FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Cartesian Product of Functions

Grzegorz Bancerek
Warsaw University
Białystok

Summary. A supplement of [3] and [2], i.e. some useful and explanatory properties of the product and also the curried and uncurried functions are shown. Besides, the functions yielding functions are considered: two different products and other operation of such functions are introduced. Finally, two facts are presented: quasi-distributivity of the power of the set to other one w.r.t. the union ($X^{\biguplus_x f(x)} \approx \prod_x X^{f(x)}$) and quasi-distributivity of the poroduct w.r.t. the raising to the power ($\prod_x f(x)^X \approx (\prod_x f(x))^X$).

MML Identifier: `FUNCT_6`.

The articles [16], [14], [8], [17], [5], [12], [9], [11], [6], [4], [13], [15], [7], [10], [2], [1], and [3] provide the notation and terminology for this paper.

Properties of Cartesian product

For simplicity we follow the rules: x, y, y_1, y_2, z, a will be arbitrary, f, g, h, h', f_1, f_2 will denote functions, i will denote a natural number, X, Y, Z, V_1, V_2 will denote sets, P will denote a permutation of X, D, D_1, D_2, D_3 will denote non-empty sets, d_1 will denote an element of D_1, d_2 will denote an element of D_2, and d_3 will denote an element of D_3. We now state a number of propositions:

(1) $x \in \prod\langle X\rangle$ if and only if there exists y such that $y \in X$ and $x = \langle y\rangle$.

(2) $z \in \prod\langle X, Y\rangle$ if and only if there exist x, y such that $x \in X$ and $y \in Y$ and $z = \langle x, y\rangle$.

(3) $a \in \prod\langle X, Y, Z\rangle$ if and only if there exist x, y, z such that $x \in X$ and $y \in Y$ and $z \in Z$ and $a = \langle x, y, z\rangle$.

(4) $\prod\langle D\rangle = D^1$.

(5) $\prod\langle D_1, D_2\rangle = \{\langle d_1, d_2\rangle\}$.

(6) $\prod\langle D, D\rangle = D^2$.

ISSN 0777–4028

(7) $\prod\langle D_1, D_2, D_3\rangle = \{\langle d_1, d_2, d_3\rangle\}$.

(8) $\prod\langle D, D, D\rangle = D^3$.

(9) $\prod(i \longmapsto D) = D^i$.

(10) $\prod f \subseteq (\bigcup f)^{\mathrm{dom}\, f}$.

Curried and uncurried functions of some functions

The following propositions are true:

(11) If $x \in \mathrm{dom} \curvearrowleft f$, then there exist y, z such that $x = \langle y, z\rangle$.

(12) $\curvearrowleft([\!: X, Y :\!] \longmapsto z) = [\!: Y, X :\!] \longmapsto z$.

(13) $\mathrm{curry}\, f = \mathrm{curry}' \curvearrowleft f$ and $\mathrm{uncurry}\, f = \curvearrowleft \mathrm{uncurry}'\, f$.

(14) If $[\!: X, Y :\!] \neq \emptyset$, then $\mathrm{curry}([\!: X, Y :\!] \longmapsto z) = X \longmapsto (Y \longmapsto z)$ and $\mathrm{curry}'([\!: X, Y :\!] \longmapsto z) = Y \longmapsto (X \longmapsto z)$.

(15) $\mathrm{uncurry}(X \longmapsto (Y \longmapsto z)) = [\!: X, Y :\!] \longmapsto z$ and $\mathrm{uncurry}'(X \longmapsto (Y \longmapsto z)) = [\!: Y, X :\!] \longmapsto z$.

(16) If $x \in \mathrm{dom}\, f$ and $g = f(x)$, then $\mathrm{rng}\, g \subseteq \mathrm{rng}\, \mathrm{uncurry}\, f$ and $\mathrm{rng}\, g \subseteq \mathrm{rng}\, \mathrm{uncurry}'\, f$.

