FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Category Ens

Czesław Byliński
Warsaw University
Białystok

Summary. If V is any non-empty set of sets, we define $\mathbf{Ens}_V$ to be the category with the objects of all sets $X \in V$, morphisms of all mappings from X into Y, with the ususal composition of mappings. By a mapping we mean a triple $\langle X, Y, f \rangle$ where f is a function from X into Y. The notations and concepts included correspond to those presented in [11,9]. We also introduce representable functors to illustrate properties of the category **Ens**.

MML Identifier: `ENS_1`.

The notation and terminology used here are introduced in the following papers: [15], [16], [13], [2], [3], [7], [5], [1], [14], [10], [12], [4], [8], and [6].

Mappings

In the sequel V denotes a non-empty set and A, B denote elements of V. Let us consider V. The functor $\mathrm{Funcs}\, V$ yielding a non-empty set of functions is defined by:

(Def.1) $\mathrm{Funcs}\, V = \bigcup\{B^A\}$.

We now state three propositions:

(1) For an arbitrary f holds $f \in \mathrm{Funcs}\, V$ if and only if there exist A, B such that if $B = \emptyset$, then $A = \emptyset$ but f is a function from A into B.

(2) $B^A \subseteq \mathrm{Funcs}\, V$.

(3) For every non-empty subset W of V holds $\mathrm{Funcs}\, W \subseteq \mathrm{Funcs}\, V$.

In the sequel f is an element of $\mathrm{Funcs}\, V$. Let us consider V. The functor $\mathrm{Maps}\, V$ yielding a non-empty set is defined as follows:

(Def.2) $\mathrm{Maps}\, V = \{\langle\langle A, B\rangle, f\rangle : (B = \emptyset \Rightarrow A = \emptyset) \wedge f \text{ is a function from } A \text{ into } B\}$.

In the sequel m, m_1, m_2, m_3 are elements of $\mathrm{Maps}\, V$. One can prove the following four propositions:

ISSN 0777–4028

(4) There exist f, A, B such that $m = \langle\langle A, B\rangle, f\rangle$ but if $B = \emptyset$, then $A = \emptyset$ and f is a function from A into B.

(5) For every function f from A into B such that if $B = \emptyset$, then $A = \emptyset$ holds $\langle\langle A, B\rangle, f\rangle \in \mathrm{Maps}\, V$.

(6) $\mathrm{Maps}\, V \subseteq [: [: V, V :], \mathrm{Funcs}\, V :]$.

(7) For every non-empty subset W of V holds $\mathrm{Maps}\, W \subseteq \mathrm{Maps}\, V$.

We now define three new functors. Let us consider V, m. The functor $\mathrm{graph}(m)$ yields a function and is defined as follows:

(Def.3) $\mathrm{graph}(m) = m_{\mathbf{2}}$.

The functor $\mathrm{dom}\, m$ yields an element of V and is defined by:

(Def.4) $\mathrm{dom}\, m = (m_{\mathbf{1}})_{\mathbf{1}}$.

The functor $\mathrm{cod}\, m$ yielding an element of V is defined by:

(Def.5) $\mathrm{cod}\, m = (m_{\mathbf{1}})_{\mathbf{2}}$.

The following three propositions are true:

(8) $m = \langle\langle \mathrm{dom}\, m, \mathrm{cod}\, m\rangle, \mathrm{graph}(m)\rangle$.

(9) $\mathrm{cod}\, m \neq \emptyset$ or $\mathrm{dom}\, m = \emptyset$ but $\mathrm{graph}(m)$ is a function from $\mathrm{dom}\, m$ into $\mathrm{cod}\, m$.

(10) For every function f and for all sets A, B such that $\langle\langle A, B\rangle, f\rangle \in \mathrm{Maps}\, V$ holds if $B = \emptyset$, then $A = \emptyset$ but f is a function from A into B.

Let us consider V, A. The functor $\mathrm{id}(A)$ yields an element of $\mathrm{Maps}\, V$ and is defined by:

(Def.6) $\mathrm{id}(A) = \langle\langle A, A\rangle, \mathrm{id}_A\rangle$.

The following proposition is true

(11) $\mathrm{graph}(\mathrm{id}(A)) = \mathrm{id}_A$ and $\mathrm{dom}\,\mathrm{id}(A) = A$ and $\mathrm{cod}\,\mathrm{id}(A) = A$.

