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The Euclidean Space

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Summary. The general definition of Euclidean Space.

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The papers [14], [6], [9], [8], [12], [1], [5], [10], [3], [13], [4], [15], [16], [7], [11], and [2] provide the notation and terminology for this paper. In the sequel k, n denote natural numbers and r denotes a real number. Let us consider n. The functor $\mathcal{R}^n$ yields a non-empty set of finite sequences of $\mathbb{R}$ and is defined as follows:

(Def.1) $\mathcal{R}^n = \mathbb{R}^n$.

In the sequel x will denote a finite sequence of elements of $\mathbb{R}$. The function $|\Box|_{\mathbb{R}}$ from $\mathbb{R}$ into $\mathbb{R}$ is defined as follows:

(Def.2) for every r holds $|\Box|_{\mathbb{R}}(r) = |r|$.

Let us consider x. The functor $|x|$ yields a finite sequence of elements of $\mathbb{R}$ and is defined as follows:

(Def.3) $|x| = |\Box|_{\mathbb{R}} \cdot x$.

Let us consider n. The functor $\langle\underbrace{0,\ldots,0}_{n}\rangle$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by:

(Def.4) $\langle\underbrace{0,\ldots,0}_{n}\rangle = n \longmapsto 0$ **qua** a real number .

Let us consider n. Then $\langle\underbrace{0,\ldots,0}_{n}\rangle$ is an element of $\mathcal{R}^n$.

In the sequel x, x_1, x_2, y denote elements of $\mathcal{R}^n$. One can prove the following proposition

(1) x is an element of $\mathbb{R}^n$.

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Let us consider n, x. Then $-x$ is an element of $\mathcal{R}^n$.

Let us consider n, x, y. Then $x+y$ is an element of $\mathcal{R}^n$. Then $x-y$ is an element of $\mathcal{R}^n$.

Let us consider n, r, x. Then $r\cdot x$ is an element of $\mathcal{R}^n$.

Let us consider n, x. Then $|x|$ is an element of $\mathbb{R}^n$.

Let us consider n, x. Then ${}^2x$ is an element of $\mathbb{R}^n$.

Let x be a finite sequence of elements of $\mathbb{R}$. The functor $|x|$ yielding a real number is defined by:

(Def.5) $|x| = \sqrt{\sum{}^2|x|}$.

Next we state a number of propositions:

(2) $\operatorname{len} x = n$.

(3) $\operatorname{dom} x = \operatorname{Seg} n$.

(4) If $k \in \operatorname{Seg} n$, then $x(k) \in \mathbb{R}$.

(5) If for every k such that $k \in \operatorname{Seg} n$ holds $x_1(k) = x_2(k)$, then $x_1 = x_2$.

(6) If $k \in \operatorname{Seg} n$ and $r = x(k)$, then $|x|(k) = |r|$.

(7) $|\langle\underbrace{0,\ldots,0}_{n}\rangle| = n \longmapsto 0$ **qua** a real number .

(8) $|-x| = |x|$.

(9) $|r\cdot x| = |r|\cdot|x|$.

(10) $|\langle\underbrace{0,\ldots,0}_{n}\rangle| = 0$.

(11) If $|x| = 0$, then $x = \langle\underbrace{0,\ldots,0}_{n}\rangle$.

(12) $|x| \geq 0$.

(13) $|-x| = |x|$.

(14) $|r\cdot x| = |r|\cdot|x|$.

(15) $|x_1+x_2| \leq |x_1|+|x_2|$.

(16) $|x_1-x_2| \leq |x_1|+|x_2|$.

(17) $|x_1|-|x_2| \leq |x_1+x_2|$.

(18) $|x_1|-|x_2| \leq |x_1-x_2|$.

(19) $|x_1-x_2| = 0$ if and only if $x_1 = x_2$.

(20) If $x_1 \neq x_2$, then $|x_1-x_2| > 0$.

(21) $|x_1-x_2| = |x_2-x_1|$.

(22) $|x_1-x_2| \leq |x_1-x|+|x-x_2|$.