(17) $\mathrm{dom}\, \mathrm{uncurry}(X \longmapsto f) = [\!: X, \mathrm{dom}\, f :\!]$ and $\mathrm{rng}\, \mathrm{uncurry}(X \longmapsto f) \subseteq \mathrm{rng}\, f$ and $\mathrm{dom}\, \mathrm{uncurry}'(X \longmapsto f) = [\!: \mathrm{dom}\, f, X :\!]$ and $\mathrm{rng}\, \mathrm{uncurry}'(X \longmapsto f) \subseteq \mathrm{rng}\, f$.

(18) If $X \neq \emptyset$, then $\mathrm{rng}\, \mathrm{uncurry}(X \longmapsto f) = \mathrm{rng}\, f$ and $\mathrm{rng}\, \mathrm{uncurry}'(X \longmapsto f) = \mathrm{rng}\, f$.

(19) If $[\!: X, Y :\!] \neq \emptyset$ and $f \in Z^{[\!: X, Y :\!]}$, then $\mathrm{curry}\, f \in (Z^Y)^X$ and $\mathrm{curry}'\, f \in (Z^X)^Y$.

(20) If $f \in (Z^Y)^X$, then $\mathrm{uncurry}\, f \in Z^{[\!: X, Y :\!]}$ and $\mathrm{uncurry}'\, f \in Z^{[\!: Y, X :\!]}$.

(21) If $\mathrm{curry}\, f \in (Z^Y)^X$ or $\mathrm{curry}'\, f \in (Z^X)^Y$ but $\mathrm{dom}\, f \subseteq [\!: V_1, V_2 :\!]$, then $f \in Z^{[\!: X, Y :\!]}$.

(22) If $\mathrm{uncurry}\, f \in Z^{[\!: X, Y :\!]}$ or $\mathrm{uncurry}'\, f \in Z^{[\!: Y, X :\!]}$ but $\mathrm{rng}\, f \subseteq V_1 \dot{\rightarrow} V_2$ and $\mathrm{dom}\, f = X$, then $f \in (Z^Y)^X$.

(23) If $f \in [\!: X, Y :\!] \dot{\rightarrow} Z$, then $\mathrm{curry}\, f \in X \dot{\rightarrow} (Y \dot{\rightarrow} Z)$ and $\mathrm{curry}'\, f \in Y \dot{\rightarrow} (X \dot{\rightarrow} Z)$.

(24) If $f \in X \dot{\rightarrow} (Y \dot{\rightarrow} Z)$, then $\mathrm{uncurry}\, f \in [\!: X, Y :\!] \dot{\rightarrow} Z$ and $\mathrm{uncurry}'\, f \in [\!: Y, X :\!] \dot{\rightarrow} Z$.

(25) If $\mathrm{curry}\, f \in X \dot{\rightarrow} (Y \dot{\rightarrow} Z)$ or $\mathrm{curry}'\, f \in Y \dot{\rightarrow} (X \dot{\rightarrow} Z)$ but $\mathrm{dom}\, f \subseteq [\!: V_1, V_2 :\!]$, then $f \in [\!: X, Y :\!] \dot{\rightarrow} Z$.

(26) If $\mathrm{uncurry}\, f \in [\!: X, Y :\!] \dot{\rightarrow} Z$ or $\mathrm{uncurry}'\, f \in [\!: Y, X :\!] \dot{\rightarrow} Z$ but $\mathrm{rng}\, f \subseteq V_1 \dot{\rightarrow} V_2$ and $\mathrm{dom}\, f \subseteq X$, then $f \in X \dot{\rightarrow} (Y \dot{\rightarrow} Z)$.

Functions yielding functions

Let X be a set. The functor $\mathrm{Sub_f}\, X$ is defined as follows:

(Def.1) $x \in \mathrm{Sub_f}\, X$ if and only if $x \in X$ and x is a function.

Next we state four propositions:

(27) $\mathrm{Sub_f}\, X \subseteq X$.