Let us consider V, m_1, m_2. Let us assume that $\mathrm{cod}\, m_1 = \mathrm{dom}\, m_2$. The functor $m_2 \cdot m_1$ yields an element of $\mathrm{Maps}\, V$ and is defined as follows:

(Def.7) $m_2 \cdot m_1 = \langle\langle \mathrm{dom}\, m_1, \mathrm{cod}\, m_2\rangle, \mathrm{graph}(m_2) \cdot \mathrm{graph}(m_1)\rangle$.

One can prove the following propositions:

(12) If $\mathrm{dom}\, m_2 = \mathrm{cod}\, m_1$, then $\mathrm{graph}((m_2 \cdot m_1)) = \mathrm{graph}(m_2) \cdot \mathrm{graph}(m_1)$ and $\mathrm{dom}(m_2 \cdot m_1) = \mathrm{dom}\, m_1$ and $\mathrm{cod}(m_2 \cdot m_1) = \mathrm{cod}\, m_2$.

(13) If $\mathrm{dom}\, m_2 = \mathrm{cod}\, m_1$ and $\mathrm{dom}\, m_3 = \mathrm{cod}\, m_2$, then $m_3 \cdot (m_2 \cdot m_1) = m_3 \cdot m_2 \cdot m_1$.

(14) $m \cdot \mathrm{id}(\mathrm{dom}\, m) = m$ and $\mathrm{id}(\mathrm{cod}\, m) \cdot m = m$.

Let us consider V, A, B. The functor $\mathrm{Maps}(A, B)$ yields a set and is defined by:

(Def.8) $\mathrm{Maps}(A, B) = \{\langle\langle A, B\rangle, f\rangle : \langle\langle A, B\rangle, f\rangle \in \mathrm{Maps}\, V\}$, where f ranges over elements of $\mathrm{Funcs}\, V$.

The following propositions are true:

(15) For every function f from A into B such that if $B = \emptyset$, then $A = \emptyset$ holds $\langle\langle A, B\rangle, f\rangle \in \mathrm{Maps}(A, B)$.

(16) If $m \in \mathrm{Maps}(A, B)$, then $m = \langle\langle A, B\rangle, \mathrm{graph}(m)\rangle$.

(17) $\mathrm{Maps}(A, B) \subseteq \mathrm{Maps}\,V$.

(18) $\mathrm{Maps}\,V = \bigcup\{\mathrm{Maps}(A, B)\}$.

(19) $m \in \mathrm{Maps}(A, B)$ if and only if $\mathrm{dom}\,m = A$ and $\mathrm{cod}\,m = B$.

(20) If $m \in \mathrm{Maps}(A, B)$, then $\mathrm{graph}(m) \in B^A$.

Let us consider V, m. We say that m is a surjection if and only if:

(Def.9) $\mathrm{rng}\,\mathrm{graph}(m) = \mathrm{cod}\,m$.

Category Ens

We now define four new functors. Let us consider V. The functor Dom_V yields a function from $\mathrm{Maps}\,V$ into V and is defined by:

(Def.10) for every m holds $\mathrm{Dom}_V(m) = \mathrm{dom}\,m$.

The functor Cod_V yields a function from $\mathrm{Maps}\,V$ into V and is defined as follows:

(Def.11) for every m holds $\mathrm{Cod}_V(m) = \mathrm{cod}\,m$.

The functor $\cdot_V$ yields a partial function from $[\!:\mathrm{Maps}\,V, \mathrm{Maps}\,V:\!]$ to $\mathrm{Maps}\,V$ and is defined as follows:

(Def.12) for all m_2, m_1 holds $\langle m_2, m_1\rangle \in \mathrm{dom}(\cdot_V)$ if and only if $\mathrm{dom}\,m_2 = \mathrm{cod}\,m_1$ and for all m_2, m_1 such that $\mathrm{dom}\,m_2 = \mathrm{cod}\,m_1$ holds $\cdot_V(\langle m_2, m_1\rangle) = m_2 \cdot m_1$.

The functor Id_V yields a function from V into $\mathrm{Maps}\,V$ and is defined by:

(Def.13) for every A holds $\mathrm{Id}_V(A) = \mathrm{id}(A)$.

Let us consider V. The functor $\mathbf{Ens}_V$ yields a category structure and is defined by:

(Def.14) $\mathbf{Ens}_V = \langle V, \mathrm{Maps}\,V, \mathrm{Dom}_V, \mathrm{Cod}_V, \cdot_V, \mathrm{Id}_V\rangle$.

