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Mostowski's Fundamental Operations - Part II

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Summary. The article consists of two parts. The first part is translation of chapter II.3 of [18]. A section of $D_H(a)$ determined by f (symbolically $S_H(a,f)$) and a notion of predicative closure of a class are defined. It is proved that if following assumptions are satisfied: (o) $A = \bigcup_\xi A_\xi$, (i) $A_\xi \subset A_\eta$ for $\xi < \eta$, (ii) $A_\lambda = \bigcup_{\xi<\lambda} A_\lambda$ (λ is a limit number), (iii) $A_\xi \in A$, (iv) A_ξ is transitive, (v) $(x, y \in A) \to (x \cap y \in A)$, (vi) A is predicatively closed, then the axiom of power sets and the axiom of substitution are valid in A. The second part is continuation of [17]. It is proved that if a non-void transitive class is closed under the operations $A_1 - A_7$ then it is predicatively closed. At last sufficient criteria for a class to be a model of ZF-theory are formulated: if A_ξ satisfies o – iv and A is closed under the operations $A_1 - A_7$ then A is a model of ZF.

MML Identifier: ZF_FUND2.

The papers [21], [20], [3], [14], [15], [16], [8], [6], [7], [9], [12], [2], [1], [5], [11], [13], [19], [4], [10], [22], and [17] provide the terminology and notation for this paper. For simplicity we adopt the following rules: H will denote a ZF-formula, M, E will denote non-empty sets, e will denote an element of E, m will denote an element of M, v will denote a function from VAR into M, and f will denote a function from VAR into E. Let us consider H, M, v. The functor $\mathrm{S}_v(H)$ yields a subset of M and is defined by:

(Def.1) (i) $\mathrm{S}_v(H) = \{m : M, v(\frac{x_0}{m}) \models H\}$ if $x_0 \in \mathrm{Free}\, H$,

(ii) $\mathrm{S}_v(H) = \emptyset$, otherwise.

Let us consider M. We say that M is predicatively closed if and only if:

(Def.2) for all H, E, f such that $E \in M$ holds $\mathrm{S}_f(H) \in M$.

We now state the proposition

(1) If E is transitive, then $\mathrm{S}_{f(\frac{x_1}{e})}(\forall_{x_2}(x_2 \epsilon(x_0) \Rightarrow x_2 \epsilon(x_1))) = E \cap 2^e$.

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For simplicity we adopt the following convention: W denotes a universal class, Y denotes a subclass of W, a, b denote ordinals of W, and L denotes a transfinite sequence of non-empty sets from W. We now state several propositions:

(2) If for all a, b such that $a \in b$ holds $L(a) \subseteq L(b)$ and for every a holds $L(a) \in \bigcup L$ and $L(a)$ is transitive and $\bigcup L$ is predicatively closed, then $\bigcup L \models$ the axiom of power sets.

(3) Suppose that

(i) $\omega \in W$,

(ii) for all a, b such that $a \in b$ holds $L(a) \subseteq L(b)$,

(iii) for every a such that $a \neq \mathbf{0}$ and a is a limit ordinal number holds $L(a) = \bigcup(L \restriction a)$,

(iv) for every a holds $L(a) \in \bigcup L$ and $L(a)$ is transitive,

(v) $\bigcup L$ is predicatively closed.

Then for every H such that $\{x_0, x_1, x_2\}$ misses Free H holds $\bigcup L \models$ the axiom of substitution for H.

(4) $\mathrm{S}_v(H) = \{m : \{\langle \mathbf{0}, m \rangle\} \cup (v \cdot \mathrm{decode}) \restriction (\mathrm{code}(\mathrm{Free}\, H) \setminus \{\mathbf{0}\}) \in \mathrm{D}_M(H)\}$.

(5) If Y is closed w.r.t. A1-A7 and Y is transitive, then Y is predicatively closed.

(6) Suppose that

(i) $\omega \in W$,

(ii) for all a, b such that $a \in b$ holds $L(a) \subseteq L(b)$,

(iii) for every a such that $a \neq \mathbf{0}$ and a is a limit ordinal number holds $L(a) = \bigcup(L \restriction a)$,

(iv) for every a holds $L(a) \in \bigcup L$ and $L(a)$ is transitive,

(v) $\bigcup L$ is closed w.r.t. A1-A7.

Then $\bigcup L$ is a model of ZF.

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