Let us consider n. The functor ρ^n yields a function from $[:\mathcal{R}^n, \mathcal{R}^n:]$ into $\mathbb{R}$ and is defined by:

(Def.6) for all elements x, y of $\mathcal{R}^n$ holds $\rho^n(x, y) = |x-y|$.

Next we state two propositions:

(23) ${}^2(x-y) = {}^2(y-x)$.

(24) ρ^n is a metric of $\mathcal{R}^n$.

Let us consider n. The functor $\mathcal{E}^n$ yields a metric space and is defined by:

(Def.7) $\mathcal{E}^n = \langle \mathcal{R}^n, \rho^n \rangle$.

Let us consider n. The functor $\mathcal{E}^n_{\mathrm{T}}$ yielding a topological space is defined by:

(Def.8) $\mathcal{E}^n_{\mathrm{T}} = \mathcal{E}^n_{\mathrm{top}}$.

We adopt the following rules: p, p_1, p_2, p_3 will denote points of $\mathcal{E}^n_{\mathrm{T}}$ and x, x_1, x_2, y, y_1, y_2 will denote real numbers. One can prove the following four propositions:

(25) The carrier of $\mathcal{E}^n_{\mathrm{T}} = \mathcal{R}^n$.

(26) p is a function from $\mathrm{Seg}\, n$ into $\mathbb{R}$.

(27) p is a finite sequence of elements of $\mathbb{R}$.

(28) For every finite sequence f such that $f = p$ holds $\mathrm{len}\, f = n$.

Let us consider n. The functor $0_{\mathcal{E}^n_{\mathrm{T}}}$ yielding a point of $\mathcal{E}^n_{\mathrm{T}}$ is defined by:

(Def.9) $0_{\mathcal{E}^n_{\mathrm{T}}} = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

Let us consider n, p_1, p_2. The functor $p_1 + p_2$ yields a point of $\mathcal{E}^n_{\mathrm{T}}$ and is defined as follows:

(Def.10) for all elements p_1', p_2' of $\mathcal{R}^n$ such that $p_1' = p_1$ and $p_2' = p_2$ holds $p_1 + p_2 = p_1' + p_2'$.

One can prove the following propositions:

(29) $p_1 + p_2 = p_2 + p_1$.

(30) $p_1 + p_2 + p_3 = p_1 + (p_2 + p_3)$.

(31) $0_{\mathcal{E}^n_{\mathrm{T}}} + p = p$ and $p + 0_{\mathcal{E}^n_{\mathrm{T}}} = p$.

Let us consider x, n, p. The functor $x \cdot p$ yields a point of $\mathcal{E}^n_{\mathrm{T}}$ and is defined as follows:

(Def.11) for every element p' of $\mathcal{R}^n$ such that $p' = p$ holds $x \cdot p = x \cdot p'$.

Next we state several propositions:

(32) $x \cdot 0_{\mathcal{E}^n_{\mathrm{T}}} = 0_{\mathcal{E}^n_{\mathrm{T}}}$.

(33) $1 \cdot p = p$ and $0 \cdot p = 0_{\mathcal{E}^n_{\mathrm{T}}}$.

(34) $x \cdot y \cdot p = x \cdot (y \cdot p)$.

(35) If $x \cdot p = 0_{\mathcal{E}^n_{\mathrm{T}}}$, then $x = 0$ or $p = 0_{\mathcal{E}^n_{\mathrm{T}}}$.

(36) $x \cdot (p_1 + p_2) = x \cdot p_1 + x \cdot p_2$.

(37) $(x + y) \cdot p = x \cdot p + y \cdot p$.

(38) If $x \cdot p_1 = x \cdot p_2$, then $x = 0$ or $p_1 = p_2$.

Let us consider n, p. The functor $-p$ yields a point of $\mathcal{E}^n_{\mathrm{T}}$ and is defined as follows:

(Def.12) for every element p' of $\mathcal{R}^n$ such that $p' = p$ holds $-p = -p'$.

We now state several propositions:

(39) $--p = p$.