(28) $x \in f^{-1}\, \mathrm{Sub_f}\, \mathrm{rng}\, f$ if and only if $x \in \mathrm{dom}\, f$ and $f(x)$ is a function.

(29) $\mathrm{Sub_f}\, \emptyset = \emptyset$ and $\mathrm{Sub_f}\{f\} = \{f\}$ and $\mathrm{Sub_f}\{f, g\} = \{f, g\}$ and $\mathrm{Sub_f}\{f, g, h\} = \{f, g, h\}$.

(30) If $Y \subseteq \mathrm{Sub_f}\, X$, then $\mathrm{Sub_f}\, Y = Y$.

We now define three new functors. Let f be a function. The functor $\mathrm{dom}_\kappa\, f(\kappa)$ yielding a function is defined by:

(Def.2) $\mathrm{dom}(\mathrm{dom}_\kappa\, f(\kappa)) = f^{-1}\, \mathrm{Sub_f}\, \mathrm{rng}\, f$ and for every x such that $x \in f^{-1}\, \mathrm{Sub_f}\, \mathrm{rng}\, f$ holds $(\mathrm{dom}_\kappa\, f(\kappa))(x) = \pi_1(f(x))$.

The functor $\mathrm{rng}_\kappa\, f(\kappa)$ yields a function and is defined as follows:

(Def.3) $\mathrm{dom}(\mathrm{rng}_\kappa\, f(\kappa)) = f^{-1}\, \mathrm{Sub_f}\, \mathrm{rng}\, f$ and for every x such that $x \in f^{-1}\, \mathrm{Sub_f}\, \mathrm{rng}\, f$ holds $(\mathrm{rng}_\kappa\, f(\kappa))(x) = \pi_2(f(x))$.

The functor $\bigcap f$ is defined as follows:

(Def.4) $\bigcap f = \bigcap \mathrm{rng}\, f$.

Next we state a number of propositions:

(31) If $x \in \mathrm{dom}\, f$ and $g = f(x)$, then $x \in \mathrm{dom}(\mathrm{dom}_\kappa\, f(\kappa))$ and $(\mathrm{dom}_\kappa\, f(\kappa))(x) = \mathrm{dom}\, g$ and $x \in \mathrm{dom}(\mathrm{rng}_\kappa\, f(\kappa))$ and $(\mathrm{rng}_\kappa\, f(\kappa))(x) = \mathrm{rng}\, g$.

(32) $\mathrm{dom}_\kappa\, \square(\kappa) = \square$ and $\mathrm{rng}_\kappa\, \square(\kappa) = \square$.

(33) $\mathrm{dom}_\kappa \langle f \rangle(\kappa) = \langle \mathrm{dom}\, f \rangle$ and $\mathrm{rng}_\kappa \langle f \rangle(\kappa) = \langle \mathrm{rng}\, f \rangle$.

(34) $\mathrm{dom}_\kappa \langle f, g \rangle(\kappa) = \langle \mathrm{dom}\, f, \mathrm{dom}\, g \rangle$ and $\mathrm{rng}_\kappa \langle f, g \rangle(\kappa) = \langle \mathrm{rng}\, f, \mathrm{rng}\, g \rangle$.

(35) $\mathrm{dom}_\kappa \langle f, g, h \rangle(\kappa) = \langle \mathrm{dom}\, f, \mathrm{dom}\, g, \mathrm{dom}\, h \rangle$ and $\mathrm{rng}_\kappa \langle f, g, h \rangle(\kappa) = \langle \mathrm{rng}\, f, \mathrm{rng}\, g, \mathrm{rng}\, h \rangle$.

(36) $\mathrm{dom}_\kappa(X \longmapsto f)(\kappa) = X \longmapsto \mathrm{dom}\, f$ and $\mathrm{rng}_\kappa(X \longmapsto f)(\kappa) = X \longmapsto \mathrm{rng}\, f$.

