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Relations of Tolerance [1]

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Summary. Introduces notions of relations of tolerance, tolerance set and neighbourhood of an element. The basic properties of relations of tolerance are proved.

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The notation and terminology used here have been introduced in the following papers: [2], [3], [4], [5], and [1]. We adopt the following rules: X, Y, Z denote sets, x, y are arbitrary, and R denotes a relation between X and X. The following propositions are true:

(1) field $\varnothing = \emptyset$.

(2) $\varnothing$ is pseudo reflexive.

(3) $\varnothing$ is symmetric.

(4) $\varnothing$ is irreflexive.

(5) $\varnothing$ is antisymmetric.

(6) $\varnothing$ is asymmetric.

(7) $\varnothing$ is connected.

(8) $\varnothing$ is strongly connected.

(9) $\varnothing$ is transitive.

Let us consider X. The functor ∇_X yielding a relation between X and X is defined by:

(Def.1) $\nabla_X = [\!: X, X :\!]$.

Let us consider X, R, Y. Then $R \,|^2\, Y$ is a relation between Y and Y.

The following propositions are true:

(10) For every relation R between X and X holds $R = \nabla_X$ if and only if $R = [\!: X, X :\!]$.

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(11) $\nabla_X = [: X, X :]$.

(12) $\operatorname{dom} \nabla_X = X$.

(13) $\operatorname{rng} \nabla_X = X$.

(14) $\operatorname{field} \nabla_X = X$.

(15) For all x, y such that $x \in X$ and $y \in X$ holds $\langle x, y \rangle \in \nabla_X$.

(16) For all x, y such that $x \in \operatorname{field} \nabla_X$ and $y \in \operatorname{field} \nabla_X$ holds $\langle x, y \rangle \in \nabla_X$.

(17) ∇_X is pseudo reflexive.

(18) ∇_X is symmetric.

(19) ∇_X is strongly connected.

(20) ∇_X is transitive.

(21) ∇_X is connected.

Let us consider X. A relation between X and X is said to be a tolerance of X if:

(Def.2) it is pseudo reflexive and it is symmetric and field it $= X$.

In the sequel T, R denote tolerances of X. The following propositions are true:

(23)[2] For every tolerance R of X holds R is pseudo reflexive and R is symmetric and $\operatorname{field} R = X$.

(24) For every tolerance T of X holds $\operatorname{dom} T = X$.

(25) For every tolerance T of X holds $\operatorname{rng} T = X$.

(26) For every tolerance T of X holds $\operatorname{field} T = X$.

(27) For every tolerance T of X holds $x \in X$ if and only if $\langle x, x \rangle \in T$.

(28) For every tolerance T of X holds T is reflexive in X.

(29) For every tolerance T of X holds T is symmetric in X.

(30) For every tolerance T of X such that $\langle x, y \rangle \in T$ holds $\langle y, x \rangle \in T$.

(31) For every tolerance T of X and for all x, y such that $\langle x, y \rangle \in T$ holds $x \in X$ and $y \in X$.

(32) For every relation R between X and Y such that R is symmetric holds $R \mid^2 Z$ is symmetric.

Let us consider X, T, and let Y be a subset of X. Then $T \mid^2 Y$ is a tolerance of Y.

Next we state the proposition

(33) If $Y \subseteq X$, then $T \mid^2 Y$ is a tolerance of Y.

Let us consider X, and let T be a tolerance of X. A set is called a set of mutually elements w.r.t. T if:

(Def.3) for all x, y such that $x \in$ it and $y \in$ it holds $\langle x, y \rangle \in T$.

We now state the proposition

(34) $\emptyset$ is a set of mutually elements w.r.t. T.

[2]The proposition (22) was either repeated or obvious.

Let us consider X, and let T be a tolerance of X. A set of mutually elements w.r.t. T is called a tolerance class of T if:

(Def.4) for every x such that $x \notin$ it and $x \in X$ there exists y such that $y \in$ it and $\langle x, y \rangle \notin T$.

Next we state a number of propositions:

(36)[3] Y is a set of mutually elements w.r.t. T if and only if for all x, y such that $x \in Y$ and $y \in Y$ holds $\langle x, y \rangle \in T$.

(38)[4] For every tolerance T of X such that $\emptyset$ is a tolerance class of T holds $T = \varnothing$.

(39) $\varnothing$ is a tolerance of $\emptyset$.

(40) For all x, y such that $\langle x, y \rangle \in T$ holds $\{x, y\}$ is a set of mutually elements w.r.t. T.

(41) For every x such that $x \in X$ holds $\{x\}$ is a set of mutually elements w.r.t. T.

(42) For all Y, Z such that Y is a set of mutually elements w.r.t. T and Z is a set of mutually elements w.r.t. T holds $Y \cap Z$ is a set of mutually elements w.r.t. T.

(43) If Y is a set of mutually elements w.r.t. T, then $Y \subseteq X$.

(44) If Y is a tolerance class of T, then $Y \subseteq X$.

(45) For every set Y of mutually elements w.r.t. T there exists a tolerance class Z of T such that $Y \subseteq Z$.