(40) $p + -p = 0_{\mathcal{E}^n_{\rm T}}$ and $-p + p = 0_{\mathcal{E}^n_{\rm T}}$.

(41) If $p_1 + p_2 = 0_{\mathcal{E}^n_{\rm T}}$, then $p_1 = -p_2$ and $p_2 = -p_1$.

(42) $-(p_1 + p_2) = -p_1 + -p_2$.

(43) $-p = (-1) \cdot p$.

(44) $-x \cdot p = (-x) \cdot p$ and $-x \cdot p = x \cdot -p$.

Let us consider n, p_1, p_2. The functor $p_1 - p_2$ yields a point of $\mathcal{E}^n_{\rm T}$ and is defined by:

(Def.13) for all elements p'_1, p'_2 of $\mathcal{R}^n$ such that $p'_1 = p_1$ and $p'_2 = p_2$ holds $p_1 - p_2 = p'_1 - p'_2$.

One can prove the following propositions:

(45) $p_1 - p_2 = p_1 + -p_2$.

(46) $p - p = 0_{\mathcal{E}^n_{\rm T}}$.

(47) If $p_1 - p_2 = 0_{\mathcal{E}^n_{\rm T}}$, then $p_1 = p_2$.

(48) $-(p_1 - p_2) = p_2 - p_1$ and $-(p_1 - p_2) = -p_1 + p_2$.

(49) $p_1 + (p_2 - p_3) = (p_1 + p_2) - p_3$.

(50) $p_1 - (p_2 + p_3) = p_1 - p_2 - p_3$.

(51) $p_1 - (p_2 - p_3) = (p_1 - p_2) + p_3$.

(52) $p = (p + p_1) - p_1$ and $p = (p - p_1) + p_1$.

(53) $x \cdot (p_1 - p_2) = x \cdot p_1 - x \cdot p_2$.

(54) $(x - y) \cdot p = x \cdot p - y \cdot p$.

In the sequel p, p_1, p_2 will be points of $\mathcal{E}^2_{\rm T}$. Next we state the proposition

(55) There exist x, y such that $p = \langle x, y \rangle$.

We now define two new functors. Let us consider p. The functor $p_{\mathbf{1}}$ yields a real number and is defined by:

(Def.14) for every finite sequence f such that $p = f$ holds $p_{\mathbf{1}} = f(1)$.

The functor $p_{\mathbf{2}}$ yielding a real number is defined by:

(Def.15) for every finite sequence f such that $p = f$ holds $p_{\mathbf{2}} = f(2)$.

Let us consider x, y. The functor $[x, y]$ yields a point of $\mathcal{E}^2_{\rm T}$ and is defined as follows:

(Def.16) $[x, y] = \langle x, y \rangle$.

The following propositions are true:

(56) $[x, y]_{\mathbf{1}} = x$ and $[x, y]_{\mathbf{2}} = y$.

(57) $p = [p_{\mathbf{1}}, p_{\mathbf{2}}]$.

(58) $0_{\mathcal{E}^2_{\rm T}} = [0, 0]$.

(59) $p_1 + p_2 = [p_{1\mathbf{1}} + p_{2\mathbf{1}}, p_{1\mathbf{2}} + p_{2\mathbf{2}}]$.

(60) $[x_1, y_1] + [x_2, y_2] = [x_1 + x_2, y_1 + y_2]$.

(61) $x \cdot p = [x \cdot p_{\mathbf{1}}, x \cdot p_{\mathbf{2}}]$.

(62) $x \cdot [x_1, y_1] = [x \cdot x_1, x \cdot y_1]$.

(63) $-p = [-p_{\mathbf{1}}, -p_{\mathbf{2}}]$.

(64) $-[x_1, y_1] = [-x_1, -y_1]$.

(65) $p_1 - p_2 = [p_{1\mathbf{1}} - p_{2\mathbf{1}}, p_{1\mathbf{2}} - p_{2\mathbf{2}}]$.

(66) $[x_1, y_1] - [x_2, y_2] = [x_1 - x_2, y_1 - y_2]$.

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