(37) If $f \neq \square$, then $x \in \bigcap f$ if and only if for every y such that $y \in \mathrm{dom}\, f$ holds $x \in f(y)$.

(38) $\bigcup \square = \emptyset$ and $\bigcap \square = \emptyset$.

(39) $\bigcup \langle X \rangle = X$ and $\bigcap \langle X \rangle = X$.

(40) $\bigcup \langle X, Y \rangle = X \cup Y$ and $\bigcap \langle X, Y \rangle = X \cap Y$.

(41) $\bigcup \langle X, Y, Z \rangle = X \cup Y \cup Z$ and $\bigcap \langle X, Y, Z \rangle = X \cap Y \cap Z$.

(42) $\bigcup (\emptyset \longmapsto Y) = \emptyset$ and $\bigcap (\emptyset \longmapsto Y) = \emptyset$.

(43) If $X \neq \emptyset$, then $\bigcup (X \longmapsto Y) = Y$ and $\bigcap (X \longmapsto Y) = Y$.

Let f be a function, and let x, y be arbitrary. The functor $f(x)(y)$ is defined by:

(Def.5) $f(x)(y) = (\mathrm{uncurry}\, f)(\langle\!\langle x, y \rangle\!\rangle)$.

We now state several propositions:

(44) If $x \in \mathrm{dom}\, f$ and $g = f(x)$ and $y \in \mathrm{dom}\, g$, then $f(x)(y) = g(y)$.

(45) If $x \in \operatorname{dom} f$, then $\langle f\rangle(1)(x) = f(x)$ and $\langle f, g\rangle(1)(x) = f(x)$ and $\langle f, g, h\rangle(1)(x) = f(x)$.

(46) If $x \in \operatorname{dom} g$, then $\langle f, g\rangle(2)(x) = g(x)$ and $\langle f, g, h\rangle(2)(x) = g(x)$.

(47) If $x \in \operatorname{dom} h$, then $\langle f, g, h\rangle(3)(x) = h(x)$.

(48) If $x \in X$ and $y \in \operatorname{dom} f$, then $(X \longmapsto f)(x)(y) = f(y)$.

Cartesian product of functions with the same domain

Let f be a function. The functor $\prod^* f$ yielding a function is defined as follows:

(Def.6) $\prod^* f = \operatorname{curry}(\operatorname{uncurry}' f \restriction [: \bigcap(\operatorname{dom}_\kappa f(\kappa)), \operatorname{dom} f :])$.

We now state several propositions:

(49) $\operatorname{dom} \prod^* f = \bigcap(\operatorname{dom}_\kappa f(\kappa))$ and $\operatorname{rng} \prod^* f \subseteq \prod(\operatorname{rng}_\kappa f(\kappa))$.

(50) If $x \in \operatorname{dom} \prod^* f$, then $(\prod^* f)(x)$ is a function.

(51) If $x \in \operatorname{dom} \prod^* f$ and $g = (\prod^* f)(x)$, then $\operatorname{dom} g = f^{-1} \operatorname{Sub_f} \operatorname{rng} f$ and for every y such that $y \in \operatorname{dom} g$ holds $\langle y, x\rangle \in \operatorname{dom} \operatorname{uncurry} f$ and $g(y) = (\operatorname{uncurry} f)(\langle y, x\rangle)$.

(52) If $x \in \operatorname{dom} \prod^* f$, then for every g such that $g \in \operatorname{rng} f$ holds $x \in \operatorname{dom} g$.

(53) If $g \in \operatorname{rng} f$ and for every g such that $g \in \operatorname{rng} f$ holds $x \in \operatorname{dom} g$, then $x \in \operatorname{dom} \prod^* f$.

(54) If $x \in \operatorname{dom} f$ and $g = f(x)$ and $y \in \operatorname{dom} \prod^* f$ and $h = (\prod^* f)(y)$, then $g(y) = h(x)$.