We now state the proposition

(21) $\langle V, \mathrm{Maps}\,V, \mathrm{Dom}_V, \mathrm{Cod}_V, \cdot_V, \mathrm{Id}_V\rangle$ is a category.

Let us consider V. Then $\mathbf{Ens}_V$ is a category.

In the sequel a, b are objects of $\mathbf{Ens}_V$. Next we state the proposition

(22) A is an object of $\mathbf{Ens}_V$.

Let us consider V, A. The functor ${}^{@}A$ yielding an object of $\mathbf{Ens}_V$ is defined as follows:

(Def.15) ${}^{@}A = A$.

One can prove the following proposition

(23) a is an element of V.

Let us consider V, a. The functor ${}^{@}a$ yields an element of V and is defined by:

(Def.16) ${}^{@}a = a$.

In the sequel f, g denote morphisms of $\mathbf{Ens}_V$. The following proposition is true

(24) m is a morphism of $\mathbf{Ens}_V$.

Let us consider V, m. The functor ${}^@m$ yields a morphism of $\mathbf{Ens}_V$ and is defined as follows:

(Def.17) ${}^@m = m$.

One can prove the following proposition

(25) f is an element of $\operatorname{Maps} V$.

Let us consider V, f. The functor ${}^@f$ yields an element of $\operatorname{Maps} V$ and is defined as follows:

(Def.18) ${}^@f = f$.

One can prove the following propositions:

(26) $\operatorname{dom} f = \operatorname{dom}({}^@f)$ and $\operatorname{cod} f = \operatorname{cod}({}^@f)$.

(27) $\hom(a, b) = \operatorname{Maps}({}^@a, {}^@b)$.

(28) If $\operatorname{dom} g = \operatorname{cod} f$, then $g \cdot f = ({}^@g) \cdot ({}^@f)$.

(29) $\mathrm{id}_a = \mathrm{id}({}^@a)$.

(30) If $a = \emptyset$, then a is an initial object.

(31) If $\emptyset \in V$ and a is an initial object, then $a = \emptyset$.

(32) For every universal class W and for every object a of $\mathbf{Ens}_W$ such that a is an initial object holds $a = \emptyset$.

(33) If there exists arbitrary x such that $a = \{x\}$, then a is a terminal object.

(34) If $V \neq \{\emptyset\}$ and a is a terminal object, then there exists arbitrary x such that $a = \{x\}$.

(35) For every universal class W and for every object a of $\mathbf{Ens}_W$ such that a is a terminal object there exists arbitrary x such that $a = \{x\}$.

(36) f is monic if and only if $\operatorname{graph}(({}^@f))$ is one-to-one.

(37) If f is epi and there exists A and there exist arbitrary x_1, x_2 such that $x_1 \in A$ and $x_2 \in A$ and $x_1 \neq x_2$, then ${}^@f$ is a surjection.

(38) If ${}^@f$ is a surjection, then f is epi.

(39) For every universal class W and for every morphism f of $\mathbf{Ens}_W$ such that f is epi holds ${}^@f$ is a surjection.

(40) For every non-empty subset W of V holds $\mathbf{Ens}_W$ is full subcategory of $\mathbf{Ens}_V$.

Representable Functors

We follow a convention: C will be a category, a, b, c will be objects of C, and f, g, h, f', g' will be morphisms of C. Let us consider C. The functor $\operatorname{Hom}(C)$ yields a non-empty set and is defined as follows:

(Def.19) $\operatorname{Hom}(C) = \{\hom(a, b)\}$.

We now state two propositions:

(41) $\hom(a, b) \in \operatorname{Hom}(C)$.

(42) If $\hom(a, \operatorname{cod} f) = \emptyset$, then $\hom(a, \operatorname{dom} f) = \emptyset$ but if $\hom(\operatorname{dom} f, a) = \emptyset$, then $\hom(\operatorname{cod} f, a) = \emptyset$.

We now define two new functors. Let us consider C, a, f. The functor $\hom(a, f)$ yielding a function from $\hom(a, \operatorname{dom} f)$ into $\hom(a, \operatorname{cod} f)$ is defined by:

(Def.20) for every g such that $g \in \hom(a, \operatorname{dom} f)$ holds $(\hom(a, f))(g) = f \cdot g$.

The functor $\hom(f, a)$ yields a function from $\hom(\operatorname{cod} f, a)$ into $\hom(\operatorname{dom} f, a)$ and is defined by:

(Def.21) for every g such that $g \in \hom(\operatorname{cod} f, a)$ holds $(\hom(f, a))(g) = g \cdot f$.

