

Calculus of Quantifiers. Deduction Theorem

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Summary. Some tautologies of the Classical Quantifier Calculus.
The deduction theorem is also proved.

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The papers [11], [13], [8], [2], [5], [3], [12], [10], [9], [1], [6], [4], and [7] provide the terminology and notation for this paper. For simplicity we adopt the following convention: X will denote a subset of CQC–WFF, F, G, p, q, r will denote elements of CQC–WFF, s, h will denote formulae, and x, y will denote bound variables. Next we state a number of propositions:

- (1) If $\vdash p \Rightarrow (q \Rightarrow r)$, then $\vdash p \wedge q \Rightarrow r$.
- (2) If $\vdash p \Rightarrow (q \Rightarrow r)$, then $\vdash q \wedge p \Rightarrow r$.
- (3) If $\vdash p \wedge q \Rightarrow r$, then $\vdash p \Rightarrow (q \Rightarrow r)$.
- (4) If $\vdash p \wedge q \Rightarrow r$, then $\vdash q \Rightarrow (p \Rightarrow r)$.
- (5) $y \in \text{snb}(\forall_x s)$ if and only if $y \in \text{snb}(s)$ and $y \neq x$.
- (6) $y \in \text{snb}(\exists_x s)$ if and only if $y \in \text{snb}(s)$ and $y \neq x$.
- (7) $y \in \text{snb}(s \Rightarrow h)$ if and only if $y \in \text{snb}(s)$ or $y \in \text{snb}(h)$.
- (8) $y \in \text{snb}(\neg s)$ if and only if $y \in \text{snb}(s)$.
- (9) $y \in \text{snb}(s \wedge h)$ if and only if $y \in \text{snb}(s)$ or $y \in \text{snb}(h)$.
- (10) $y \in \text{snb}(s \vee h)$ if and only if $y \in \text{snb}(s)$ or $y \in \text{snb}(h)$.
- (11) $x \notin \text{snb}(\forall_{x,y} s)$ and $y \notin \text{snb}(\forall_{x,y} s)$.
- (12) $x \notin \text{snb}(\exists_{x,y} s)$ and $y \notin \text{snb}(\exists_{x,y} s)$.
- (13) If F is closed, then $x \notin \text{snb}(F)$.
- (14) $s \Rightarrow h(x) = (s(x)) \Rightarrow (h(x))$.
- (15) $s \vee h(x) = (s(x)) \vee (h(x))$.

- (16) $\exists_x p(x) = \exists_x p.$
- (17) If $x \neq y$, then $\exists_x p(y) = \exists_x (p(y)).$
- (18) $\vdash p \Rightarrow \exists_x p.$
- (19) If $\vdash p$, then $\vdash \exists_x p.$
- (20) $\vdash \forall_x p \Rightarrow \exists_x p.$
- (21) $\vdash \forall_x p \Rightarrow \exists_y p.$
- (22) If $\vdash p \Rightarrow q$ and $x \notin \text{snb}(q)$, then $\vdash (\exists_x p) \Rightarrow q.$
- (23) If $x \notin \text{snb}(p)$, then $\vdash (\exists_x p) \Rightarrow p.$
- (24) If $x \notin \text{snb}(p)$ and $\vdash \exists_x p$, then $\vdash p.$
- (25) If $p = h(x)$ and $q = h(y)$ and $y \notin \text{snb}(h)$, then $\vdash p \Rightarrow \exists_y q.$
- (26) If $\vdash p$, then $\vdash \forall_x p.$
- (27) If $x \notin \text{snb}(p)$, then $\vdash p \Rightarrow \forall_x p.$
- (28) If $p = h(x)$ and $q = h(y)$ and $x \notin \text{snb}(h)$, then $\vdash \forall_x p \Rightarrow q.$
- (29) If $y \notin \text{snb}(p)$, then $\vdash \forall_x p \Rightarrow \forall_y p.$
- (30) If $p = h(x)$ and $q = h(y)$ and $x \notin \text{snb}(h)$ and $y \notin \text{snb}(p)$, then $\vdash \forall_x p \Rightarrow \forall_y q.$
- (31) If $x \notin \text{snb}(p)$, then $\vdash (\exists_x p) \Rightarrow \exists_y p.$

One can prove the following propositions:

- (32) If $p = h(x)$ and $q = h(y)$ and $x \notin \text{snb}(q)$ and $y \notin \text{snb}(h)$, then $\vdash (\exists_x p) \Rightarrow \exists_y q.$
- (34)¹ $\vdash \forall_x (p \Rightarrow q) \Rightarrow (\forall_x p \Rightarrow \forall_x q).$
- (35) If $\vdash \forall_x (p \Rightarrow q)$, then $\vdash \forall_x p \Rightarrow \forall_x q.$
- (36) $\vdash \forall_x (p \Leftrightarrow q) \Rightarrow (\forall_x p \Leftrightarrow \forall_x q).$
- (37) If $\vdash \forall_x (p \Leftrightarrow q)$, then $\vdash \forall_x p \Leftrightarrow \forall_x q.$
- (38) $\vdash \forall_x (p \Rightarrow q) \Rightarrow ((\exists_x p) \Rightarrow \exists_x q).$
- (39) If $\vdash \forall_x (p \Rightarrow q)$, then $\vdash (\exists_x p) \Rightarrow \exists_x q.$
- (40) $\vdash \forall_x (p \wedge q) \Rightarrow \forall_x p \wedge \forall_x q$ and $\vdash \forall_x p \wedge \forall_x q \Rightarrow \forall_x (p \wedge q).$
- (41) $\vdash \forall_x (p \wedge q) \Leftrightarrow \forall_x p \wedge \forall_x q.$
- (42) $\vdash \forall_x (p \wedge q)$ if and only if $\vdash \forall_x p \wedge \forall_x q.$
- (43) $\vdash \forall_x p \vee \forall_x q \Rightarrow \forall_x (p \vee q).$
- (44) $\vdash (\exists_x p \vee q) \Rightarrow (\exists_x p) \vee \exists_x q$ and $\vdash (\exists_x p) \vee \exists_x q \Rightarrow \exists_x p \vee q.$
- (45) $\vdash (\exists_x p \vee q) \Leftrightarrow (\exists_x p) \vee \exists_x q.$
- (46) $\vdash \exists_x p \vee q$ if and only if $\vdash (\exists_x p) \vee \exists_x q.$
- (47) $\vdash (\exists_x p \wedge q) \Rightarrow (\exists_x p) \wedge \exists_x q.$
- (48) If $\vdash \exists_x p \wedge q$, then $\vdash (\exists_x p) \wedge \exists_x q.$
- (49) $\vdash \forall_x \neg p \Rightarrow \forall_x p$ and $\vdash \forall_x p \Rightarrow \forall_x \neg p.$
- (50) $\vdash \forall_x \neg p \Leftrightarrow \forall_x p.$
- (51) $\vdash (\exists_x \neg p) \Rightarrow \exists_x p$ and $\vdash (\exists_x p) \Rightarrow \exists_x \neg p.$

¹The proposition (33) was either repeated or obvious.

- (52) $\vdash (\exists x \neg \neg p) \Leftrightarrow \exists x p.$
(53) $\vdash \neg \exists x \neg p \Rightarrow \forall x p$ and $\vdash \forall x p \Rightarrow \neg \exists x \neg p.$
(54) $\vdash \neg \exists x \neg p \Leftrightarrow \forall x p.$
(55) $\vdash \neg \forall x p \Rightarrow \exists x \neg p$ and $\vdash (\exists x \neg p) \Rightarrow \neg \forall x p.$
(56) $\vdash \neg \forall x p \Leftrightarrow \exists x \neg p.$
(57) $\vdash \neg \exists x p \Rightarrow \forall x \neg p$ and $\vdash \forall x \neg p \Rightarrow \neg \exists x p.$
(58) $\vdash \forall x \neg p \Leftrightarrow \neg \exists x p.$
(59) $\vdash \forall x \forall y p \Rightarrow \forall y \forall x p$ and $\vdash \forall x, y p \Rightarrow \forall y, x p.$
(60) If $p = h(x)$ and $q = h(y)$ and $y \notin \text{snb}(h)$, then $\vdash \forall x \forall y q \Rightarrow \forall x p.$
(61) $\vdash (\exists x \exists y p) \Rightarrow \exists y \exists x p$ and $\vdash (\exists x, y p) \Rightarrow (\exists y, x p).$
(62) If $p = h(x)$ and $q = h(y)$ and $y \notin \text{snb}(h)$, then $\vdash (\exists x p) \Rightarrow (\exists x, y q).$

We now state a number of propositions:

- (63) $\vdash (\exists x \forall y p) \Rightarrow \forall y \exists x p.$
(64) $\vdash \exists x p \Leftrightarrow p.$
(65) $\vdash (\exists x p \Rightarrow q) \Rightarrow (\forall x p \Rightarrow \exists x q)$ and $\vdash (\forall x p \Rightarrow \exists x q) \Rightarrow \exists x p \Rightarrow q.$
(66) $\vdash (\exists x p \Rightarrow q) \Leftrightarrow (\forall x p \Rightarrow \exists x q).$
(67) $\vdash \exists x p \Rightarrow q$ if and only if $\vdash \forall x p \Rightarrow \exists x q.$
(68) $\vdash \forall x (p \wedge q) \Rightarrow p \wedge \forall x q.$
(69) $\vdash \forall x (p \wedge q) \Rightarrow \forall x p \wedge q.$
(70) If $x \notin \text{snb}(p)$, then $\vdash p \wedge \forall x q \Rightarrow \forall x (p \wedge q).$
(71) If $x \notin \text{snb}(p)$ and $\vdash p \wedge \forall x q$, then $\vdash \forall x (p \wedge q).$
(72) If $x \notin \text{snb}(p)$, then $\vdash p \vee \forall x q \Rightarrow \forall x (p \vee q)$ and $\vdash \forall x (p \vee q) \Rightarrow p \vee \forall x q.$
(73) If $x \notin \text{snb}(p)$, then $\vdash p \vee \forall x q \Leftrightarrow \forall x (p \vee q).$
(74) If $x \notin \text{snb}(p)$, then $\vdash p \vee \forall x q$ if and only if $\vdash \forall x (p \vee q).$
(75) If $x \notin \text{snb}(p)$, then $\vdash p \wedge \exists x q \Rightarrow \exists x p \wedge q$ and $\vdash (\exists x p \wedge q) \Rightarrow p \wedge \exists x q.$
(76) If $x \notin \text{snb}(p)$, then $\vdash p \wedge \exists x q \Leftrightarrow \exists x p \wedge q.$
(77) If $x \notin \text{snb}(p)$, then $\vdash p \wedge \exists x q$ if and only if $\vdash \exists x p \wedge q.$
(78) If $x \notin \text{snb}(p)$, then $\vdash \forall x (p \Rightarrow q) \Rightarrow (p \Rightarrow \forall x q)$ and $\vdash (p \Rightarrow \forall x q) \Rightarrow \forall x (p \Rightarrow q).$
(79) If $x \notin \text{snb}(p)$, then $\vdash (p \Rightarrow \forall x q) \Leftrightarrow \forall x (p \Rightarrow q).$
(80) If $x \notin \text{snb}(p)$, then $\vdash \forall x (p \Rightarrow q)$ if and only if $\vdash p \Rightarrow \forall x q.$
(81) If $x \notin \text{snb}(q)$, then $\vdash (\exists x p \Rightarrow q) \Rightarrow (\forall x p \Rightarrow q).$
(82) $\vdash (\forall x p \Rightarrow q) \Rightarrow \exists x p \Rightarrow q.$
(83) If $x \notin \text{snb}(q)$, then $\vdash \forall x p \Rightarrow q$ if and only if $\vdash \exists x p \Rightarrow q.$
(84) If $x \notin \text{snb}(q)$, then $\vdash ((\exists x p) \Rightarrow q) \Rightarrow \forall x (p \Rightarrow q)$ and $\vdash \forall x (p \Rightarrow q) \Rightarrow ((\exists x p) \Rightarrow q).$
(85) If $x \notin \text{snb}(q)$, then $\vdash ((\exists x p) \Rightarrow q) \Leftrightarrow \forall x (p \Rightarrow q).$
(86) If $x \notin \text{snb}(q)$, then $\vdash (\exists x p) \Rightarrow q$ if and only if $\vdash \forall x (p \Rightarrow q).$
(87) If $x \notin \text{snb}(p)$, then $\vdash (\exists x p \Rightarrow q) \Rightarrow (p \Rightarrow \exists x q).$

- (88) $\vdash (p \Rightarrow \exists_x q) \Rightarrow \exists_x p \Rightarrow q$.
 (89) If $x \notin \text{snb}(p)$, then $\vdash (p \Rightarrow \exists_x q) \Leftrightarrow \exists_x p \Rightarrow q$.
 (90) If $x \notin \text{snb}(p)$, then $\vdash p \Rightarrow \exists_x q$ if and only if $\vdash \exists_x p \Rightarrow q$.
 (91) $\{p\} \vdash p$.
 (92) $\text{Cn}(\{p\} \cup \{q\}) = \text{Cn}\{p \wedge q\}$.
 (93) $\{p, q\} \vdash r$ if and only if $\{p \wedge q\} \vdash r$.

The following propositions are true:

- (94) If $X \vdash p$, then $X \vdash \forall_x p$.
 (95) If $x \notin \text{snb}(p)$, then $X \vdash \forall_x (p \Rightarrow q) \Rightarrow (p \Rightarrow \forall_x q)$.
 (96) If F is closed and $X \cup \{F\} \vdash G$, then $X \vdash F \Rightarrow G$.

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