(55) If $x \in \operatorname{dom} f$ and $f(x)$ is a function and $y \in \operatorname{dom} \prod^* f$, then $f(x)(y) = (\prod^* f)(y)(x)$.

Cartesian product of functions

Let f be a function. The functor $\prod^\circ f$ yielding a function is defined by the conditions (Def.7).

(Def.7) (i) $\operatorname{dom} \prod^\circ f = \prod(\operatorname{dom}_\kappa f(\kappa))$,

(ii) for every g such that $g \in \prod(\operatorname{dom}_\kappa f(\kappa))$ there exists h such that $(\prod^\circ f)(g) = h$ and $\operatorname{dom} h = f^{-1} \operatorname{Sub_f} \operatorname{rng} f$ and for every x such that $x \in \operatorname{dom} h$ holds $h(x) = (\operatorname{uncurry} f)(\langle x, g(x)\rangle)$.

The following propositions are true:

(56) If $g \in \prod(\operatorname{dom}_\kappa f(\kappa))$ and $x \in \operatorname{dom} g$, then $(\prod^\circ f)(g)(x) = f(x)(g(x))$.

(57) If $x \in \operatorname{dom} f$ and $g = f(x)$ and $h \in \prod(\operatorname{dom}_\kappa f(\kappa))$ and $h' = (\prod^\circ f)(h)$, then $h(x) \in \operatorname{dom} g$ and $h'(x) = g(h(x))$ and $h' \in \prod(\operatorname{rng}_\kappa f(\kappa))$.

(58) $\operatorname{rng} \prod^\circ f = \prod(\operatorname{rng}_\kappa f(\kappa))$.

(59) If $\square \notin \operatorname{rng} f$, then $\prod^\circ f$ is one-to-one if and only if for every g such that $g \in \operatorname{rng} f$ holds g is one-to-one.

Properties of Cartesian products of functions

The following propositions are true:

(60) $\prod^* \square = \square$ and $\prod^\circ \square = \{\square\} \longmapsto \square$.

(61) $\operatorname{dom} \prod^* \langle h \rangle = \operatorname{dom} h$ and for every x such that $x \in \operatorname{dom} h$ holds $(\prod^* \langle h \rangle)(x) = \langle h(x) \rangle$.

(62) $\operatorname{dom} \prod^* \langle f_1, f_2 \rangle = \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ and for every x such that $x \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ holds $(\prod^* \langle f_1, f_2 \rangle)(x) = \langle f_1(x), f_2(x) \rangle$.

(63) If $X \neq \emptyset$, then $\operatorname{dom} \prod^*(X \longmapsto f) = \operatorname{dom} f$ and for every x such that $x \in \operatorname{dom} f$ holds $(\prod^*(X \longmapsto f))(x) = X \longmapsto f(x)$.

(64) $\operatorname{dom} \prod^\circ \langle h \rangle = \prod \langle \operatorname{dom} h \rangle$ and $\operatorname{rng} \prod^\circ \langle h \rangle = \prod \langle \operatorname{rng} h \rangle$ and for every x such that $x \in \operatorname{dom} h$ holds $(\prod^\circ \langle h \rangle)(\langle x \rangle) = \langle h(x) \rangle$.

(65) (i) $\operatorname{dom} \prod^\circ \langle f_1, f_2 \rangle = \prod \langle \operatorname{dom} f_1, \operatorname{dom} f_2 \rangle$,

(ii) $\operatorname{rng} \prod^\circ \langle f_1, f_2 \rangle = \prod \langle \operatorname{rng} f_1, \operatorname{rng} f_2 \rangle$,

(iii) for all x, y such that $x \in \operatorname{dom} f_1$ and $y \in \operatorname{dom} f_2$ holds $(\prod^\circ \langle f_1, f_2 \rangle)(\langle x, y \rangle) = \langle f_1(x), f_2(y) \rangle$.

