not sumable.

(103) For all functions F_1, F_2 from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F_1 is non-negative holds if for every natural number n holds $F_1(n) \leq F_2(n)$, then if F_2 is sumable, then F_1 is sumable.

(104) For all functions F_1, F_2 from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F_1 is non-negative holds if for every natural number n holds $F_1(n) \leq F_2(n)$, then if F_1 is not sumable, then F_2 is not sumable.

(105) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F is non-negative for every natural number n such that for every natural number r such that $n \leq r$ holds $F(r) = 0_{\overline{\mathbb{R}}}$ holds $\sum F = \operatorname{Ser} F(n)$.

[6] The proposition (92) was either repeated or obvious.

[7] The proposition (101) was either repeated or obvious.

(106) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that for every natural number n holds $F(n) \in \mathbb{R}$ for every natural number n holds $\operatorname{Ser} F(n) \in \mathbb{R}$.

(107) For every function F from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that F is non-negative holds if there exists a natural number n such that for every natural number k such that $n \leq k$ holds $F(k) = 0_{\overline{\mathbb{R}}}$ and for every natural number k such that $k \leq n$ holds $F(k) \neq +\infty$, then F is sumable.

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