We now state several propositions:

(43) $\hom(a, \mathrm{id}_c) = \mathrm{id}_{\hom(a,c)}$.

(44) $\hom(\mathrm{id}_c, a) = \mathrm{id}_{\hom(c,a)}$.

(45) If $\operatorname{dom} g = \operatorname{cod} f$, then $\hom(a, g \cdot f) = \hom(a, g) \cdot \hom(a, f)$.

(46) If $\operatorname{dom} g = \operatorname{cod} f$, then $\hom(g \cdot f, a) = \hom(f, a) \cdot \hom(g, a)$.

(47) $\langle\langle \hom(a, \operatorname{dom} f), \hom(a, \operatorname{cod} f)\rangle, \hom(a, f)\rangle$ is an element of $\mathrm{Maps}\,\mathrm{Hom}(C)$.

(48) $\langle\langle \hom(\operatorname{cod} f, a), \hom(\operatorname{dom} f, a)\rangle, \hom(f, a)\rangle$ is an element of $\mathrm{Maps}\,\mathrm{Hom}(C)$.

We now define two new functors. Let us consider C, a. The functor $\hom(a, -)$ yields a function from the morphisms of C into $\mathrm{Maps}\,\mathrm{Hom}(C)$ and is defined as follows:

(Def.22) for every f holds $(\hom(a, -))(f) = \langle\langle \hom(a, \operatorname{dom} f), \hom(a, \operatorname{cod} f)\rangle, \hom(a, f)\rangle$.

The functor $\hom(-, a)$ yields a function from the morphisms of C into $\mathrm{Maps}\,\mathrm{Hom}(C)$ and is defined as follows:

(Def.23) for every f holds $(\hom(-, a))(f) = \langle\langle \hom(\operatorname{cod} f, a), \hom(\operatorname{dom} f, a)\rangle, \hom(f, a)\rangle$.

The following propositions are true:

(49) If $\mathrm{Hom}(C) \subseteq V$, then $\hom(a, -)$ is a functor from C to $\mathbf{Ens}_V$.

(50) If $\mathrm{Hom}(C) \subseteq V$, then $\hom(-, a)$ is a contravariant functor from C into $\mathbf{Ens}_V$.

(51) If $\hom(\operatorname{dom} f, \operatorname{cod} f') = \emptyset$, then $\hom(\operatorname{cod} f, \operatorname{dom} f') = \emptyset$.

Let us consider C, f, g. The functor $\hom(f, g)$ yielding a function from $\hom(\operatorname{cod} f, \operatorname{dom} g)$ into $\hom(\operatorname{dom} f, \operatorname{cod} g)$ is defined by:

(Def.24) for every h such that $h \in \hom(\operatorname{cod} f, \operatorname{dom} g)$ holds $(\hom(f, g))(h) = g \cdot h \cdot f$.

We now state several propositions:

(52) $\langle\langle \hom(\operatorname{cod} f, \operatorname{dom} g), \hom(\operatorname{dom} f, \operatorname{cod} g)\rangle, \hom(f, g)\rangle$ is an element of $\mathrm{Maps}\,\mathrm{Hom}(C)$.

(53) $\hom(\mathrm{id}_a, f) = \hom(a, f)$ and $\hom(f, \mathrm{id}_a) = \hom(f, a)$.

(54) $\hom(\mathrm{id}_a, \mathrm{id}_b) = \mathrm{id}_{\hom(a,b)}$.

(55) $\hom(f, g) = \hom(\mathrm{dom}\, f, g) \cdot \hom(f, \mathrm{dom}\, g)$.

(56) If $\mathrm{cod}\, g = \mathrm{dom}\, f$ and $\mathrm{dom}\, g' = \mathrm{cod}\, f'$, then $\hom(f \cdot g, g' \cdot f') = \hom(g, g') \cdot \hom(f, f')$.

Let us consider C. The functor $\hom_C(-,-)$ yielding a function from the morphisms of $[\![C, C]\!]$ into Maps $\mathrm{Hom}(C)$ is defined as follows:

(Def.25) for all f, g holds $(\hom_C(-,-))(\langle f, g \rangle) = \langle \langle \hom(\mathrm{cod}\, f, \mathrm{dom}\, g), \hom(\mathrm{dom}\, f, \mathrm{cod}\, g) \rangle, \hom(f, g) \rangle$.