(46) For all x, y such that $\langle x, y \rangle \in T$ there exists a tolerance class Z of T such that $x \in Z$ and $y \in Z$.

(47) For every x such that $x \in X$ there exists a tolerance class Z of T such that $x \in Z$.

Let us consider X. Then $\triangle_X$ is a tolerance of X.

We now state three propositions:

(48) ∇_X is a tolerance of X.

(49) $T \subseteq \nabla_X$.

(50) $\triangle_X \subseteq T$.

The scheme *ToleranceEx* concerns a set $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists a tolerance T of $\mathcal{A}$ such that for all x, y such that $x \in \mathcal{A}$ and $y \in \mathcal{A}$ holds $\langle x, y \rangle \in T$ if and only if $\mathcal{P}[x, y]$

provided the parameters satisfy the following conditions:

- for every x such that $x \in \mathcal{A}$ holds $\mathcal{P}[x, x]$,
- for all x, y such that $x \in \mathcal{A}$ and $y \in \mathcal{A}$ and $\mathcal{P}[x, y]$ holds $\mathcal{P}[y, x]$.

One can prove the following propositions:

[3] The proposition (35) was either repeated or obvious.

[4] The proposition (37) was either repeated or obvious.

(51) For every Y there exists a tolerance T of $\bigcup Y$ such that for every Z such that $Z \in Y$ holds Z is a set of mutually elements w.r.t. T.

(52) Let Y be a set. Let T, R be tolerances of $\bigcup Y$. Then if for all x, y holds $\langle x, y \rangle \in T$ if and only if there exists Z such that $Z \in Y$ and $x \in Z$ and $y \in Z$ and for all x, y holds $\langle x, y \rangle \in R$ if and only if there exists Z such that $Z \in Y$ and $x \in Z$ and $y \in Z$, then $T = R$.

(53) For all tolerances T, R of X such that for every Z holds Z is a tolerance class of T if and only if Z is a tolerance class of R holds $T = R$.

Let us consider X, and let T be a tolerance of X, and let us consider x. The functor neighbourhood(x, T) yielding a set is defined by:

(Def.5) for every y holds $y \in$ neighbourhood(x, T) if and only if $\langle x, y \rangle \in T$.

One can prove the following propositions:

(54) For every tolerance T of X and for every x and for every set Y holds $Y =$ neighbourhood(x, T) if and only if for every y holds $y \in Y$ if and only if $\langle x, y \rangle \in T$.

(55) For every tolerance T of X holds $y \in$ neighbourhood(x, T) if and only if $\langle x, y \rangle \in T$.

(56) If $x \in X$, then $x \in$ neighbourhood(x, T).

(57) neighbourhood$(x, T) \subseteq X$.

(58) For every Y such that for every set Z holds $Z \in Y$ if and only if $x \in Z$ and Z is a tolerance class of T holds neighbourhood$(x, T) = \bigcup Y$.

(59) For every Y such that for every Z holds $Z \in Y$ if and only if $x \in Z$ and Z is a set of mutually elements w.r.t. T holds neighbourhood$(x, T) = \bigcup Y$.

We now define two new functors. Let us consider X, and let T be a tolerance of X. The functor TolSets T yields a set and is defined by:

(Def.6) for every Y holds $Y \in$ TolSets T if and only if Y is a set of mutually elements w.r.t. T.

The functor TolClasses T yields a set and is defined by:

(Def.7) for every Y holds $Y \in$ TolClasses T if and only if Y is a tolerance class of T.

The following propositions are true:

(60) For every set Y and for every tolerance T of X holds $Y =$ TolSets T if and only if for every Z holds $Z \in Y$ if and only if Z is a set of mutually elements w.r.t. T.

(61) For every tolerance T of X and for every Z holds $Z \in$ TolSets T if and only if Z is a set of mutually elements w.r.t. T.

(62) For every set Y and for every tolerance T of X holds $Y =$ TolClasses T if and only if for every Z holds $Z \in Y$ if and only if Z is a tolerance class of T.

(63) For every tolerance T of X holds $Z \in$ TolClasses T if and only if Z is a tolerance class of T.

(64) If TolClasses $R \subseteq$ TolClasses T, then $R \subseteq T$.

(65) For all tolerances T, R of X such that TolClasses T = TolClasses R holds $T = R$.

(66) $\bigcup$(TolClasses T) $= X$.

(67) $\bigcup$(TolSets T) $= X$.

(68) If for every x such that $x \in X$ holds neighbourhood(x, T) is a set of mutually elements w.r.t. T, then T is transitive.

(69) If T is transitive, then for every x such that $x \in X$ holds
neighbourhood(x, T)
is a tolerance class of T.

(70) For every x and for every tolerance class Y of T such that $x \in Y$ holds $Y \subseteq$ neighbourhood(x, T).

(71) TolSets $R \subseteq$ TolSets T if and only if $R \subseteq T$.

(72) TolClasses $T \subseteq$ TolSets T.

(73) If for every x such that $x \in X$ holds
neighbourhood$(x, R) \subseteq$ neighbourhood(x, T),
then $R \subseteq T$.

(74) $T \subseteq T \cdot T$.

(75) If $T = T \cdot T$, then T is transitive.

References

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