(66) $\operatorname{dom} \prod^\circ(X \longmapsto f) = (\operatorname{dom} f)^X$ and $\operatorname{rng} \prod^\circ(X \longmapsto f) = (\operatorname{rng} f)^X$ and for every g such that $g \in (\operatorname{dom} f)^X$ holds $(\prod^\circ(X \longmapsto f))(g) = f \cdot g$.

(67) If $x \in \operatorname{dom} f_1$ and $x \in \operatorname{dom} f_2$, then for all y_1, y_2 holds $\langle f_1, f_2 \rangle(x) = \langle y_1, y_2 \rangle$ if and only if $(\prod^* \langle f_1, f_2 \rangle)(x) = \langle y_1, y_2 \rangle$.

(68) If $x \in \operatorname{dom} f_1$ and $y \in \operatorname{dom} f_2$, then for all y_1, y_2 holds $[\!: f_1, f_2 :\!](\langle x, y \rangle) = \langle y_1, y_2 \rangle$ if and only if $(\prod^\circ \langle f_1, f_2 \rangle)(\langle x, y \rangle) = \langle y_1, y_2 \rangle$.

(69) If $\operatorname{dom} f = X$ and $\operatorname{dom} g = X$ and for every x such that $x \in X$ holds $f(x) \approx g(x)$, then $\prod f \approx \prod g$.

(70) If $\operatorname{dom} f = \operatorname{dom} h$ and $\operatorname{dom} g = \operatorname{rng} h$ and h is one-to-one and for every x such that $x \in \operatorname{dom} h$ holds $f(x) \approx g(h(x))$, then $\prod f \approx \prod g$.

(71) If $\operatorname{dom} f = X$, then $\prod f \approx \prod(f \cdot P)$.

Function yielding powers

Let us consider f, X. The functor X^f yielding a function is defined by:

(Def.8) $\operatorname{dom}(X^f) = \operatorname{dom} f$ and for every x such that $x \in \operatorname{dom} f$ holds $X^f(x) = X^{f(x)}$.

We now state several propositions:

(72) If $\emptyset \notin \operatorname{rng} f$, then $\emptyset^f = \operatorname{dom} f \longmapsto \emptyset$.

(73) $X^\square = \square$.

(74) $Y^{\langle X \rangle} = \langle Y^X \rangle$.

(75) $Z^{\langle X, Y \rangle} = \langle Z^X, Z^Y \rangle$.

(76) $Z^{X \longmapsto Y} = X \longmapsto Z^Y$.

(77) $X^{\bigcup \operatorname{disjoin} f} \approx \prod(X^f)$.

Let us consider X, f. The functor f^X yielding a function is defined by:

(Def.9) $\mathrm{dom}(f^X) = \mathrm{dom}\, f$ and for every x such that $x \in \mathrm{dom}\, f$ holds $f^X(x) = f(x)^X$.

Next we state several propositions:

(78) $f^{\emptyset} = \mathrm{dom}\, f \longmapsto \{\Box\}$.

(79) $\Box^X = \Box$.

(80) $\langle Y \rangle^X = \langle Y^X \rangle$.

(81) $\langle Y, Z \rangle^X = \langle Y^X, Z^X \rangle$.

(82) $(Y \longmapsto Z)^X = Y \longmapsto Z^X$.

(83) $\prod(f^X) \approx (\prod f)^X$.

References

[1] Grzegorz Bancerek. Cardinal arithmetics. *Formalized Mathematics*, 1(**3**):543–547, 1990.

[2] Grzegorz Bancerek. Curried and uncurried functions. *Formalized Mathematics*, 1(**3**):537–541, 1990.

[3] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.

[4] Grzegorz Bancerek. Zermelo theorem and axiom of choice. *Formalized Mathematics*, 1(**2**):265–267, 1990.

[5] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[6] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.

[7] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[8] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[9] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[10] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.

[11] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[12] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.

[13] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[14] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.

[15] Andrzej Trybulec. Function domains and Frænkel operator. *Formalized Mathematics*, 1(**3**):495–500, 1990.

[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[17] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

Received September 30, 1991