The following two propositions are true:

(57) $\hom(a, -) = (\mathrm{curry}(\hom_C(-,-)))(\mathrm{id}_a)$ and $\hom(-, a) = (\mathrm{curry}'(\hom_C(-,-)))(\mathrm{id}_a)$.

(58) If $\mathrm{Hom}(C) \subseteq V$, then $\hom_C(-,-)$ is a functor from $[\![C^{\mathrm{op}}, C]\!]$ to $\mathbf{Ens}_V$.

We now define two new functors. Let us consider V, C, a. Let us assume that $\mathrm{Hom}(C) \subseteq V$. The functor $\hom_V(a, -)$ yields a functor from C to $\mathbf{Ens}_V$ and is defined by:

(Def.26) $\hom_V(a, -) = \hom(a, -)$.

The functor $\hom_V(-, a)$ yields a contravariant functor from C into $\mathbf{Ens}_V$ and is defined as follows:

(Def.27) $\hom_V(-, a) = \hom(-, a)$.

Let us consider V, C. Let us assume that $\mathrm{Hom}(C) \subseteq V$. The functor $\hom_V^C(-,-)$ yielding a functor from $[\![C^{\mathrm{op}}, C]\!]$ to $\mathbf{Ens}_V$ is defined as follows:

(Def.28) $\hom_V^C(-,-) = \hom_C(-,-)$.

One can prove the following propositions:

(59) If $\mathrm{Hom}(C) \subseteq V$, then $(\hom_V(a, -))(f) = \langle \langle \hom(a, \mathrm{dom}\, f), \hom(a, \mathrm{cod}\, f) \rangle, \hom(a, f) \rangle$.

(60) If $\mathrm{Hom}(C) \subseteq V$, then $(\mathrm{Obj}(\hom_V(a, -)))(b) = \hom(a, b)$.

(61) If $\mathrm{Hom}(C) \subseteq V$, then $(\hom_V(-, a))(f) = \langle \langle \hom(\mathrm{cod}\, f, a), \hom(\mathrm{dom}\, f, a) \rangle, \hom(f, a) \rangle$.

(62) If $\mathrm{Hom}(C) \subseteq V$, then $(\mathrm{Obj}(\hom_V(-, a)))(b) = \hom(b, a)$.

(63) If $\mathrm{Hom}(C) \subseteq V$, then $(\hom_V^C(-,-))(\langle f^{\mathrm{op}}, g \rangle) = \langle \langle \hom(\mathrm{cod}\, f, \mathrm{dom}\, g), \hom(\mathrm{dom}\, f, \mathrm{cod}\, g) \rangle, \hom(f, g) \rangle$.

(64) If $\mathrm{Hom}(C) \subseteq V$, then $(\mathrm{Obj}(\hom_V^C(-,-)))(\langle a^{\mathrm{op}}, b \rangle) = \hom(a, b)$.

(65) If $\mathrm{Hom}(C) \subseteq V$, then $(\hom_V^C(-,-))(a^{\mathrm{op}}, -) = \hom_V(a, -)$.

(66) If $\mathrm{Hom}(C) \subseteq V$, then $(\hom_V^C(-,-))(-, a) = \hom_V(-, a)$.

Acknowledgements

I would like to thank Andrzej Trybulec for his useful sugestions and valuable comments.

References

[1] Grzegorz Bancerek. Curried and uncurried functions. *Formalized Mathematics*, 1(**3**):537–541, 1990.

[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[4] Czesław Byliński. Introduction to categories and functors. *Formalized Mathematics*, 1(**2**):409–420, 1990.

[5] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.

[6] Czesław Byliński. Opposite categories and contravariant functors. *Formalized Mathematics*, 2(**3**):419–424, 1991.

[7] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[8] Czesław Byliński. Subcategories and products of categories. *Formalized Mathematics*, 1(**4**):725–732, 1990.

[9] Sunders MacLane. *Categories for the working mathematician.* Springer, Berlin/Heilderberg/New York, 1972.

[10] Bogdan Nowak and Grzegorz Bancerek. Universal classes. *Formalized Mathematics*, 1(**3**):595–600, 1990.

[11] Zbigniew Semadeni and Antoni Wiweger. *Wstęp do teorii kategorii i funktorów.* Volume 45 of *Biblioteka Matematyczna*, PWN, Warszawa, 1978.

[12] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[13] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[14] Andrzej Trybulec. Function domains and Frænkel operator. *Formalized Mathematics*, 1(**3**):495–500, 1990.

[15] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[16] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

Received August 1, 1991